Allowing Two Moves in Succession
Increases the Game’s Bias: a Theorem

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Allowing two moves in succession should not increase the game’s bias, but it does: a seeming paradox. Some games are biased in the sense that whoever starts the game has an advantage. For example, in chess, it is widely believed that a player who plays the whites (and who, therefore, has the first move) have an advantage over the player who plays the blacks.

In such games, in some sense, each move of the 1st player “adds” some “bias” to the “value” of the position. With this interpretation in mind, if we consider, e.g., the sequence of four moves, the resulting bias should be the same whether we allow the players to play as usual: 1st, 2nd, 1st, 2nd, or whether we allow each player to make two moves at the same time: in both cases, the first player makes exactly two moves, so, informally, in both cases, the result is biased “twice”.

However, in practice (e.g., in chess), if we allow each player to make two moves in succession, the bias noticeably increases. This empirical fact seems to be in contradiction with the above understanding of bias.

In this paper, we explain this seeming paradox, and prove that for four-move games, allowing players to make two moves in succession does increase the game’s bias.

Motivations for the following definitions: formalizing bias. In order to formulate the problem in precise terms, let us describe what bias means.

We are considering zero-sum games like chess, in which one person’s win is another person’s loss (see, e.g., [1, 2]). So, to describe the payoffs, it is sufficient to describe the payoffs of the first player: the payoffs for the second player would be exactly opposite. The first player wants to maximize his payoff, the second player, accordingly, wants to maximize his payoffs, i.e., equivalently, minimize the first player’s payoffs.

In this paper, we will follow traditional game theory and assume that both
players can come up with the optimal strategies (for a possible formalism that
enables us to take into account the fact that players are not perfect optimizers,
see, e.g., [3]).

Our ultimate goal is to consider the first four moves of the game. Before we
do that, let us first handle the first two moves, $m_1$ and $m_2$. Normally, the first
player makes the move $m_1$, and then the second player makes the move $m_2$. For
each sequence of moves $(m_1, m_2)$, let $P_2(m_1, m_2)$ denote the expected payoff
to the first player. After the first player makes a move $m_1$, the second player
selects a move that would maximize his reward, i.e., equivalently, minimize the
first player’s payoff $P_2(m_1, m_2)$. So, after each move $m_1$, the expected payoff
of the first player is equal to

$$\min_{m_2} P(m_1, m_2).$$

The first player wants to maximize his payoff, so he chooses a move $m_1$ for
which this quantity takes the largest possible value. His resulting payoff is

$$\max_{m_1} \min_{m_2} P_2(m_1, m_2). \quad (1)$$

If the players switch turns, i.e., if the second player started and the first player
made the next move, the resulting payoff for the first player would be, similarly,
equal to

$$\min_{m_1} \max_{m_2} P_2(m_1, m_2).$$

For such two-move sequences, the fact that the game is favorably biased towards
the first player means that the first player gets a better payoff if he starts first,
i.e.,

$$\max_{m_1} \min_{m_2} P_2(m_1, m_2) > \min_{m_1} \max_{m_2} P_2(m_1, m_2). \quad (2)$$

Let us now consider four moves. Let $P_4(m_1, m_2, m_3, m_4)$ denote the expected
payoff of the first player after the four moves $m_i$. The moves are, normally, done
in turn, so that the 1st player makes the move $m_1$, the 2nd player makes the
move $m_2$, the 1st player makes the move $m_3$, and then the 2nd player makes the
move $m_4$. We want to define the “bias” by saying that in any position, if the
first player starts a two-move sequence, he gets a better payoff. In particular, we
can consider the positions after a first move $m_1$. In this case, after a two-move
sequence $(m_2, m_3)$, it is the second player’s turn, so the expected payoff after
the first three moves is

$$P_3(m_1, m_2, m_3) = \min_{m_4} P_4(m_1, m_2, m_3, m_4). \quad (3)$$

In our case, the move $m_1$ is fixed, so we can use the idea of the formula (2),
with

$$\tilde{P}_2(m_2, m_3) = P_3(m_1, m_2, m_3), \quad (4)$$

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to describe the fact that the game is favorably biased towards the first player:

$$\max_{m_2} \min_{m_4} \hat{P}_2(m_2, m_3) > \min_{m_2} \max_{m_3} \hat{P}_2(m_2, m_3). \quad (5)$$

**Motivations for the following definitions:** describing the payoff for the normal game and for the game which allows two moves in succession. Similarly to the way we derived the formula (1), we can conclude that for the normal game, the payoff for the first player is equal to

$$\max_{m_1} \min_{m_2} \max_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \quad (6)$$

Using the same idea, we can conclude that in the game in which the first player makes the first two moves and the second player makes the next two moves, the expected payoff of the first player is equal to

$$\max_{m_1} \max_{m_2} \min_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \quad (7)$$

Now, we are ready for the formal result.

**Definition 1.** By a four-move game (or simply game, for short), we mean a triple $G = (M, M, P_4)$, where:

- $M$ be a finite set; its elements are called moves;
- $M \subseteq M^4$ be a set of four-move sequences;
- $P : M \rightarrow R$ is a function from the set $M$ to the set of all real numbers.

**Definition 2.** For every game $G$:

- by an expected payoff for the first player, we mean a number
  $$E(G) = \max_{m_1} \min_{m_2} \max_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4); \quad (6a)$$

- by an expected payoff for the first player when we allow two moves in succession, we mean a number
  $$E_2(G) = \max_{m_1} \max_{m_2} \min_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \quad (7a)$$
**Definition 3.** We say that a game is favorably biased towards the first player (or simply biased, for short), if for every $m_1$, we have
\[
\max_{m_2} \min_{m_3} \tilde{P}_2(m_2, m_3) > \min_{m_2} \max_{m_3} \tilde{P}_2(m_2, m_3),
\]
where $\tilde{P}_2(m_2, m_3)$ is defined by the formula
\[
\tilde{P}_2(m_2, m_3) = P_3(m_1, m_2, m_3),
\]
and $P_3$ is defined by the formula
\[
P_3(m_1, m_2, m_3) = \min_{m_4} P_4(m_1, m_2, m_3, m_4).
\]

**Proposition.** If a game is favorably biased towards the first player, then the expected payoff for the first player increases if we allow two moves in succession (i.e., $E_2(G) > E(G)$).

**Comment.** This result is in good accordance with common sense, according to which, when fighting against a mightier opponent, it is better to respond after each attack than to wait. This is often true in sports, this is often true in debate, this is often true in international policy.

**Proof.** If we substitute the definition (4) of $\tilde{P}_2$ into the formula (5), we conclude that the fact that the game is biased towards the first player means that for every move $m_1$, we have
\[
\max_{m_2} \min_{m_3} P_3(m_1, m_2, m_3) > \min_{m_2} \max_{m_3} P_3(m_1, m_2, m_3).
\]
Substituting the expression (3) for $P_3$ into this formula, we conclude that
\[
\max_{m_2} \min_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4) > \min_{m_2} \max_{m_3} \min_{m_4} P_4(m_1, m_2, m_3, m_4). \tag{8}
\]
This inequality is true for each of finitely many possible first moves, so, the largest possible value of the left-hand sides must be larger than the larger possible value of the right-hand sides. According to Definition 2, the largest possible value of the left-hand side is $E_2(G)$, while the largest possible value of the right-hand sides is $E(G)$. Thus, $E_2(G) > E(G)$. The proposition is proven.

**Acknowledgments.** This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grant No. MDA904-98-1-0361.

The author are thankful to Michael Gelfond and Ilya Molchanov for valuable discussions.
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