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Approximate Nature of Traditional Fuzzy Methodology Naturally Leads to Complex-Valued Fuzzy Degrees

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Abstract—**In the traditional fuzzy logic, the experts' degrees of confidence in their statements is described by numbers from the interval $[0, 1]$. These degree have a clear intuitive meaning. Somewhat surprisingly, in some applications, it turns out to be useful to also consider different numerical degrees – e.g., complex-valued degrees. While these complex-valued degrees are helpful in solving practical problems, their intuitive meaning is not clear. In this paper, we provide a possible explanation for the success of complex-valued degrees which makes their use more intuitively understandable – namely, we show that these degrees naturally appear due to the approximate nature of the traditional fuzzy methodology.**

I. Formulation of the Problem

Fact: complex-valued fuzzy degrees are sometimes useful in practice. One of the main motivations for fuzzy logic was to take into account that experts are often not 100% sure about their statements, especially about statements which use imprecise ("fuzzy") words from a natural language, such as "John is young".

Traditionally, in fuzzy logic, to describe the expert's degree of certainty in a statement, we use numbers from the interval $[0, 1]$; see, e.g., [4], [7], [8]. These numerical degrees have a clear intuitive sense. For example, we can ask the expert to mark, on a scale from 0 to 10, to what extend he or she believes that 29 years old is young. If the expert marks 8 on a scale of 0 to 10, we take $8/10 = 0.8$ as the desired degree. In general, if an expert marks m on a scale from 0 to n, we take m/n as the desired degree.

Empirically, it is sometimes helpful to use numerical degrees which are *not* from the interval $[0, 1]$ – e.g., complex-valued degrees; see, e.g., [1], [2], [3], [5], [6].

Challenge: why complex-valued fuzzy degrees? From the pragmatic viewpoint, this is good, since the use of complex numbers helps in solving practical problems. However, from the intuitive viewpoint, the meaning of such degrees is not clear.

One of the advantages of fuzzy logic (as opposed to, e.g., neural networks) is the intuitively clear nature of its operations and results. It is therefore desirable to come up with an intuitive explanation for complex-valued fuzzy degrees.

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What we do in this paper. In this paper, we deal with the above challenge by proposing a possible explanation for the use of complex-valued degrees.

The structure of this paper. To provide this explanation, we first recall the motivation behind "and" and "or" operations (t-norms and t-conorms) and remind readers that these operations provide only an approximate description of the expert's confidence in the corresponding composite statements. We then show that this approximate character naturally leads to the use of complex-valued degrees.

II. Fuzzy Degrees and "And" and "Or"-Operations: A Brief Reminder

Need for degrees. Nowadays, many of our activities depend on computers: computers regulate temperature in our homes, computers – via automatic transmissions – provide control over our cars, computers help us select best routes, etc. In view of this ubiquity of computers, it is desirable to teach the computers everything we know. A large part of our knowledge is, however, formulated in terms of imprecise ("fuzzy") words from natural language, such as "slow", "fast", "close", etc.

Precise properties like "speed is larger than 50 km/h" are easier to describe in computer terms, since for each possible value of the corresponding quantity, each such property is either true or false – and in the computer, "true" is usually represented as 1, and "false" as 0. In contrast, for imprecise properties, we are often not 100% sure whether a given values satisfies this property. A 20-year old is most probably young, a 40-year old is probably not, but how about 30? 29? 28? At some point, most people become unsure.

To describe this uncertainty, L. A. Zadeh proposed to use numbers between 0 ("false") and 1 ("true"): 0 corresponds to false, 1 corresponds to "true", and intermediate values correspond to intermediate degrees of certainty. Such a number can be obtained, e.g., if we ask an expert to mark his or her degree of certainty in a given statement on a scale from 0 to some integer n. If the expert marks his or her degree of certainty by a mark m on a scale from 0 to n, we take $d = m/n$ as the degree of expert's confidence in this particular statement.

Need for "and" and "or" operations. One of the main reasons for storing the corresponding knowledge is that this stored knowledge can enable computers to use expert rules which use such properties. For example, when we design an automatic transmission, we may want to incorporate experts rules such as "if the car is going fast and the road becomes somewhat slippery, change to a different transmission level". Similarly to this example, in general, the condition of each such rule does not simply consist of a *single* natural-language property, this condition is usually a propositional combination of *several* such properties.

To be able to property apply the corresponding rules, it is therefore important not only to describe our degrees of belief in the original statements $S_1, \ldots, S_n$, it is also necessary to describe our degree of belief in their propositional combinations such as $S_i \,\&\, S_j$, $S_i \vee S_j$, $S_i \,\&\, (S_j \vee \neg S_k)$, etc.

In the ideal world, we should be able to ask the expert to estimate his/her degree of confidence in each such combination. However, in practice, this is not possible. Indeed, for each of n statements $S_1, \ldots, S_n$, we can consider either this statement $S_i^+ \stackrel{\text{def}}{=} S_i$ or its negation $S_i^- \stackrel{\text{def}}{=} \neg S_i$. By combining statements and their negations, we get 2^n possible propositional combinations $S_1^{\varepsilon_1} \,\&\, \ldots \,\&\, S_n^{\varepsilon_n}$, where $\varepsilon_i \in \{-,+\}$. Even for reasonable number of statements $n \approx 300$, the number 2^n is astronomically large: it is larger than the number of particles in the Universe, so there is no way we can ask an expert that many questions.

Since we cannot directly ask the expert about his/her degree of belief in all possible propositional combinations, we need to *estimate* these degrees based on whatever information we have. In particular, we need to be able, given the expert's degrees of confidence $d(A)$ and $d(B)$ in statements A and B, to generate a degree of belief in the composite statement $A \,\&\, B$. In other words, we need an algorithm that, given the values $d(A)$ and $d(B)$, computes the estimate for $d(A \,\&\, B)$. Let us denote the function computed by this algorithm by $f_\&$; then the resulting estimate takes the form $f_\&(d(A), d(B))$.

We know that $A \,\&\, B$ is equivalent to $B \,\&\, A$. It is therefore reasonable to require that the estimates $f_\&(d(A), d(B))$ and $f_\&(d(B), d(A))$ corresponding to these two equivalent expressions coincide. This should be true for all possible values $a = d(A)$ and $b = d(B)$. Thus, in mathematical terms, the operation $f_\&(a, b)$ should be *commutative*, i.e., satisfy the condition $f_\&(a, b) = f_\&(b, a)$.

Similarly, we know that $A \,\&\, (B \,\&\, C) = (A \,\&\, B) \,\&\, C$.

- If we use the first expression to estimate the expert's degree of confidence in this composite statement, we first estimate $d(B \,\&\, C)$ as $f_\&(d(B), d(C))$ and then estimate the desired degree as
$$f_\&(d(A), d(B \,\&\, C)) = f_\&(d(A), f_\&(d(B), d(C))).$$
- Similarly, the second expression leads to the estimate
$$f_\&(f_\&(d(A), d(B)), d(C)).$$

It is reasonable to require that these estimates coincide, i.e., that
$$f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$$
for all a, b, and c. In mathematical terms, this means that the operation $f_\&(a, b)$ should be *associative*.

After adding natural requirements of monotonicity, continuity, etc., we get the usual definition of an "and"-operation (also known as a *t-norm*).

From the purely mathematical viewpoint, there are many possible t-norms. In applications, we try to use a t-norm which provide the best approximation to the actual expert reasoning, i.e., for which, in general, the estimate $f_\&(d(A), d(B))$ is the closest to the actual expert's degree $d(A \,\&\, B)$. We can do it by presenting several pairs of statements (A_k, B_k) to the expert and asking the expert, for each k, to estimate his/her degree of confidence in A_k, B_k, and $A_k \,\&\, B_k$. Once we get these estimates $d(A_k)$, $d(B_k)$, and $d(A_k \,\&\, B_k)$, we find a t-norm $f_\&(a, b)$ for which
$$d(A_k \,\&\, B_k) \approx f_\&(d(A_k), d(B_k))$$
for all k. For example, we can use the Least Squares method and find a t-norm $f_\&(a, b)$ for which the sum
$$\sum_k (d(A_k \,\&\, B_k) - f_\&(d(A_k), d(B_k)))^2$$
is the smallest possible.

A similar analysis of the need to estimate the expert's degree of belief in a statement $A \vee B$ leads to the need to consider a commutative, associative, monotonic, and continuous "or"-operation (t-conorm) $f_\vee(a, b)$.

Reminder: "and"- and "or"-operations provide only an approximate description of the expert's degree of confidence. For our analysis, it is important to keep in mind that "and"- and "or"-operations provide only an approximate description of the actual expert's confidence. Indeed, when we use, e.g., an "and"-operation to find the estimates for the expert's degree of certainty in $A \,\&\, B$, we only use the expert's degrees of certainty $d(A)$ and $d(B)$. In reality, the expert's degree of certainty in $A \,\&\, B$ depends not only on $d(A)$ and $d(B)$, but also on some additional information that the expert may have about the relation between A and B. For example, if $d(A) = d(B)$, then we may have two different situations:

- it may be that B coincides with A; in this case, $A \,\&\, B$ is equivalent to A, and so,
$$d(A \,\&\, B) = d(A);$$
- it may be that B is independent from A; in this case, it is reasonable to expect that the expert's degree of belief that both A and B are true should be smaller than the expert's degree of belief that only A is true, i.e., that
$$d(A \,\&\, B) < d(A).$$

In these two cases, the values $d(A)$ and $d(B)$ are the same, but the values $d(A \,\&\, B)$ are different. Therefore, no matter what value we choose as the estimate $f_\&(d(A), d(B))$, at least in one of these two cases, the actual expert's degree of belief $d(A \,\&\, B)$ will be different from this estimate.

III. Need for Complex-Valued Degrees

Ideal case: reminder. In the above text, we described an ideal case when we know the expert's degrees of belief $d_1, \ldots, d_n$ in the basic statements $S_1, \ldots, S_n$, and we use "and"- and "or"-operations to estimate the expert's degree of confidence in different propositional combinations of the basic statements.

In practice, the situation may be somewhat more complicated. Sometimes, instead of knowing the expert's degree of belief in the basic statements, we only know the expert's degree of belief in some propositional combinations of the basic statements. In this case:

- first, we need to recover the degrees $d_1, \ldots, d_n$ from the available information;
- then, we use the recovered values $d_1, \ldots, d_n$ to estimate the expert's degree of belief in other propositional combinations.

If "and"- and "or"-operations were exact, this procedure would always succeed. In the ideal case, when the expert's degree of belief in $A \,\&\, B$ is exactly equal to $f_\&(d(A), d(B))$ and the expert's degree of belief in $A \vee B$ is exactly equal to $f_\vee(d(A), d(B))$, we can indeed recover the desired degrees by solving the corresponding system of equations.

Example. Suppose that we use the algebraic product $f_\&(a, b) = a \cdot b$ as an "and"-operation and

$$f_\vee(a, b) = a + b - a \cdot b$$

as an "or"-operation. Suppose that instead of the actual values $d_1 = d(S_1)$ and $d_2 = d(S_2)$ we only know the degrees

$$d(S_1 \,\&\, S_2) = f_\&(d_1, d_2) = d_1 \cdot d_2$$

and

$$d(S_1 \vee S_2) = f_\vee(d_1, d_2) = d_1 + d_2 - d_1 \cdot d_2.$$

In particular, if we actual (unknown) values of d_1 and d_2 are $d_1 = 0.4$ and $d_2 = 0.6$, then

$$d(S_1 \,\&\, S_2) = 0.4 \cdot 0.6 = 0.24$$

and

$$d(S_1 \vee S_2) = 0.6 + 0.4 - 0.6 \cdot 0.4 = 0.76.$$

These two numbers $d(S_1 \,\&\, S_2) = 0.24$ and $d(S_1 \vee S_2) = 0.76$ are the only information that we have about the expert's degrees d_1 and d_2. Based on these numbers, we want to recover the values d_1 and d_2.

Since we assumed that the t-norm $a \cdot b$ and the t-conorm $a + b - a \cdot b$ describe the expert's belief in composite statements, we form two equations for the two unknowns d_1 and d_2: $d_1 \cdot d_2 = 0.24$ and $d_1 + d_2 - d_1 \cdot d_2 = 0.76$. After adding these two equations, we get $d_1 + d_2 = 1$, hence $d_2 = 1 - d_1$. Substituting $d_2 = 1 - d_1$ into the first equation, we get

$$d_1 \cdot (1 - d_1) = 0.24.$$

After opening parentheses and moving all the terms to the right-hand side, we get the equation

$$d_1^2 - d_1 + 0.24 = 0.$$

By using the known formula for solving quadratic equations, we get

$$d_1 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 0.24} =$$

$$0.5 \pm \sqrt{0.25 - 0.24} = 0.5 \pm \sqrt{0.01} = 0.5 \pm 0.1.$$

Thus, $d_1 = 0.4$ or $d_1 = 0.6$, i.e., (almost) exactly the expert's original estimates.

In this case, due to symmetry, we cannot distinguish between d_1 and d_2, but we can make this distinction if we have additional information.

What happens in practical cases, when the "and"- and "or"-operations are only approximate? Let us now analyze what will happen if we take into account that in reality, "and"- and "or"-operations provide only an approximate description of the expert's degrees of belief. As an example, let us assume that in general, the expert's reasoning is best described by the same "and"- and "or"-operations $f_\&(a, b) = a \cdot b$ and $f_\vee(a, b) = a + b - a \cdot b$. The fact that these operations are the best "on average" does not necessarily mean that these operations always exactly describe the expert's degree of belief in composite statements.

For example, as we have mentioned earlier, if the statements S_1 and S_2 coincide, then

$$d(S_1 \,\&\, S_2) = d(S_1 \vee S_2) = d(S_1).$$

For such two statements with $d(S_1) = d(S_2) = 0.5$, we will get $d(S_1 \,\&\, S_2) = 0.5$ and $d(S_1 \vee S_2) = 0.5$.

Let us see what happens if we try to apply, to these two values $d(A \,\&\, B) = 0.5$ and $d(A \vee B) = 0.5$, the above procedure of reconstructing d_1 and d_2. Specifically, we form two equations: $d_1 \cdot d_2 = 0.5$ and

$$d_1 + d_2 - d_1 \cdot d_2 = 0.5,$$

and we try to find d_1 and d_2 by solving this system of two equations. After adding the two equations, we get $d_1 + d_2 = 1$ and thus, $d_2 = 1 - d_1$. Substituting $d_2 = 1 - d_1$ into the first equation, we get

$$d_1 \cdot (1 - d_1) = 0.5.$$

After opening parentheses and moving all the terms to the right-hand side, we get the equation

$$d_1^2 - d_1 + 0.5 = 0.$$

The determinant of this equation is negative

$$(-1)^2 - 4 \cdot 1 \cdot 0.5 = 1 - 2 = -1 < 0$$

and thus, this equation does not have any real solution – and hence, no solutions with $d_1 \in [0, 1]$.

Natural idea leads to complex-valued degrees. Since we cannot get the degrees from the interval $[0, 1]$, a natural idea is to extend real numbers so that the corresponding equation (or system of equations) has a solution. In principle, we could get any solutions, so it is desirable to make sure that all (or at least almost all) equations (and systems of equations) have a solution. The need to consider quadratic equations immediately leads to the appearance of the imaginary unit $\mathrm{i} = \sqrt{-1}$, which is a solution of the equation $x^2+1=0$, and to the appearance of general complex numbers as solutions of generic quadratic equations.

Good news is that nothing else needs to be added to take care of cubic and higher order equations: a so-called main theorem of algebra states that every polynomial equation has a complex-valued solution (unless this equation has the form $c = 0$ with a constant c which is different from 0).

Thus, we arrive at the need to use complex-valued degrees.

Complex-valued degree may be a solution form the viewpoint of abstract mathematics, but how good are they in practice? Our goal is to describe expert knowledge, so it is reasonable to check how well the corresponding complex-valued degrees describe the expert knowledge.

Example 1. Let us check which complex numbers appear in the above example. By using the known formula for solving quadratic equations, we get

$$d_1 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 0.5} =$$

$$0.5 \pm \sqrt{0.25 - 0.5} = 0.5 \pm \sqrt{-0.25} = 0.5 \pm 0.5 \cdot \mathrm{i}.$$

Of course, it is difficult to interpret complex-valued degrees (or, for that purpose, any degrees outside the interval $[0,1]$). So, it is natural, for each such complex-valued degree, to take the closest value from the interval $[0,1]$.

For complex numbers, the natural distance is Euclidean distance

$$d(a_1 + a_2 \cdot \mathrm{i}, b_1 + b_2 \cdot \mathrm{i}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

It is easy to see that for a complex number $a_1 + a_2 \cdot \mathrm{i}$, the closest point on the real line is its real part a_1, and the closest point on $[0,1]$ is:

- the same value a_1 is $a_1 \in [0,1]$;
- the value 0 is $a_1 < 0$, and
- the value 1 if $a_1 > 1$.

Thus, for the complex numbers $0.5 + 0.5 \cdot \mathrm{i}$ and $0.5 - 0.5 \cdot \mathrm{i}$, the closest numbers from the interval $[0,1]$ are 0.5 and 0.5 – exactly the values that the expert assigned!

Example 2. Let us consider a slightly more general example, with the same "and"- and "or"-operations and with $S_1 = S_2$, but this time, with an arbitrary value $d \in [0,1]$ for which

$$d(S_1) = d(S_2) = d.$$

In this case, we get

$$d(S_1 \,\&\, S_2) = d$$

and

$$d(S_1 \vee S_2) = d.$$

These two values

$$d(S_1 \,\&\, S_2) = d$$

and

$$d(S_1 \vee S_2) = d$$

are all we get from the expert. Based on these two values, we want to reconstruct d_1 and d_2.

In this example, we get a system of equations $d_1 \cdot d_2 = d$ and $d_1 + d_2 - d_1 \cdot d_2 = d$. After adding these two equations, we get $d_1 + d_2 = 2d$, hence $d_2 = 2d - d_1$. Substituting $d_2 = 2d - d_1$ into the first equation, we get

$$d_1 \cdot (2d - d_1) = d.$$

After opening parentheses and moving all the terms to the right-hand side, we get the equation

$$d_1^2 - 2d \cdot d_1 + d = 0.$$

By using the known formula for solving quadratic equations, we get

$$d_1 = d \pm \sqrt{d^2 - d}.$$

Here,

$$d^2 - d = (d-1) \cdot d \le 0,$$

so

$$d_1 = d \pm \sqrt{d - d^2} \cdot \mathrm{i}.$$

For both complex values $d + \sqrt{d - d^2} \cdot \mathrm{i}$ and $d - \sqrt{d - d^2} \cdot \mathrm{i}$, the closest number from the interval $[0,1]$ is the value d – also exactly what the experts assigned.

Example 3: complex numbers are not a panacea. To avoid a false impression that complex numbers also lead to perfect results, let us consider another example in which general "and"- and "or"-operations may not be applicable: an example when S_2 implies S_1. In this case, $S_1 \,\&\, S_2$ is simply equivalent to S_2, and $S_1 \vee S_2$ is equivalent to S_1. So, for example, for $d_1 = 0.6$ and $d_2 = 0.4$, we get

$$d(S_1 \,\&\, S_2) = 0.4$$

and

$$d(S_1 \vee S_2) = 0.6.$$

These two values

$$d(S_1 \,\&\, S_2) = 0.4$$

and

$$d(S_1 \vee S_2) = 0.6$$

are all we get from the expert. Based on these two values, we want to reconstruct d_1 and d_2.

In this example, we get a system of equations $d_1 \cdot d_2 = 0.4$ and $d_1 + d_2 - d_1 \cdot d_2 = 0.6$. After adding these two equations, we get $d_1 + d_2 = 1$, hence $d_2 = 1 - d_1$. Substituting $d_2 = 1 - d_1$ into the first equation, we get

$$d_1 \cdot (1 - d_1) = 0.4.$$

After opening parentheses and moving all the terms to the right-hand side, we get the equation

$$d_1^2 - d_1 + 0.4 = 0.$$

By using the known formula for solving quadratic equations, we get

$$d_1 = 0.5 \pm \sqrt{0.25 - 0.4} = 0.5 \pm \sqrt{-0.15} =$$

$$0.5 \pm \sqrt{0.15} \cdot \mathrm{i}.$$

For both complex values $0.5 + \sqrt{0.15} \cdot \mathrm{i}$ and $0.5 - \sqrt{0.15} \cdot \mathrm{i}$, the closest number from the interval $[0, 1]$ is the value 0.5, which is somewhat different from the original expert values 0.4 and 0.6 (bit still rather close to these values).

IV. Conclusion

Traditionally, fuzzy logic uses degree from the interval $[0, 1]$. These degrees have a clear intuitive sense. Recently, it turned out that in some practical situations, it is beneficial to use complex-valued degrees. While complex-valued degrees are practically useful, their intuitive meaning is not clear. In this paper, we show that an approximate character of "and"-operations $f_\&(a, b)$ and "or"-operations $f_\vee(a, b)$ (also known as t-norms and t-conorms) naturally leads to complex-valued degrees.

Specifically, in some situations, we know the expert's degree of belief $d(S_1 \,\&\, S_2)$ and $d(S_1 \vee S_2)$ in composite statements like $S_1 \,\&\, S_2$ and $S_1 \vee S_2$, and we want to use these degrees to estimate the expert's degrees of belief $d(S_1)$ and $d(S_2)$ in the original statements S_1 and S_2. For this reconstruction, we form a system of equations $d(S_1 \,\&\, S_2) = f_\&(d(S_1), d(S_2))$ and $d(S_1 \vee S_2) = f_\vee(d(S_1), d(S_2))$. In the ideal case, when the expert's degrees of belief in $S_1 \,\&\, S_2$ and $S_1 \vee S_2$ are exactly equal to the results of applying "and"- and "or"-operations to $d(S_1)$ and $d(S_2)$, these equations indeed allows us to reconstruct the desired degrees $d(S_1)$ and $d(S_2)$. However, in reality, the expert's degrees of belief in $S_1 \,\&\, S_2$ and $S_1 \vee S_2$ are somewhat different from the estimates obtained by using "and"- and "or"-operations. As a result, the corresponding system of equations sometimes does not have solutions from the interval $[0, 1]$ – only complex-valued solutions.

On several examples, we show that these complex-valued degree make sense, in the sense that for each of these estimated degrees $\widetilde{d}(S_i)$, the closest real number from the interval $[0, 1]$ is indeed close to (or even equal to) the original expert's degree $d(S_i)$.

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References

[1] S. Aghakhani and S. Dick, "An on-line learning algorithm for complex fuzzy logic", *Proceedings of the 2010 International IEEE Conference on Fuzzy Sets and Systems FUZZ-IEEE'2010*, pp. 1–7.

[2] Z. Chen, S. Aghakhani, J. Man, and S. Dick, "ANCFIS: A Neurofuzzy Architecture Employing Complex Fuzzy Sets", *IEEE Transactions on Fuzzy Systems*, 2011, Vol. 19, No. 2, pp. 305–322.

[3] S. Dick, "Towards complex fuzzy logic", *IEEE Transactions on Fuzzy Systems*, 2005, Vol. 13, No. 3, pp. 405–414.

[4] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[5] O. Kosheleva, V. Kreinovich, and T. Ngamsantivong, "Why Complex-Valued Fuzzy? Why Complex Values in General? A Computational Explanation", *Proceedings of the Joint World Congress of the International Fuzzy Systems Association and Annual Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS'2013*, Edmonton, Canada, June 24–28, 2013, pp. 1233–1236.

[6] H. T. Nguyen, V. Kreinovich, and V. Shekhter, "On the Possibility of Using Complex Values in Fuzzy Logic For Representing Inconsistencies", *International Journal of Intelligent Systems*, 1998, Vol. 13, No. 8, pp. 683–714.

[7] H. T. Nguyen and E. A. Walker, *First Course In Fuzzy Logic*, CRC Press, Boca Raton, Florida, 2006.

[8] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.