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Product of Partially Ordered Sets (Posets), with Potential Applications to Uncertainty Logic and Space-Time Geometry

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Abstract

One of the main objectives of science and engineering is to help people select the most beneficial decisions. To make these decisions,

- we must know people's preferences,
- we must have the information about different events – possible consequences of different decisions, and
- since information is never absolutely accurate and precise, we must also have information about the degree of certainty.

All these types of information naturally lead to partial orders:

- For preferences, $a \prec b$ means that b is preferable to a . This relation is used in decision theory.
- For events, $a \prec b$ means that a can influence b . This causality relation is used in space-time physics.
- For uncertain statements, $a \prec b$ means that a is less certain than b . This relation is used in logics describing uncertainty such as fuzzy logic.

In many practical situations, we are analyzing a complex system that consists of several subsystems. Each subsystem can be described as a separate ordered space. To get a description of the system as a whole, we must therefore combine these ordered spaces into a single space that describes the whole system.

In this paper, we consider the general problem of how to combine two ordered spaces A_1 and A_2 into one. We also analyze which properties of orders are preserved under the resulting products.

1 Formulation of the Problem.

Partially ordered sets (posets) in space-time geometry. Starting from general relativity, space-time models are usually formulated in terms of appropriate physical fields, e.g., a metric field; see, e.g., [5]. These fields assume that the space-time is smooth. However, there are important situations of non-smoothness:

- *singularities* like the Big Bang or a black hole, and
- *quantum fluctuations*.

According to modern physics (see, e.g., [5]), a proper description of the corresponding non-smooth space-time models means that we no longer have a metric field, we only have a *causality* relation \preceq between events – a partial order; see, e.g., [1, 4, 8, 9].

Comment. We will use the standard notation $a \prec b$ meaning that $a \preceq b$ and $a \neq b$, i.e., equivalently, that $a \preceq b$ and $b \not\preceq a$.

Products of space-time posets. Sometimes, we need to consider *pairs* of events – e.g., in situations like quantum entanglement, situations of importance to quantum computing [7]. How to extend partial orders on posets A_1 and A_2 to a partial order on the set $A_1 \times A_2$ of all such pairs?

Posets in uncertainty logic: need for products. A similar partial order \preceq is useful in describing degrees of expert's certainty, where $a \preceq a'$ means that a corresponds to less certainty than a' ; see, e.g., [2, 6].

Sometimes, two (or more) experts evaluate a statement S . Then, our certainty in S is described by a *pair* (a_1, a_2) , where $a_i \in A_i$ is the i -th expert's degree of certainty. When our certainty in S is described by a *pair* $(a_1, a_2) \in A_1 \times A_2$, we must define a *partial order* on the set $A_1 \times A_2$ of all pairs.

What we do in this paper. In this paper, we consider the general problem of how to combine two ordered spaces A_1 and A_2 into one. We also analyze what properties of orders are preserved under the resulting products.

Comment. Some of our results were presented at the 14th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN2010, Lyon, France, September 27-30, 2010 [10].

2 Products of Partially Ordered Sets: What Is Known

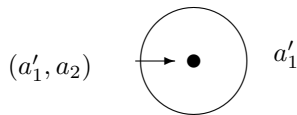
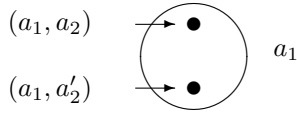
Known examples of product operations. At present, two product operations are known [1, 8]:

- *Cartesian product:* $(a_1, a_2) \preceq (a'_1, a'_2) \Leftrightarrow (a_1 \preceq_1 a'_1 \ \& \ a_2 \prec_2 a'_2)$, and
- *lexicographic product*

$$(a_1, a_2) \preceq (a'_1, a'_2) \Leftrightarrow ((a_1 \preceq a'_1 \ \& \ a_1 \neq a'_1) \vee (a_1 = a'_1 \ \& \ a_2 \preceq a'_2)).$$

Physical meaning of lexicographic order. For space-time models, a possible meaning of a lexicographic product $A_1 \times A_2$ is that A_1 is *macroscopic* space-time, and A_2 is *microscopic* space-time. When a'_1 macroscopically precedes a_1 , i.e., when $a'_1 \prec_1 a_1$, then, of course, the microscopic events should not matter – and we should have $(a'_1, a'_2) \preceq (a_1, a_2)$.

On the other hand, when $a'_1 = a_1$, i.e., when, from the macroscopic viewpoint, the two events a'_1 and a_1 are indistinguishable, we need to go to the microscopic level to see which of these two events causally influences another one, i.e., $(a_1, a'_2) \preceq (a_1, a_2) \Leftrightarrow a'_2 \preceq_2 a_2$.



Logical meaning of Cartesian product. The Cartesian product means that our confidence in S is higher than in S' if and only if it is higher for both experts. In other words, the Cartesian product corresponds to a *maximally cautious* approach.

Logical meaning of lexicographic product. In contrast, a lexicographic product means that we have *absolute confidence* in the first expert, and we only use the opinion of the 2nd expert when, to the 1st expert, the degrees of certainty are equivalent.

3 Products of Partially Ordered Sets: Towards a General Description

A natural question. A natural question is: what other operations are possible?

What we prove in this section. In this section, we prove that every non-degenerate product operation satisfying the above properties coincides with one of these two products.

Reasonable assumptions on the product. It is reasonable to assume that the validity of the relation $(a_1, a_2) \preceq (a'_1, a'_2)$ depends only on whether $a_1 \preceq_1 a'_1$, $a'_1 \preceq_1 a_1$, $a_2 \preceq_2 a'_2$, and/or $a'_2 \preceq_2 a_2$.

It is also reasonable to assume that if $a_1 \preceq_1 a'_1$ and $a_2 \preceq_2 a'_2$ then

$$(a_1, a_2) \preceq (a'_1, a'_2).$$

Definition 1.

- By a product operation, we mean a Boolean function $P : \{T, F\}^4 \rightarrow \{T, F\}$.
- For every two partially ordered sets A_1 and A_2 , we define the following relation on $A_1 \times A_2$:

$$(a_1, a_2) \preceq (a'_1, a'_2) \stackrel{\text{def}}{=} P(a_1 \preceq_1 a'_1, a'_1 \preceq_1 a_1, a_2 \preceq_2 a'_2, a'_2 \preceq_2 a_2).$$

- We say that a product operation is consistent if \preceq is always a partial order, and

$$(a_1 \preceq_1 a'_1 \ \& \ a_2 \preceq_2 a'_2) \Rightarrow (a_1, a_2) \preceq (a'_1, a'_2).$$

Theorem 1. Every consistent product operation is the Cartesian or the lexicographic product.

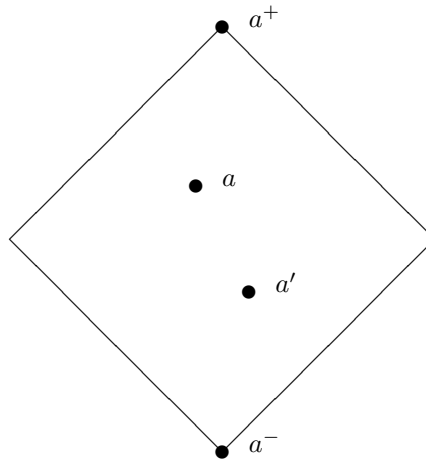
Comment. For reader's convenience, the proofs of all the results are placed in a special Proofs section.

4 Auxiliary Results: General Idea and First Example

General idea. For each property of an ordered set A , we analyze which properties need to be satisfied for A_1 and A_2 so that the corresponding property is satisfied in $A_1 \times A_2$.

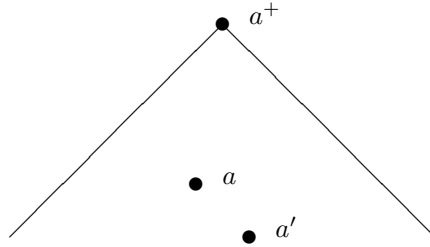
First example: connectedness property. As a first example, let us consider the following

- *Connectedness property (CP):* for every two points $a, a' \in A$, there exists an interval that contains a and a' : $\forall a \forall a' \exists a^- \exists a^+ (a^- \preceq a, a' \preceq a^+)$.

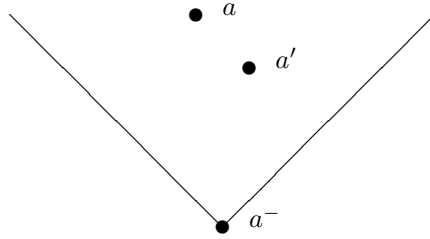


Connectedness property: analysis. One can easily see that a partially ordered set A satisfies the connectedness property if and only if it satisfies the following two properties:

- A is *upward-directed*: $\forall a \forall a' \exists a^+ (a, a' \preceq a^+)$;



- A is downward-directed: $\forall a \forall a' \exists a^- (a^- \preceq a, a')$.



So, to check when the product satisfies the connectivity property, it is sufficient to check when the product is upward- and downward-directed.

Results for Cartesian product. For both \preceq - and \prec -Cartesian products, we get the following results.

Proposition 1. *A Cartesian product $A_1 \times A_2$ is upward-directed if and only if both A_1 and A_2 are upward-directed.*

Proposition 2. *A Cartesian product $A_1 \times A_2$ is downward-directed if and only if both A_1 and A_2 are downward-directed.*

Results for lexicographic product. For the lexicographic product, we get the following results:

Definition 2.

- An element $\bar{a} \in A$ is called maximal if there are no elements a with $\bar{a} \prec a$.
- An element $\underline{a} \in A$ is called minimal if there are no elements a with $a \prec \underline{a}$.

Proposition 3. A lexicographic product $A_1 \times A_2$ is upward-directed \Leftrightarrow the following two conditions hold:

- the set A_1 is upward-directed, and
- if A_1 has a maximal element \bar{a}_1 , then A_2 is upward-directed.

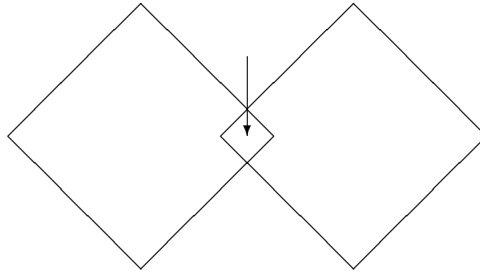
Proposition 4. A lexicographic product $A_1 \times A_2$ is downward-directed \Leftrightarrow the following two conditions hold:

- the set A_1 is downward-directed, and
- if A_1 has a minimal element \underline{a}_1 , then A_2 is downward-directed.

5 Auxiliary Result: Second Example

Second example: intersection property. As a second example, let us consider the following

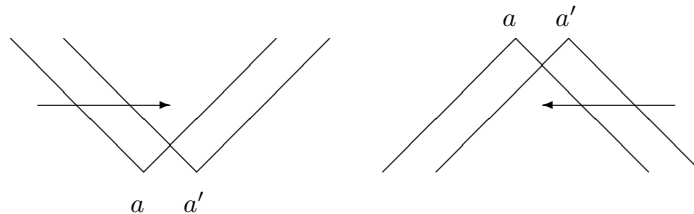
- *Intersection property:* the intersection of every two intervals is an interval.



Comment. This property is satisfied for intervals on the real line.

Intersection property: analysis. Similarly to the connectivity property, the intersection property can also be reduced to two properties:

- the intersection of every two future cones $Q_a^+ \stackrel{\text{def}}{=} \{b : a \preceq b\}$ is a future cone;
- the intersection of every two past cones $Q_a^- \stackrel{\text{def}}{=} \{b : b \preceq a\}$ is a past cone.



If both properties hold, then a non-empty intersection of every two intervals $[a, b] = Q_a^+ \cap Q_b^-$ is an interval.

Definition 3.

- An ordered set for which the intersection $Q_a^+ \cap Q_{a'}^+$ of every two future cones Q_a^+ and $Q_{a'}^+$ is a future cone is called an upper semi-lattice.
- For every two elements a, a' , the element a'' for which $Q_a^+ \cap Q_{a'}^+ = Q_{a''}^+$ is called a join of a and a' and is denoted by $a \vee a'$.
- An ordered set for which the intersection $Q_a^- \cap Q_{a'}^-$ of every two past cones Q_a^- and $Q_{a'}^-$ is a past cone is called a lower semi-lattice.
- For every two elements a, a' , the element a'' for which $Q_a^- \cap Q_{a'}^- = Q_{a''}^-$ is called a meet of a and a' and is denoted by $a \wedge a'$.

What we plan to do. We plan to analyze when the Cartesian and lexicographic products are upper and lower semi-lattices.

Proposition 5. A Cartesian product $A_1 \times A_2$ is an upper semi-lattice if and only if both A_1 and A_2 are upper semi-lattices.

Proposition 6. A Cartesian product $A_1 \times A_2$ is a lower semi-lattice if and only if both A_1 and A_2 are lower semi-lattices.

To describe when a lexicographic product is an upper semi-lattice, we need to introduce the following auxiliary notions:

Definition 4.

- We say that an ordered set A is linearly (totally) ordered if for every two elements $a, a' \in A$, we have either $a \preceq a'$ or $a' \preceq a$.
- We say that an ordered set is a conditional upper semi-lattice if for all a and a' for which the future cones Q_a^+ and $Q_{a'}^+$ intersect, this intersection is also a future cone.
- We say that an ordered set is a conditional lower semi-lattice if for all a and a' for which the past cones Q_a^- and $Q_{a'}^-$ intersect, this intersection is also a past cone.
- We say that an element a^+ is the next element to a if for every a' , the condition $a \prec a'$ is equivalent to $a^+ \preceq a'$.
- We say that an ordered set is sequential up if every element has a next one.

- We say that an element a^- is the previous element to a if for every a' , the condition $a' \prec a$ is equivalent to $a' \preceq a^-$.
- We say that an ordered set is sequential down if every element has a previous one.

Proposition 7. *The lexicographic product $A_1 \times A_2$ is an upper semi-lattice if and only if A_1 is an upper semi-lattice and one of the following conditions holds:*

- A_1 is linearly ordered and A_2 is an upper semi-lattice;
- A_2 is an upper semi-lattice that has the smallest element;
- A_1 is sequential up, A_2 is a conditional upper semi-lattice, and A_2 has the smallest element.

Proposition 8. *The lexicographic product $A_1 \times A_2$ is a lower semi-lattice if and only if A_1 is a lower semi-lattice and one of the following conditions holds:*

- A_1 is linearly ordered and A_2 is a lower semi-lattice;
- A_2 is a lower semi-lattice that has the largest element;
- A_1 is sequential down, A_2 is a conditional lower semi-lattice, and A_2 has the largest element.

6 Proofs

Proof of Theorem 1.

1°. According to the definition, whether $(a_1, a_2) \preceq (a'_1, a'_2)$ depends on the two relations: the relation between a_1 and a'_1 and on the relation between a_2 and a'_2 . For each pair a_i and a'_i , we have four possible relations:

- the relation $a_i \prec_i a'_i$; we will denote this case by +;
- the relation $a'_i \prec_i a_i$; we will denote this case by -;
- the relation $a_i = a'_i$; we will denote this relation by =; and
- the relation $a_i \not\prec_i a'_i$ and $a'_i \not\prec_i a_i$; we will denote this relation by ||.

The case when we have relation R_1 for a_1 and a'_1 and relation R_2 for a_2 and a'_2 will be denoted by R_1R_2 . So, we have 16 possible pairs of relations: ++, +-, + =, + ||, -+, --, etc. To describe the product, it is sufficient to describe which of these 16 pairs correspond to $(a_1, a_2) \preceq (a'_1, a'_2)$.

Due to the consistency requirement, pairs ++, + =, = +, and == always result in \preceq , so it is sufficient to classify the remaining 12 pairs. If only these four pairs result in \preceq , then we have the Cartesian product. So, to prove our theorem, it is sufficient to prove that if at least one other pair leads to \preceq , then

we get a lexicographic product. To prove this, let us consider the remaining 12 pairs one by one.

2°. Let us first consider pairs that contain $-$.

2.1°. Let us prove that the pair $--$ cannot lead to \preceq . Indeed, when both A_1 and A_2 are real lines \mathbb{R} with the usual order, due to the fact that $++$ leads to \preceq , we get $(0,0) \preceq (1,1)$, while due to the fact that $--$ leads to \preceq , we get $(1,1) \preceq (0,0)$. Hence, we have $(0,0) \preceq (1,1)$ and $(1,1) \preceq (0,0)$ but $(0,0) \neq (1,1)$ – a contradiction to antisymmetry.

2.2°. Similarly, the pair $- =$ cannot lead to \preceq because otherwise, for the same example $A_1 = A_2 = \mathbb{R}$, we would get $(0,0) \preceq (1,0)$ and $(1,0) \preceq (0,0)$ but $(0,0) \neq (1,0)$ – also a contradiction to antisymmetry.

2.3°. Let us now consider the pair $- \parallel$.

To prove that it cannot lead to \preceq , we consider $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$ with Cartesian order. In this case,

$$(0,0) \parallel_2 (1,-2)$$

and $(1,-2) \parallel_2 (-1,-1)$. Thus, if $- \parallel$ leads to \preceq , we have $(0,(0,0)) \preceq (-1,(1,-2))$ and $(-1,(1,-2)) \preceq (-2,(-1,-1))$. Thus, due to transitivity of \preceq , we get $(0,(0,0)) \preceq (-2,(-1,-1))$. On the other hand, due to consistency, from $-2 \preceq_1 0$ and $(-1,-1) \preceq_2 (0,0)$, we conclude that $(-2,(-1,-1)) \preceq (0,(0,0))$ – a contradiction with antisymmetry.

2.4°. Similarly, pairs $= -$ and $\parallel -$ cannot lead to \preceq . Thus, the only pairs containing $-$ that can potentially lead to \preceq are pairs containing a $+$.

3°. Let us prove a similar property for pairs containing \parallel . We already know that pairs $\parallel -$ and $- \parallel$ cannot lead to \preceq , so it is sufficient to consider pairs $\parallel =$, $= \parallel$, and $\parallel \parallel$.

3.1°. To prove that the pair $= \parallel$ cannot lead to \preceq , let us consider the same case $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. In this case, due to

$$(0,0) \parallel_2 (1,-2)$$

and $(1,-2) \parallel_2 (-1,-1)$, if $= \parallel$ leads to \preceq , we have $(0,(0,0)) \preceq (0,(1,-2))$ and $(0,(1,-2)) \preceq (0,(-1,-1))$. Thus, due to transitivity of \preceq , we get

$$(0,(0,0)) \preceq (0,(-1,-1)).$$

On the other hand, due to consistency, from $0 \preceq_1 0$ and $(-1,-1) \preceq_2 (0,0)$, we conclude that $(0,(-1,-1)) \preceq (0,(0,0))$ – a contradiction with antisymmetry.

3.2°. Similarly, it is possible to prove that the pair $\parallel =$ cannot lead to \preceq .

3.3°. To prove that the pair $\parallel \parallel$ cannot lead to \preceq , let us consider the case when $A_1 = A_2 = \mathbb{R} \times \mathbb{R}$. In this case, due to

$$(0,0) \parallel_i (1,-2)$$

and $(1, -2) \parallel_i (-1, -1)$, if \parallel leads to \preceq , we have $((0, 0), (0, 0)) \preceq ((1, -2), (1, -2))$ and $((1, -2), (1, -2)) \preceq ((-1, -1), (-1, -1))$. Thus, due to transitivity of \preceq , we get $((0, 0), (0, 0)) \preceq ((-1, -1), (-1, -1))$. On the other hand, due to consistency, from $(-1, -1) \preceq_i (0, 0)$, we conclude that $((-1, -1), (-1, -1)) \preceq ((0, 0), (0, 0))$ – a contradiction with antisymmetry.

4°. Thus, due to Part 2 and 3 of this proof, the only additional pairs that can, in principle, lead to \preceq are pairs containing $+$, i.e., pairs $+ -$, $+ \parallel$, $- +$, and $- \parallel$.

5°. Let us prove that the pair $+ -$ leads to \preceq if and only if the pair $+ \parallel$ leads to \preceq .

5.1°. Let us first prove that if the pair $+ -$ leads to \preceq , then the pair $+ \parallel$ also leads to \preceq .

Indeed, let us consider the case when $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. If $+ -$ leads to \preceq , then $0 \prec_1 1$ and $(-1, -1) \prec_2 (0, 0)$ imply $(0, (0, 0)) \preceq (1, (-1, -1))$. Due to consistency, $1 \preceq_1 1$ and $(-1, -1) \preceq_2 (-1, 1)$ lead to $(1, (-1, -1)) \preceq (1, (-1, 1))$. Due to transitivity of \preceq , we get $(0, (0, 0)) \preceq (1, (-1, 1))$. In this case, \preceq holds for a pair for which $0 \prec_1 1$ and $(0, 0) \parallel_2 (-1, 1)$, i.e., for a pair of type $+ \parallel$. By our definition of an order on the product, this means that \preceq must hold for all pairs of this type, i.e., that the pair $+ \parallel$ indeed leads to \preceq .

5.2°. Let us now prove that if the pair $+ \parallel$ leads to \preceq , then the pair $+ -$ also leads to \preceq .

Let us consider the same case $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. If $+ \parallel$ leads to \preceq , then $0 \prec_1 1$ and $(1, -2) \parallel_2 (-1, -1)$ imply $(0, (0, 0)) \preceq (1, (1, -2))$, and $1 \prec_1 2$ and $(0, 0) \parallel_2 (1, -2)$ imply $(1, (1, -2)) \preceq (2, (-1, -1))$. Due to transitivity of \preceq , we get $(0, (0, 0)) \preceq (2, (-1, -1))$. In this case, \preceq holds for a pair for which $0 \prec_1 2$ and $(-1, -1) \prec_2 (0, 0)$, i.e., for a pair of type $+ -$. By our definition of an order on the product, this means that \preceq must hold for all pairs of this type, i.e., that the pair $+ -$ indeed leads to \preceq .

6°. Similarly, we can prove that the pair $- +$ leads to \preceq if and only if the pair $- \parallel$ leads to \preceq . Thus, adding $+ -$ is equivalent to adding $+ \parallel$, and adding $- +$ is equivalent to adding $- \parallel$.

If we add $+ -$ (and hence $+ \parallel$), we get the lexicographic product $A_1 \times A_2$. If we add $- +$ (and hence $- \parallel$), we get the lexicographic product $A_2 \times A_1$. Thus, to complete the proof, it is sufficient to show that we cannot simultaneously add $+ -$ and $- +$.

7°. Let us prove that $+ -$ and $- +$ cannot simultaneously lead to \preceq .

We will prove this by contradiction. Let us assume that adding both $+ -$ and $- +$ always leads to a consistent partial order. In this case, let us take $A_1 = A_2 = \mathbb{R}$. Since $+ -$ leads to \preceq , the conditions $0 \prec_1 1$ and $-2 \prec_2 0$ lead to $(0, 0) \preceq (1, -2)$. Similarly, since $- +$ leads to \preceq , from $-1 \prec_1 1$ and $-2 \prec_2 -1$, we conclude that $(1, -2) \preceq (-1, -1)$. By transitivity of \preceq , we can now conclude

that $(0,0) \preceq (-1,-1)$. However, due to consistency, $(-1,-1) \preceq (0,0)$ – a contradiction to anti-symmetry.

The statement is proven, and so is the theorem.

Proof of Proposition 1.

1°. Let us assume that the \preceq -Cartesian product $A_1 \times A_2$ is upward-directed. We want to prove that A_1 is upward-directed. (For A_2 , the proof of similar.)

In other words, we want to prove that for every $a_1, a'_1 \in A_1$, there exists an element $a_1^+ \in A_1$ for which $a_1, a'_1 \preceq_1 a_1^+$. Let us take any $a_1, a'_1 \in A_1$, and any $a_2 \in A_2$. Then, since the product $A_1 \times A_2$ is upward-directed, there exists an element $a^+ = (a_1^+, a_2^+) \in A_1 \times A_2$ for which

$$(a_1, a_2) \preceq a^+ = (a_1^+, a_2^+) \text{ and } (a'_1, a_2) \preceq a^+ = (a_1^+, a_2^+).$$

By definition of an order on $A_1 \times A_2$, we thus conclude that $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$. Thus, A_1 is indeed upward-directed.

2°. Let us now assume that both A_i are upward-directed. We want to prove that $A_1 \times A_2$ is upward-directed, i.e., that for any two elements $a = (a_1, a_2) \in A_1 \times A_2$ and $a' = (a'_1, a'_2) \in A_1 \times A_2$, there exists an element a^+ for which $a \preceq a^+$ and $a' \preceq a^+$.

Indeed, since the set A_1 is upward-directed, there exists an element a_1^+ for which $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$. Similarly, since the set A_2 is upward-directed, there exists an element a_2^+ for which $a_2 \preceq_2 a_2^+$ and $a'_2 \preceq_2 a_2^+$. By definition of the order on the Cartesian product, we can now conclude that $(a_1, a_2) \preceq (a_1^+, a_2^+)$ and $(a'_1, a'_2) \preceq (a_1^+, a_2^+)$. Thus, the set $A_1 \times A_2$ is upward-directed.

Proof of Proposition 2 is similar to the proof of Proposition 1.

Proof of Proposition 3.

1°. Let us assume that $A_1 \times A_2$ is upward-directed.

1.1°. Let us prove that A_1 is upward-directed, i.e., that for every $a_1 \in A_1$ and $a'_1 \in A_1$, there exists an element a_1^+ for which $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$.

Indeed, since the product $A_1 \times A_2$ is upward-directed, for any $a_2 \in A_2$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which $(a_1, a_2) \preceq a^+$ and $(a'_1, a_2) \preceq a^+$. By definition of the lexicographic product, this implies that $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$. Thus, the set A_1 is upward-directed.

1.2°. Let us now assume that, in addition to $A_1 \times A_2$ being upward-directed, the set A_1 has a maximal element \bar{a}_1 . Let us prove that under these assumptions, the set A_2 is also upward-directed, i.e., for every $a_2, a'_2 \in A_2$, there exists an element a_2^+ for which $a_2 \preceq_2 a_2^+$ and $a'_2 \preceq_2 a_2^+$.

Indeed, since the product $A_1 \times A_2$ is upward-related, there exists an element $a^+ = (a_1^+, a_2^+) \in A_1 \times A_2$ for which $(\bar{a}_1, a_2) \preceq a^+$ and $(\bar{a}_1, a'_2) \preceq a^+$. By

definition of the lexicographic product, this implies that $\bar{a}_1 \preceq_1 a_1^+$, i.e., that either $\bar{a}_1 \prec_1 a_1^+$ or $\bar{a}_1 = a_1^+$. Since the element \bar{a}_1 is maximal, we cannot have $\bar{a}_1 \prec_1 a_1^+$, so we have $a_1^+ = \bar{a}_1$. In this case, $(\bar{a}_1, a_2) \preceq a^+ = (\bar{a}_1, a_2^+)$, so by definition of the lexicographic order, we get $a_2 \preceq_2 a_2^+$. Similarly, we get $a_2' \preceq_2 a_2^+$, so the set A_2 is indeed upward-directed.

2°. Let us now assume that A_1 is upward-directed, and that if A_1 has a maximal element, then A_2 is upward-directed. Let us prove that $A_1 \times A_2$ is upward-directed.

Indeed, let us take any two elements $a = (a_1, a_2)$ and $a' = (a_1', a_2')$ from $A_1 \times A_2$, and let us show that there exists an element a^+ for which $a \preceq a^+$ and $a' \preceq a^+$.

Since A_1 is upward-directed, there exists an element $a_1^+ \in A_1$ for which $a_1 \preceq_1 a_1^+$ and $a_1' \preceq_1 a_1^+$.

Since $a_1 \preceq_1 a_1^+$, we have either $a_1 \prec_1 a_1^+$ or $a_1 = a_1^+$. Similarly, since $a_1' \preceq_1 a_1^+$, we have either $a_1' \prec_1 a_1^+$ or $a_1' = a_1^+$. Thus, by considering both a_1 and a_1' , we have four possible situations:

- situation when $a_1 \prec_1 a_1^+$ and $a_1' \prec_1 a_1^+$;
- situation when $a_1 \prec_1 a_1^+$ and $a_1' = a_1^+$;
- situation when $a_1 = a_1^+$ and $a_1' \prec_1 a_1^+$; and
- situation when $a_1 = a_1^+$ and $a_1' = a_1^+$,

Let us consider these four situations one by one.

When $a_1 \prec_1 a_1^+$ and $a_1' \prec_1 a_1^+$, then, by the definition of lexicographic order, we have $(a_1, a_2) \preceq (a_1^+, a_2')$ and $(a_1', a_2) \preceq (a_1^+, a_2')$. So, in this case, we can take (a_1^+, a_2') as the desired element a^+ .

When $a_1 \prec_1 a_1^+$ and $a_1' = a_1^+$, then, by the definition of lexicographic order, we have $(a_1, a_2) \preceq (a_1^+, a_2')$. We also have $(a_1', a_2) = (a_1^+, a_2)$ hence $(a_1', a_2) \preceq (a_1^+, a_2')$. So, in this case, we can also take (a_1^+, a_2') as the desired element a^+ .

When $a_1 = a_1^+$ and $a_1' \prec_1 a_1^+$, then, by the definition of lexicographic order, we have $(a_1', a_2) \preceq (a_1^+, a_2)$. We also have $(a_1, a_2) = (a_1^+, a_2)$ hence $(a_1, a_2) \preceq (a_1^+, a_2)$. So, in this case, we can also take (a_1^+, a_2) as the desired element a^+ .

Finally, let us consider the situation when $a_1 = a_1^+ = a_1'$. In this situation, there are two possible sub-situations: when this element a_1 is maximal and when it is not maximal.

If the element a_1 is not maximal, then there exists a value a_1^+ for which $a_1 \prec_1 a_1^+$. In this case, by definition of the lexicographic product, we have $(a_1, a_2) \preceq (a_1^+, a_2)$ and $(a_1', a_2) \preceq (a_1^+, a_2)$. So, we can take (a_1^+, a_2) as the desired element a^+ .

If the element a_1 is maximal, then A_2 is upward-directed. Thus, there exists an element $a_2^+ \in A_2$ for which $a_2 \preceq_2 a_2^+$ and $a_2' \preceq_2 a_2^+$. For this element, we have $(a_1, a_2) \preceq (a_1, a_2^+)$ and $(a_1, a_2') \preceq (a_1, a_2^+)$. So, we can take (a_1, a_2^+) as the desired element a^+ .

In all four situations, there exists an element a^+ for which $a \preceq a^+$ and $a' \preceq a^+$. The statement is proven, and so is the proposition.

Proof of Proposition 4 is similar to the proof of Proposition 3.

Proof of Proposition 5.

1°. Let us assume that the Cartesian product $A_1 \times A_2$ is an upper semi-lattice. Let us prove that in this case, A_1 is also an upper semi-lattice (for A_2 , the proof is the same).

We need to prove that for every two elements $a_1, a'_1 \in A_1$, there exists an element a_1^+ for which $Q_{a_1}^+ \cap Q_{a'_1}^+ = Q_{a_1^+}^+$, i.e., for which, for every element b_1 , we have

$$(a_1 \preceq_1 b_1 \ \& \ a'_1 \preceq_1 b_1) \Leftrightarrow a_1^+ \preceq_1 b_1. \quad (1)$$

To prove it, let us take an arbitrary element $a_2 \in A_2$ and consider two elements $a = (a_1, a_2)$ and $a' = (a'_1, a_2)$. Since the set $A_1 \times A_2$ is an upper semi-lattice, there exists an element $a^+ = (a_1^+, a_2^+)$ which is a join of the elements a and a' , i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \ \& \ (a'_1, a_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \quad (2)$$

Let us prove that the first component a_1^+ of this join a^+ is the desired join of a_1 and a'_1 , i.e., that for this first component, the condition (1) holds.

\Leftarrow If $a_1^+ \preceq_1 b_1$, then by the definition of the Cartesian product, we have $(a_1^+, a_2^+) \preceq (b_1, a_2^+)$. Applying the \Leftarrow part of the condition (2) with $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preceq (b_1, a_2^+)$ and $(a'_1, a_2) \preceq (b_1, a_2^+)$. By the definition of the Cartesian product, the first condition implies that $a_1 \preceq_1 b_1$, and the second condition implies that $a'_1 \preceq_1 b_1$.

\Rightarrow Vice versa, let us assume that $a_1 \preceq_1 b_1$ and $a'_1 \preceq_1 b_1$. We need to prove that $a_1^+ \preceq_1 b_1$. In this case, by the definition of the Cartesian product, we have $(a_1, a_2) \preceq (b_1, a_2)$ and $(a'_1, a_2) \preceq (b_1, a_2)$. Applying the \Rightarrow part of the condition (2) with $b_2 = a_2$, we conclude that $(a_1^+, a_2^+) \preceq (b_1, a_2)$. By the definition of the Cartesian product, this implies that $a_1^+ \preceq_1 b_1$.

2°. It is easy to prove that if A_1 and A_2 are upper semi-lattices, then their Cartesian product $A_1 \times A_2$ is also an upper semi-lattice, with $(a_1, a_2) \vee (a'_1, a'_2) = (a_1 \vee a'_1, a_2 \vee a'_2)$.

The proposition is proven.

Proof of Proposition 6 is similar to the proof of Proposition 5.

Proof of Proposition 7.

1°. Let us assume that $A_1 \times A_2$ is an upper semi-lattice. Let us prove that in this case, A_1 is an upper semi-lattices, and one of the three properties described in the formulation of Proposition 7 holds.

1.1°. Let us prove that A_1 is an upper semi-lattice, i.e., that for every two elements $a_1, a'_1 \in A_1$, there exists an element a_1^+ for which $Q_{a_1}^+ \cap Q_{a'_1}^+ = Q_{a_1^+}^+$, i.e., for which, for every element b_1 , we have

$$(a_1 \preceq_1 b_1 \ \& \ a'_1 \preceq_1 b_1) \Leftrightarrow a_1^+ \preceq_1 b_1. \quad (3)$$

Let us take an arbitrary element $a_2 \in A_2$ and consider two elements $a = (a_1, a_2)$ and $a' = (a'_1, a_2)$. Since the set $A_1 \times A_2$ is an upper semi-lattice, there exists an element $a^+ = (a_1^+, a_2^+)$ which is a join of the elements a and a' , i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \ \& \ (a'_1, a_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \quad (4)$$

Let us prove that the first component a_1^+ of this join a^+ is the desired join of a_1 and a'_1 , i.e., that for this first component, the condition (3) holds.

\Leftarrow If $a_1^+ \preceq_1 b_1$, then, by the definition of the lexicographic product, we have $(a_1^+, a_2^+) \preceq (b_1, a_2^+)$. Applying the \Leftarrow part of the condition (4) with $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preceq (b_1, a_2^+)$ and $(a'_1, a_2) \preceq (b_1, a_2^+)$. By the definition of the lexicographic product, the first condition implies that $a_1 \preceq_1 b_1$, and the second condition implies that $a'_1 \preceq_1 b_1$.

\Rightarrow Vice versa, let us assume that $a_1 \preceq_1 b_1$ and $a'_1 \preceq_1 b_1$. We need to prove that $a_1^+ \preceq_1 b_1$. To prove this statement, we will consider four possible cases:

- case when $a_1 \prec_1 b_1$ and $a'_1 \prec_1 b_1$;
- case when $a_1 \prec_1 b_1$ and $a'_1 = b_1$;
- case when $a_1 = b_1$ and $a'_1 \prec_1 b_1$; and
- case when $a_1 = b_1$ and $a'_1 = b_1$.

1) When $a_1 \prec_1 b_1$ and $a'_1 \prec_1 b_1$, then, by the definition of the lexicographic order, we have $(a_1, a_2^+) \preceq (b_1, a_2^+)$ and $(a'_1, a_2^+) \preceq (b_1, a_2^+)$. Thus, due to the \Rightarrow part of (4), with $b_2 = a_2^+$, we get $(a_1^+, a_2^+) \preceq (b_1, a_2^+)$. So, by the definition of the lexicographic product, we have $a_1^+ \preceq_1 b_1$.

2) When $a_1 \prec_1 b_1$ and $a'_1 = b_1$, we have $a_1 \prec_1 a'_1$. Here, $(a'_1, a_2) = (a'_1, a_2)$ hence $(a'_1, a_2) \preceq (a'_1, a_2)$, and $a_1 \prec_1 a'_1$ implies that $(a_1, a_2) \prec (a'_1, a_2)$. Thus, due to the \Rightarrow part of (4), with $b_1 = a'_1$ and $b_2 = a_2$, we get $(a_1^+, a_2^+) \preceq (a'_1, a_2)$. So, by the definition of the lexicographic product, we have $a_1^+ \preceq_1 a'_1 = b_1$.

3) When $a_1 = b_1$ and $a'_1 \prec_1 b_1$, we have $a'_1 \prec_1 a_1$. Here, $(a_1, a_2) = (a_1, a_2)$ hence $(a_1, a_2) \preceq (a_1, a_2)$, and $a'_1 \prec_1 a_1$ implies that $(a'_1, a_2) \prec (a_1, a_2)$. Thus, due to the \Rightarrow part of (4), with $b_1 = a_1$ and $b_2 = a_2$, we get $(a_1^+, a_2^+) \preceq (a_1, a_2)$. So, by the definition of the lexicographic product, we have $a_1^+ \preceq_1 a_1 = b_1$.

4) Finally, when $a_1 = b_1 = a'_1$, we have $a_1 = a'_1$. In this case, $(a_1, a_2) = (a'_1, a_2)$, so the condition $(a_1, a_2) \preceq (b_1, b_2)$ is simply equivalent to $(a'_1, a_2) \preceq (b_1, b_2)$. In this case, the condition (4) simply means that

$$(a_1, a_2) \preceq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \quad (5)$$

For $(b_1, b_2) = (a_1, a_2)$, the \Rightarrow part of the formula (5) implies that $(a_1^+, a_2^+) \preceq (a_1, a_2)$. For $(b_1, b_2) = (a_1^+, a_2^+)$, the \Leftarrow part the formula (5) implies that $(a_1, a_2) \preceq (a_1^+, a_2^+)$. Thus, $(a_1^+, a_2^+) = (a_1, a_2)$, and hence, $a_1^+ = a_1 (= a'_1)$. Since $a_1^+ = a_1 = a'_1$, the condition (3) is clearly satisfied.

In all four cases, we have the desired proof, so the statement is proven. So, A_1 is indeed an upper semi-lattice.

1.2°. Let us prove that if the set A_1 is not linearly ordered, then the set A_2 has the smallest element.

Indeed, the fact that the set A_1 is not linearly ordered means that there exist values a_1 and a'_1 for which $a_1 \not\prec_1 a'_1$ and $a'_1 \not\prec_1 a_1$.

Let us take an arbitrary $a_2 \in A_2$. Since $A_1 \times A_2$ is an upper semi-lattice, for $a = (a_1, a_2)$ and $a' = (a'_1, a_2)$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which $Q_a^+ \cap Q_{a'}^+ = Q_{a^+}^+$, i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \& (a'_1, a_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \quad (6)$$

Let us prove that a_2^+ is the smallest element of the set A_2 .

Indeed, the formula (6) implies, for $(b_1, b_2) = (a_1^+, a_2^+)$, we have $(a_1^+, a_2^+) = (b_1, b_2)$ and hence, $(a_1^+, a_2^+) \preceq (b_1, b_2)$. Thus, due to the \Leftarrow part of the formula (6), we conclude that $(a_1, a_2) \preceq (a_1^+, a_2^+)$ and $(a'_1, a_2) \preceq (a_1^+, a_2^+)$. By definition of the lexicographic order, we conclude that $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$.

We cannot have $a_1 = a_1^+$, since then we would have $a'_1 \preceq_1 a_1$ - which contradicts to our choice of a_1 and a'_1 . Thus, $a_1 \preceq_1 a_1^+$ implies that $a_1 \prec_1 a_1^+$.

Similarly, we cannot have $a'_1 = a_1^+$, since then we would have $a_1 \preceq_1 a'_1$ - which also contradicts to our choice of a_1 and a'_1 . Thus, $a'_1 \preceq_1 a_1^+$ implies that $a'_1 \prec_1 a_1^+$.

Let a'_2 be an arbitrary element of the set A_2 . Since $a_1 \prec_1 a_1^+$ and $a'_1 \prec_1 a_1^+$, we have $(a_1, a_2) \prec (a_1^+, a'_2)$ and $(a'_1, a_2) \prec (a_1^+, a'_2)$. Thus, by applying the \Rightarrow part of the formula (6) to $b_1 = a_1^+$ and $b_2 = a'_2$, we conclude that $(a_1^+, a_2^+) \preceq (a_1^+, a'_2)$. By the definition of the lexicographic order, this means that $a_2^+ \preceq_2 a'_2$. Thus, a_2^+ precedes all the elements of A_2 , so it is indeed the smallest element of A_2 .

1.3°. Let us now prove that A_2 is a conditional semi-lattice, i.e., that for every $a_2 \in A_2$ and $a'_2 \in A_2$ for which $a_2 \preceq_2 a''_2$ and $a'_2 \preceq_2 a''_2$ for some a''_2 , there exists an element a^{+2} for which $Q_{a_2}^+ \cap Q_{a'_2}^+ = Q_{a^{+2}}^+$, i.e., for which, for every $b_1 \in A_2$, we have

$$(a_2 \preceq_2 b_2 \& a'_2 \preceq_2 b_2) \Leftrightarrow a_2^+ \preceq_2 b_2. \quad (7)$$

Indeed, let us take an arbitrary element $a_1 \in A_1$. Since the set $A_1 \times A_2$ is an upper semi-lattice, for elements (a_1, a_2) and (a'_1, a_2) , there exists an element $a^+ = (a_1^+, a_2^+)$ for which

$$Q_{(a_1, a_2)}^+ \cap Q_{(a'_1, a_2)}^+ = Q_{(a_1^+, a_2^+)}^+.$$

In other words, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \& (a_1, a'_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \quad (8)$$

Let us show that the second component a_2^+ of this element is the desired join of a_2 and a'_2 .

For that, let us first prove that $a_1^+ = a_1$. Indeed, from $(a_1^+, a_2^+) = (a_1^+, a_2^+)$, we conclude that $(a_1^+, a_2^+) \preceq (a_1^+, a_2^+)$. So, from the \Leftarrow part of the formula (8) for $(b_1, b_2) = (a_1^+, a_2^+)$, we conclude that $(a_1, a_2) \preceq (b_1, b_2)$, i.e., that $(a_1, a_2) \preceq (a_1^+, a_2^+)$. By the definition of the lexicographic order, this implies that $a_1 \preceq_1 a_1^+$.

Now, $a_2 \preceq_2 a''_2$ and $a'_2 \preceq_2 a''_2$. So, by definition of the lexicographic order, we have $(a_1, a_2) \preceq (a_1, a''_2)$ and $(a_1, a'_2) \preceq (a_1, a''_2)$. Thus, due to the \Rightarrow part of the formula (8), with $(b_1, b_2) = (a_1, a''_2)$, we conclude that $(a_1^+, a_2^+) \preceq (a_1, a''_2)$. Thus, by definition of the lexicographic order, we conclude that $a_1^+ \preceq_1 a_1$. From $a_1 \preceq_1 a_1^+$ and $a_1^+ \preceq_1 a_1$, we conclude that $a_1^+ = a_1$. Thus, the formula (8) takes the form

$$((a_1, a_2) \preceq (b_1, b_2) \& (a_1, a'_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1, a_2^+) \preceq (b_1, b_2). \quad (9)$$

Now, we are ready to prove that $a_2^+ = a_2 \vee a'_2$, i.e., that the formula (7) holds.

\Leftarrow Let $a_2^+ \preceq_2 b_2$. Then, by definition of the lexicographic order, we have $(a_1, a_2^+) \preceq (a_1, b_2)$. Due to the \Leftarrow part of the formula (9), with $b_1 = a_1$, we get $(a_1, a_2) \preceq (b_1, b_2) = (a_1, b_2)$. By definition of the lexicographic order, this implies that $a_2 \preceq_2 b_2$. Similarly, we conclude that $a'_2 \preceq_2 b_2$.

\Rightarrow Let $a_2 \preceq_2 b_2$ and $a'_2 \preceq_2 b_2$. Then, by definition of the lexicographic order, we have $(a_1, a_2) \preceq (a_1, b_2)$ and $(a_1, a'_2) \preceq (a_1, b_2)$. Due to the \Rightarrow part of the formula (9), with $b_1 = a_1$, we get $(a_1, a_2^+) \preceq (b_1, b_2) = (a_1, b_2)$. By definition of the lexicographic order, this implies that $a_2^+ \preceq_2 b_2$.

The formula (7) is proven.

1.4°. Let us now prove that if A_2 is not an upper semi-lattice, then A_1 is sequential up and A_2 has the smallest element.

We have already proven that A_2 is a conditional upper semi-lattice, i.e., that every two elements a_2 and a'_2 for which there exists an element a''_2 with $a_2 \preceq_2 a''_2$ and $a'_2 \preceq_2 a''_2$ have a joint $a_1 \vee a'_2$. Since A_2 is not an upper semi-lattice, this means that there exist elements a_2 and a'_2 for which no element from A_2 follows both these elements.

Let us prove that A_2 has the smallest element and that A_1 is sequential up. For that, let us select any element $a_1 \in A_1$. Since $A_1 \times A_2$ is an upper

semi-lattice, for the elements $a = (a_1, a_2)$ and $a' = (a_1, a'_2)$, there exists a join $a^+ = (a_1^+, a_2^+)$, i.e., an element for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \& (a_1, a'_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \quad (10)$$

Let us show a_1^+ is the next element to a_1 and that a^+2 is the smallest element of the set A_2 . This will prove that A_1 is sequential up and that the set A_2 has the smallest element.

1.4.1°. To prove that a_1^+ is the next element to a_1 , we need to prove that for every $a'_1 \in A_1$, we have $a_1 \prec_1 a'_1 \Leftrightarrow a_1^+ \preceq_1 a'_1$.

\Rightarrow Let $a_1 \prec_1 a'_1$. Then, for an arbitrary $b_2 \in A_2$, by definition of the lexicographic order, we have $(a_1, a_2) \preceq (a'_1, b_2)$ and $(a_1, a'_2) \preceq (a'_1, b_2)$. Thus, due to the \Rightarrow part of the formula (10), with $b_1 = a'_1$, we conclude that $(a_1^+, a_2^+) \preceq (a'_1, b_2)$. Thus, by definition of the lexicographic order, we have $a_1^+ \preceq_1 a'_1$.

\Leftarrow Let $a_1^+ \preceq_1 a'_1$. Then, for $b_2 = a_2^+$, by definition of the lexicographic order, we have $(a_1^+, a_2^+) \preceq (a'_1, a_2^+)$. Thus, due to the \Leftarrow part of the formula (10), with $b_1 = a'_1$ and $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preceq (a'_1, a_2^+)$ and $(a_1, a_2) \preceq (a'_1, a_2^+)$. By definition of the lexicographic order, we thus have $a_1 \preceq_1 a'_1$, i.e., either $a_1 \prec_1 a'_1$ or $a_1 = a'_1$.

We cannot have $a'_1 = a_1$ because then, by definition of the lexicographic order, we would have $a_2 \prec_2 a_2^+$ and $a'_2 \preceq_2 a_2^+$, which contradicts to our choice of a_2 and a'_2 as the elements for which there is no elements following both. Thus, we have $a_1 \prec_1 a'_1$.

1.4.2°. To prove that a_2^+ is the smallest element of the set A_2 , we must prove that for every $b_2 \in A_2$, we have $a_2^+ \preceq_2 b_2$.

Indeed, since $a_1 \prec_1 a_1^+$, for every $b_2 \in A_2$, we have $(a_1, a_2) \preceq (a_1^+, b_2)$ and $(a_1, a'_2) \preceq (a_1^+, b_2)$. Thus, due to the \Rightarrow part of the formula (10), with $b_1 = a_1^+$, we conclude that $(a_1^+, a_2^+) \preceq (a_1^+, b_2)$. By definition of the lexicographic order, this implies that $a_2^+ \preceq_2 b_2$. The statement is proven.

2°. Let us now prove that in all three cases described in the formulation of Proposition 7, the lexicographic product $A_1 \times A_2$ is an upper semi-lattice, i.e., for every two elements $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$, there exists a join $(a_1, a_2) \vee (a'_1, a'_2)$.

In this proof, we simply describe the corresponding joins. Proving that the described elements are indeed joins is (reasonably) easy.

2.1°. Let us first consider the case when A_1 is linearly ordered and A_2 is an upper semi-lattice.

In this case, since the set A_1 is linearly ordered, for every two elements $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$, we have either $a_1 \prec_1 a'_1$, or $a'_1 \prec_1 a_1$, or $a_1 = a'_1$. Let us consider these three cases one by one.

- When $a_1 \prec_1 a'_1$, by definition of the lexicographic product, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence $(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2)$.
- When $a'_1 \prec_1 a_1$, by definition of the lexicographic product, we have $(a'_1, a'_2) \prec (a_1, a_2)$, hence $(a_1, a_2) \vee (a'_1, a'_2) = (a_1, a_2)$.
- Finally, when $a_1 = a'_1$, we have $(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2)$.

2.2°. Let us now consider the case when A_1 is an upper semi-lattice, and A_2 is an upper semi-lattice that has the smallest element \underline{a}_2 .

In this case, similarly to the previous case,

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence

$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence

$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$

- When $a_1 = a'_1$, we have

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2).$$

When $a_1 \not\prec_1 a'_1$ and $a'_1 \not\prec_1 a_1$, then, since A_1 is an upper semi-lattice, there exists an element $a_1 \vee a'_1$. It is easy to prove that this element is different from a_1 and a'_1 . We can then take $(a_1, a_2) \vee (a_1, a'_2) = (a_1 \vee a'_1, \underline{a}_2)$.

2.3°. Finally, let us consider the case when A_1 is an upper semi-lattice which is sequential up, A_2 is a conditional upper semi-lattice, and A_2 has the smallest element \underline{a}_2 .

To describe the join operation, let us consider all possible relations between a_1 and a'_1 .

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence

$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence

$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$

- When $a_1 \not\prec_1 a'_1$ and $a'_1 \not\prec_1 a_1$, we have

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1 \vee a'_1, \underline{a}_2).$$

- When $a_1 = a'_1$, and for elements $a_2, a'_2 \in A_2$, there exists an element a''_2 that follows both a_2 and a'_2 , then (since A_2 is a conditional upper semi-lattice), there exists a join $a_2 \vee a'_2$ of these two elements. So, we can take

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2).$$

- Finally, if $a_1 = a'_1$, and there are no elements a''_2 which follow both a_2 and a'_2 , then we take

$$(a_1, a_2) \vee (a_1, a'_2) = (a_1^+, \underline{a_2}),$$

where a_1^+ is the next element to a_1 .

Proof of Proposition 8 is similar to the proof of Proposition 7.

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