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Analysis of Random Metric Spaces Explains Emergence Phenomenon and Suggests Discreteness of Physical Space

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ANALYSIS OF RANDOM METRIC SPACES EXPLAINS EMERGENCE PHENOMENON AND SUGGESTS DISCRETENESS OF PHYSICAL SPACE

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In many practical situations, systems follow the pattern set by the second law of thermodynamics: they evolve from an organized inhomogeneous state into a homogeneous structure-free state. In many other practical situations, however, we observe the opposite *emergence* phenomenon: in an originally homogeneous structure-free state, an inhomogeneous structure spontaneously appears. In this paper, we show that the analysis of random metric spaces provides a possible explanation for this phenomenon. We also show that a similar analysis supports space-time models in which proper space is discrete.

1. Emergence Phenomenon: Brief Reminder

Second law of thermodynamics: brief reminder. One of the main laws of macrophysics is the second law of thermodynamics, according to which any structure and inhomogeneity in a closed system eventually decreases until the system reaches its stable homogeneous structure-less equilibrium state; see, e.g., [1].

Second law of thermodynamics: expected behavior. Of course, real-life systems are rarely closed, there is usually some interaction with the outside world, but many systems are almost closed – with a small interaction. For such systems, it is natural to expect a similar behavior: a spontaneous transition from order to chaos, from inhomogeneity to homogeneity.

In real life, we indeed observe many examples of such behavior:

- if we inject ink into a bowl of water, ink will spread throughout the whole bowl, and the originally inhomogeneous system will eventually become homogeneous;

- houses built thousand years ago and left unattended collapse and traces of them disappear, to the extent that it is sometimes difficult to find where they were.

Emergence phenomenon. In many other situations, however, an inverse process is observed: in a (practically) structure-less homogeneous system, structure and inhomogeneity spontaneously emerge.

Examples of such *emergence phenomena* can be found everywhere:

- in macrophysics, an initial homogeneous mass often dissolves into clumps;

- in astrophysics, a homogeneous dust cloud evolves into stars separated by practically empty spaces;

- in biology, a fetus which originally consists of a homogeneous group of cells spontaneously transforms into a highly inhomogeneous body, with different cells transforming into different organs;

- in social life, in a homogeneous population of initially equal farmers or craftsmen, inequality emerges and increases until some are rich and others are poor.

Comment. The emergence phenomenon does not contradict the *letter* of the second law of thermodynamics, since these systems are not closed, but it seems to violate the *spirit* of this law.

Why is emergence phenomenon ubiquitous? The fact that the emergence phenomenon can be observed in many areas makes us think that there should be a general area-independent explanation for this phenomenon – just like the second law of thermodynamics explains the opposite (homogenization) phenomena.

What we do in the first part of this paper. In the first part of this paper, we provide a possible explanation of the emergence phenomenon.

2. Emergence Phenomenon: Analysis and Possible Explanation

An informal description of the problem: reminder. What we observe is that if we start with a reasonably homogeneous group of objects, then, after a seemingly random perturbation, clusters emerge.

Informally, clusters means that objects within each cluster are, in general, more like each other than like objects from other clusters.

What is a cluster: an ideal case. In the ideal case, if two objects are close to each other, this means that they belong to the same cluster. Hence, if two objects can be connected by a chain of objects such that each object is close to the next one, then all these objects also belong to the same cluster.

If two objects are not close and cannot be connected by a chain of close objects, this means that these two objects belong to different clusters.

What is a cluster: a more realistic approach. In practice, we sometimes have objects which are similar to each other, but belong to different clusters.

For example, whales and dolphins live and swim in the water just like fish, their shape is similar to the shape of a fish, but they are mammals and thus, they belong to completely different clusters than fish.

What makes clustering meaningful is that such examples are rare, they form less than a small percentage of all possible connections, and if we exclude such rare abnormal closenesses, we get a perfect subdivision into clusters.

Towards formalizing the problem. How can we describe the above phenomenon in precise terms? A natural way to described to what extend two objects are like each other is to use a *metric*, i.e., a function that assigns to every two objects a and b, a real number $d(a, b) \geq 0$ (known as *distance*) describing how different are these objects. Intuitively, it is reasonable to require:

- that this function is symmetric $d(a, b) = d(b, a)$,

- that $d(a, b) = 0$ if and only if a and b are the same object, i.e., if and only if $a = b$, and

- that this function should satisfy the triangle inequality $d(a, c) \leq d(a, b) + d(b, c)$ for all a, b, and c.

In mathematics, such functions are known as *metrics*.

In terms of the metric, similarity between the objects a and b means that the distance $d(a, b)$ between them does not exceed a certain threshold t: $d(a, b) \leq t$. In these terms, the ideal subdivision into clusters means that we classify a and b to the same class if and only if there is a chain $a_0 = 1$, a_1, a_2, ..., $a_m = b$ for which $d(a_i, a_{i+1}) \leq t$ for all i. In other words, we consider the *transitive closure* of the symmetric relation $d(a, b) \leq t$. A transitive closure of a symmetric relation is an equivalence relation. With respect to this relation, the set of all the objects is divided into equivalence classes; these classes form clusters.

In reality, as we have mentioned, there may be some deviations from this ideal clustering: out of $\dfrac{n \cdot (n-1)}{2}$ possible distances, some may not correspond to clustering. Let $\delta > 0$ be a proportion of distances which may not correspond to clustering. Then, out of $\dfrac{n \cdot (n-1)}{2}$ distances, $\delta \cdot \dfrac{n \cdot (n-1)}{2}$. We say that two metrics on the set of n elements are δ-*close* if they differ only in $\delta \cdot \dfrac{n \cdot (n-1)}{2}$ distance values. In these terms, the actual clustering means that for some given value $\delta > 0$,

- first, we find a metric $\widetilde{d}$ which is δ-close to the original metric, and then

- we divide the objects into the equivalence classes corresponding to the transitive closure of the relation $\widetilde{d}(a, b) \leq t$.

Which metrics should we consider? Originally, all elements were similar to each other to the same degree, i.e., we have $d(a, b) = \text{const}$ for all $a \neq b$. Once we apply a random perturbation, instead of the original constant metric, we now have a metric which is random – with respect to some probability distribution.

From this viewpoint, a natural idea is to analyze random metrics.

How can we describe random metrics? To select a random metric, we need to select a value $d(a, b)$ for each pairs (a, b) in such a way that the resulting values form a metric. We would like this selection to be random in some natural sense.

When we have finitely many possible alternatives, it is natural to define a random object: just pick one of the possible alternatives with equal probability. There is no such natural solution for the case of infinitely many alternatives, so a natural idea is to do what we usually do to process infinite collections, whether it is a sum of an infinite series, or an integral:

- to consider approximate situation with finitely alternatives, and then

- consider the limit of the result when the number of alternatives tends to infinity.

In our case, we do have finitely many objects; let us denote the number of objects by n. However, for each two objects a and b, we have an infinite set of possible values of the distance $d(a, b)$: namely, the set of all positive real numbers. Let us therefore approximate this infinite set by a finite set of values 0, $\dfrac{1}{N}$, $\dfrac{2}{N}$, ..., $\dfrac{r}{N}$ for some positive integers r and N. When $N \to \infty$ and $\dfrac{r}{N} \to \infty$, these values become everywhere dense on the real line and thus, approximate any real value with any given accuracy.

For each r and N, for each of the finitely many pairs (a, b), we have a finite number of possible values of distance. Thus, we have finitely many possible metrics on a given set of n elements. In accordance with the above natural idea, we consider all possible metrics equally probable.

Which properties are we interested in? In some metrics, we may have the emergence phenomenon, in some we do not. By counting the number of metrics with and without the emergence phenomenon, we can find the probability that the emergent phenomenon occurs.

The number n of objects is usually large, so it makes sense to consider the limit value of the corresponding probability when $n \to \infty$: by definition of the limit, the probability corresponding to large n should be equal to this value. Now, the problem if formulated in precise terms:

- we fix n, r, and N;

- we consider a set of n elements;

- on this set, we consider all possible metrics for which, for every two objects a and b, we have $d(a, b) = \dfrac{k}{N}$ for some integer $k = 0, 1, \ldots, r$;

- we consider all these metrics equally probable;

- we consider the probability of different properties in the limit $n \to \infty$.

Known results. The analysis of such random metric spaces was performed in [?]. In this paper, we use only the results corresponding to odd r; this is OK, since we are interested in $r \to \infty$ anyway, so we can always restrict ourselves to only odd values of r.

For odd r, the conclusion is that for every $\delta > 0$, with probability that tends to 1 as $n \to \infty$, there exists a metric $\widetilde{d}(a, b)$ which is δ-close to the original metric and for which the initial set of n elements can be divided into ℓ subsets $S_1, \ldots, S_\ell$ such that:

- for all pairs $a \neq b$ from the same subset, we have $\dfrac{r-1}{2N} \leq \widetilde{d}(a, b) \leq \dfrac{r-1}{N}$;

- for pairs of elements from different subsets, we have $\dfrac{r+1}{2N} \leq \widetilde{d}(a, b) \leq \dfrac{r}{N}$.

In principle, we can have trivial situations:

- when $\ell = 1$, i.e., when all the objects form a single cluster, and

- when $\ell = n$, i.e., each cluster S_i consists of only one object.

However, in [?], it is proven that the probability of such situations tends to 0.

Analysis of these results. Based on the above result, we conclude that if for some pair a and b, we have $\widetilde{d}(a, b) \leq \dfrac{r-1}{2N}$, then a and b belong to the same class S_i. Let us select a threshold $t \stackrel{\text{def}}{=} \dfrac{r-1}{2N}$. Then, if a and b are connected by a chain of similar objects, then a and b still belong to the same class. Thus, each equivalence class corresponding to the transitive closure of the relation $\widetilde{d}(a, b) \leq t$ is a subset of a set S_i.

Thus, if we subdivide each set S_i into such equivalence classes, we get sets S'_1, $\ldots$, $S'_{\ell'}$ which are clusters and which, as one can easily see, satisfy the same two requirements as the original sets S_i.

In principle, we can have one class containing all objects, or we can have n classes each of which contain a single object, but, as we have mentioned, the probability of such trivial subdivision tends to 0. Thus, *we have an explanation for the empirical fact that objects naturally divide into clusters.*

Conclusion. *For every $\delta > 0$, for almost all random metrics $d(a, b)$, there exists a δ-close metric $\widetilde{d}(a, b)$ for which we get a non-trivial subdivision into clusters S'_1, $\ldots$, $S'_{\ell'}$ so that:*

- *every two close objects (i.e., objects with $\widetilde{d}(a, b) \leq t$) belong to the same cluster, and*

- *if two objects a and b are assigned to the same cluster, this means that they can be connected by a chain of elements $a_0 = a$, a_1, $\ldots$, a_{m-1}, $a_m = b$ for which, for every i, objects a_i and a_{i+1} are close to each other.*

3. Physical Space: Why Discreteness

Physical space: problem with continuity. According to modern physics, the matter consists of molecules and atoms which are, in turn, formed from elementary

particles, and the elementary particles are no longer divisible. A state of each elementary particle is uniquely determined by finitely many parameters; usually, its coordinates, momentum, and angular momentum (for particles with spin).

In relativity theory, elementary particles are point-wise. In Newton's theory, we could potentially consider particles of finite spatial size. In such a theory, immediate action-at-a-distance is possible, so if we apply a force to one point on the surface of the particle, this particle can react as a whole, thus acting as a single indivisible object.

In relativistic physics, however, effects can no longer spread instantaneously: the speed with which each effect spreads is limited by the speed of light. As a result, when an action is applied to one point on the surface of the particle, at that moment, other parts of the particle are not affected. In this sense, the particle of finite size is no longer an indivisible object – in effect, it consists of points which may arrange themselves differently. From this viewpoint, an indivisible elementary particle must be spatially point-wise; see, e.g., [1].

Point-wise particles cause problems. Let us describe a simple easy-to-describe problem with a point-wise particle. This problem appears when we try to find the overall energy of the electric field generated by a single charged particle.

The energy density $\rho(x)$ of the electric field $\vec{E}(x)$ is known to be proportional to $|\vec{E}(x)|^2$, and the electric field of a point-wise particle decreases with the distance R to the particle according to the Coulomb law $|\vec{E}(x)| \sim \dfrac{1}{R^2}$. Thus, the energy density $\rho(x)$ is proportional to $|\vec{E}(x)|^2 \sim \dfrac{1}{R^4}$.

The overall energy is equal to the integral $\int \rho(x)\, dx$ and is, thus, proportional to the integral $\int \dfrac{1}{R^4}\, dx$. In polar coordinates, after integrating over angular coordinates, we get
$$I = \int_0^\infty \frac{2\pi \cdot R^2}{R^4}\, dr = 2\pi \cdot \int_0^\infty \frac{1}{R^2}\, dr.$$
This integral is equal to
$$I = -\,2\pi \cdot \frac{1}{R}\Bigg|_0^\infty = \infty.$$

Similar physically meaningless infinities appear when we compute other quantities related to a point particle [1].

Natural idea: discrete space. Since allowing distances R as close to 0 as possible leads to physically meaningless conclusions, a reasonable idea is to assume that such distances are not physically possible, i.e., that there is a lower limit R_0 on how close spatial points can be to each other.

In other words, since the assumption that the proper space is *continuous* leads to physically meaningful results, it is reasonable to consider models in which space is discrete.

One of the possible approaches: Wheeler's geometry without geometry. One of the possible approaches to coming with discrete proper space is the approach

of geometry without geometry initiated by J. A. Wheeler; see, e.g., [4].

This approach is motivated by the fact that to properly account for physics, we must take into account quantum effects. In the first approximation, quantum mechanics means, crudely speaking, that instead of trajectories uniquely determined by the initial conditions (as in Newtonian physics), particles can follow any trajectory – with different probabilities. On the macro-level, the probability of deviating from the Newtonian trajectory is very small, but as we decrease the sizes, due to uncertainty principle energy fluctuations increase and the deviations become the norm.

In the first approximation, particles' trajectories are random, but the interaction between the particles are still described by the same laws as in Newtonian physics: e.g., the two charged particles interact according to the Coulomb's law. In other words, the corresponding fields – which are responsible for this interaction – are uniquely determined by the trajectories of the particles. In the second approximation (also known as second quantization), we take into account that, in quantum theory, the fields can also randomly deviate from their deterministic values.

At the atomic sizes, where the quantum effects start being important, the effects of the second quantization are small, but as the sizes decrease to the size of elementary particles, these effects become dominant.

What Wheeler noticed is that in general, in quantum mechanics, nothing is deterministic, all logically possible models are possible with a certain probability, and as we further decrease the size and thus, increase the energy fluctuations, all logical models become equally probable. Thus, a natural way to study the properties of, e.g., spatial metric on small distances is to consider the situation when all possible metrics are possible.

Such a random metric model is exactly what we considered in the previous section. So, let us use the results of the corresponding analysis.

Random metrics explain discreteness of proper space. According to the results from [?], with probability tending to 1, the distances are bounded from below by a number $R_0 \approx \dfrac{r}{2N}$. A similar conclusion can be derived if, instead of discrete approximations to metrics, we consider continuous metrics [3].

Thus, the Wheeler's model indeed leads to discreteness.

Comment. This conclusion is also in line with a similar lower bound that we obtained by using additional physical assumptions [2].

Conclusion. *Random metrics explain discreteness of proper space.*

Acknowledgments

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