7-2016

Geometric Symmetries Partially Explain Why Some Paleolithic Signs Are More Frequent

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Comments:
Technical Report: UTEP-CS-16-47

Geometric Symmetries Partially Explain Why Some Paleolithic Signs Are More Frequent

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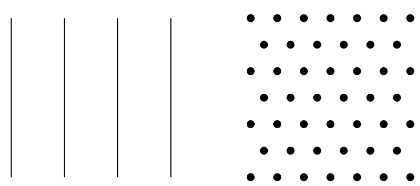
Abstract

A recent analysis of Paleolithic signs have described which signs are more frequent and which are less frequent. In this paper, we show that this relative frequency can be (at least partially) explained by the symmetries of the signs: in general, the more symmetries, the more frequent the sign.

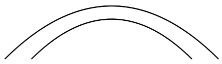
1 Different Paleolithic Signs Occur with Different Frequencies: Formulation of the Problem

Interesting empirical fact. In a recent book [1], the author analyzes Paleolithic signs. In particular, in Chapter 16, she lists seven most frequent signs:

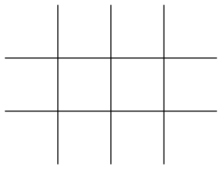
- the most frequent sign is a set of *parallel lines* (P);
- next in frequency are *dots* (D);
- then the following signs are of approximately the same frequency: *half-circles* (C), *crosshatches* (CH, see below), and *finger fluting* (F, see below);
- even rarer are *zigzags* (Z);
- finally, the least frequent of the seven symbols are *spirals* (S).



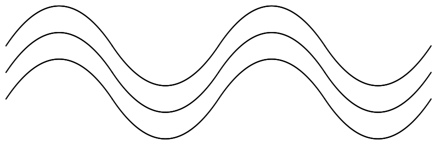
P D



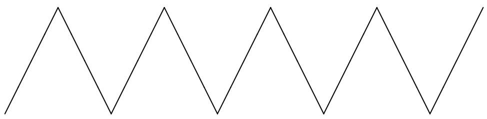
C



CH



F



Z

Open question. How can we explain the observed relative frequency?

Our idea. Our idea is to look into the symmetries of the corresponding geometric shapes. This idea is motivated by the fact that many real-life objects have symmetries, and so, symmetries are important.

Resulting explanation. As a result of our analysis, we show that most of the relative frequencies of different Paleolithic signs can be explained by their geometric symmetry: the more symmetries, the more frequent the sign.

2 Analysis of the Problem

Which geometric symmetries we consider. Since we are talking about 2-D signs, natural geometric symmetries are symmetries of the 2-D geometry: shifts and rotations.

To use symmetries, we must view each sign as a local part of a potentially unbounded image. Signs are *bounded* in space. Strictly speaking, a bounded set cannot be invariant with respect to any shift – once we allow a shift by a vector a, then for each point x from the invariant set, we will have an unbounded set of points $x + n \cdot a$ for integers $n \in Z$ contained in the same set.

In this sense, the only possible invariance is rotation-invariance – but none of the several most frequent signs is rotation-invariant.

So, to meaningfully use symmetry, we need to consider signs as local parts of a potentially *unbounded* image. This is a reasonable idea, since how else can we draw a potentially unbounded set – such as an (infinite) straight line – other than by drawing its local fragment?

How do we compare symmetries of different signs? A natural idea is to compare the *numbers* of symmetries.

Of course, for most symmetries, the number is infinite, so, strictly speaking, we only have two options:

- continuum (for continuous symmetry groups) and
- countably many (for discrete groups).

To get a more detailed comparison, let us take into account that we are only considering a bounded region – and that within this region, we only consider values with some accuracy.

Let us denote the linear size of the region by L, the accuracy by ε, the distance between parallel lines or parallel circles by d, and the circle's radius by R. For example, for a 1-D group of shifts, if we only consider shifts by $\le L$, and if we consider shifts different if they differ at least by ε, then we get $\dfrac{L}{\varepsilon}$ different shifts.

We then count symmetries for each of the type. If for some pair of signs, the first sign has significantly more symmetries when $L \to \infty$ and $\varepsilon \to 0$, then we say that the first the first sign is more symmetric. Of course, we have to be cautious here: these are approximate estimates, so we should not make serious conclusions if number of symmetries differs only by, say, a factor of two, we really need to check whether these numbers are significantly different: e.g., whether the ratio between the corresponding numbers tends to infinity.

Let us describe – and count – geometric symmetries of different Paleolithic signs. In view of the above comment, let us analyze the geometric symmetries of the seven Paleolithic signs.

Case of parallel lines. A set of equidistant parallel line segments (P) is, naturally, a local part of the set of an infinite sequence of equidistant parallel lines, with distance d between the lines.

Let us select the coordinates so that all the lines are parallel to the y-axis. Then, for this sign, we have:

- a 1-D continuous unbounded family of shifts along the lines $y \to y + y_0$, and
- a 1-D discrete unbounded family of shifts from one line to the other $x \to x + n \cdot d$, $n \in Z$.

Overall, we have $\dfrac{L}{\varepsilon}$ continuous shifts and $\dfrac{L}{d}$ discrete ones. A general symmetry is a combination of these two shifts; so overall, we have

$$S_{\rm P} = \frac{L^2}{\varepsilon \cdot d}$$

symmetries.

Case of dots. For the set of dots (D), we have:

- a 1-D discrete unbounded family of shifts in the horizontal direction,
- a 1-D discrete unbounded family of shifts in the direction which is 60° to this direction, and
- 6 rotations by angles proportional to 60°.

So, the overall number of symmetries is

$$S_{\rm D} = 6 \cdot \frac{L}{d} \cdot \frac{L}{d} = \frac{6 \cdot L^2}{d^2}.$$

Case of half-circle. A half-circle is a bounded part of a circle. For a circle C, the only symmetries are rotations, so we have a 1-D bounded family of rotations, with

$$S_{\rm C} = \frac{2\pi \cdot R}{\varepsilon}$$

possible symmetries.

Case of crosshatch. For a crosshatch CH, which is a bounded part of a two mutually orthogonal families of equidistant parallel lines, we have:

- a 1-D discrete unbounded family of shifts in x-direction,
- a 1-D discrete unbounded family of shifts in y-directions, and
- 4 rotations by angles proportional to 90°.

So, the overall number of symmetries is

$$S_{\rm CH} = 4 \cdot \frac{L}{d} \cdot \frac{L}{d} = \frac{4 \cdot L^2}{d^2}.$$

Case of finger fluting. For finger fluting F, i.e., for a family of parallel periodic lines, we have:

- a 1-D discrete unbounded family of shifts in x-direction,
- a 1-D discrete unbounded family of shifts in y-directions, and
- a rotation by 180 degrees.

So, the overall number of symmetries is

$$S_{\rm F} = 2 \cdot \frac{L}{d} \cdot \frac{L}{d} = \frac{2 \cdot L^2}{d^2}.$$

Case of a zigzag. For a zigzag (Z), there is only one 1-D discrete unbounded family of shifts plus rotation by 180 degrees; so, the overall number of symmetries is

$$S_{\rm Z} = 2 \cdot \frac{L}{d} = \frac{2 \cdot L}{d}.$$

Case of a spiral. Finally, for the spiral (S), there are no symmetries at all:

$$S_{\rm S} = 1$$

(It is worth mentioning that symmetries appear if we also allow homotheties.)

Let us compare symmetry groups of different signs. When $L \to \infty$ and $\varepsilon \to 0$, then clearly

$$\frac{S_{\rm P}}{S_{\rm D}} \to \infty, \quad \frac{S_{\rm P}}{S_{\rm CH}} \to \infty, \quad \frac{S_{\rm P}}{S_{\rm C}} \to \infty, \quad \frac{S_{\rm P}}{S_{\rm F}} \to \infty, \quad \frac{S_{\rm P}}{S_{\rm Z}} \to \infty, \text{ and } \frac{S_{\rm P}}{S_{\rm S}} \to \infty.$$

So, the set P of parallel lines is clearly more symmetric than every other sign.

Similarly,

$$\frac{S_{\rm D}}{S_{\rm Z}} \to \infty, \quad \frac{S_{\rm CH}}{S_{\rm Z}} \to \infty, \quad \frac{S_{\rm C}}{S_{\rm Z}} \to \infty, \text{ and } \frac{S_{\rm F}}{S_{\rm Z}} \to \infty.$$

Thus, the dots (D), the crosshatch (CH), the circle (C), and the finger fluting (F) are all more symmetric than the zigzag.

Finally,

$$\frac{S_{\rm Z}}{S_{\rm S}} \to \infty,$$

so the zigzag (Z) is more symmetric than the spiral (S).

3 Conclusions and Remaining Open Problems

Conclusions. By comparing the above symmetry-comparison between signs and the observed relative frequency of these signs, we can see that in many cases, symmetries explain relative frequency: signs with more symmetries are more frequent.

There are, however, two important exceptions to this general rule:

- Empirically, the dots sign (D) is more frequent than crosshatch (CH) or finger fluting (F). In our estimates, the dots indeed have more symmetries, but the difference is rather small: a factor of 1.5 or 3. So, based on our approximate model, we cannot make a definite conclusion about which sign has more symmetries.

- Empirically, the half-circle (C) is less frequent than the dots (D) and has approximately the same frequency as the crosshatch (CH) and finger fluting (F). However, based on our comparison, we cannot tell which sign has more symmetries: the results of our comparison depend on in which we order we tend L to infinity and ε to 0.

These exception motivate us to formulate the first remaining open problem.

First remaining open problem: make the above approximate model more accurate, so that it will be able to also explain the relative frequency of crosshatch and half-circle.

Discussion. There is also another open problem, motivated by the fact that the book [1] contains not only comparison between the frequencies, but also the frequencies themselves:

- parallel lines (P) occur in approximately 60% of the sites;
- the dots (D) occur in approximately 40% of the sites;
- each of the three signs crosshatch (CH), circle (C), and finger fluting (F) occur in approximately 20% of the sites;
- zigzags (Z) occur in approximately 10% of the sites; and
- the spiral (S) occurs in very few sites.

This fact motivates the following problem:

Second remaining open problem: explain the numerical values of the observed frequencies.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

References

[1] G. von Petzinger, *The First Signs: Unlocking the Mysteries of the World's Oldest Symbols*, Atria Books. New York, 2016.