8-2016

How to Determine the Stiffness of the Pavement's Upper Layer (Base) Based on the Overall Stiffness and the Stiffness of the Lower Layer (Subgrade)

Christian Servin
El Paso Community College, cservin@gmail.com

Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu

Comments:
Technical Report: UTEP-CS-16-53

How to Determine the Stiffness of the Pavement's Upper Layer (Base) Based on the Overall Stiffness and the Stiffness of the Lower Layer (Subgrade)

Christian Servin[1] and Vladik Kreinovich[2]
[1]Information Technology Department
El Paso Community College
919 Hunter, El Paso TX 79915, USA
cservin@gmail.com
[2]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
vladik@utep.edu

Abstract

In road construction, it is important to estimate difficult-measure stiffness of the pavement's upper layer based the easier-to-measure overall stiffness and the stiffness of the lower layer. In situations when the overall stiffness is not yet sufficient, it is also important to estimate how much more we need to add to the upper layer to reach the desired overall stiffness. In this paper, for the cases when a linear approximation is sufficient, we provide analytical formulas for the desired estimations.

1 Formulation of the Problems

Need for multiple-layer pavements. Usually, the soil is not stiff enough to serve as a base for the road. Two ideas are used to reach the desired stiffness:

- first, the soil is compacted, to increase it stiffness;
- second, on top of the compacted soil – which now serves as a subgrade – a stiffer layer of another material ("base") is placed (and then compacted too).

How to characterize stiffness. Stiffness describes how much the pavement is displaced under the external force: the smaller the resulting displacement, the stiffer the pavement. Thus, stiffness is usually characterized by the ratio of the stress (force per unit area) to the corresponding displacement (also known as *deflection*). In mechanics, this ratio is called *modulus*. The larger the modulus, the stiffer the pavement.

Need to measure stiffness. Road specification requires:

- that the overall stiffness is above a certain threshold (described in pavement specifications), and
- that the stiffness of the upper later is also above some threshold – already described in the pavement specifications.

Thus, to make sure that the pavement satisfies the specifications, we need to measure the corresponding stiffness values.

The overall stiffness can be measured directly. The overall pavement stiffness can be measured directly, e.g., by using a light weight deflectormeter (LWD): a falling weight hits the pavement, and we measure the deflection caused by this fall. In this experiment, we know the force, and we know the contact area, so we can compute the stress. Thus, we can find the value of the desired overall modulus by dividing this stress by the observed deflection.

The stiffness of the upper layer is not easy to measure directly. In contrast to the overall stiffness, the stiffness of the upper layer is not easy to measure directly: whatever force we apply, the resulting deflection is determined not only by the upper layer (base), but also by the elastic properties of the lower layer (subbase).

Main problem: determine the stiffness of the pavement's upper layer based on the overall stiffness and the stiffness of the lower layer. Since we cannot easily measure the desired stiffness directly, a natural idea is to measure it *indirectly*, i.e., measure related quantities and try to reconstruct the desired stiffness based on the measurement results.

In our case, the only reasonably-easy-to-measure quantity is the overall stiffness. Good news is that we can measure it both *after* the upper layer is placed and compacted and *before* the upper layer is placed and compacted – in which case we measure stiffness of the lower layer. Thus, we arrive at the following problem:

- given the overall stiffness and the stiffness of the lower layer,
- estimate the stiffness of the upper layer.

Auxiliary problem. When the measured overall stiffness is above the desired thresholds, this means that the pavement is ready. But what if the measurements show that we have not yet reached the desired stiffness value? In this case, it is desirable to estimate how much more of the base we should add to reach the desired stiffness value; see, e.g., [1, 2, 3, 4, 5].

What we do in this paper. In this paper, we consider a linear approximation model, and we show, how to solve – in this approximation – the above two problems, i.e.:

- to determine the stiffness of the upper layer based on the overall stiffness and the stiffness of the lower layer, and
- if needed, to estimate how much more of the upper layer we need to reach the desired overall stiffness.

Comment. There exist several techniques for performing similar estimates in the general case, when we consider the actual non-linear model instead of a linearized approximation; see, e.g., [2, 5].

2 Analysis of the First Problem

Starting point for our analysis: Boussinesq solution. In the first approximation, we can ignore the depth of the pavement, so we can consider the pavement as an infinite isotropic half-space $z \geq 0$.

When LWDs estimate the stiffness of the pavement, they apply the force in an area of a small radius r_0. This radius is much smaller than the size of the road, so, from the mathematical viewpoint, we can safely describe the effect of the LWD as a point force.

For the point force applied to an infinite isotropic half-space, there is an explicit solution first provided by Boussinesq. According to this solution, the vertical displacement u caused by the vertical force F at a point (x, y, z) has the form

$$u(x,y,z) = \frac{F}{2\pi} \cdot \frac{1+\nu}{E} \cdot \left(\frac{2(1-\nu)}{r} + \frac{z^2}{r^3} \right), \quad (1)$$

where $r \stackrel{\text{def}}{=} \sqrt{x^2 + y^2 + z^2}$, ν is the Poisson ratio, and E is the Young modulus.

We are interested in the deflection at the point of contact. Strictly speaking, for the point force, the above formula leads to an infinite displacement. To make the answer realistic, we need to take into account that the LWD is actually not a point, it is a small disk. Thus, to estimate the deflection cause by the LWD, it is reasonable to take the deflection corresponding to some "average" distance r_0 from the LWD's center. Substituting $r = \sqrt{r_0^2 + z^2}$ into the above formula (1), we get

$$u(z) = \frac{F}{2\pi} \cdot \frac{1+\nu}{E} \cdot \left(\frac{2(1-\nu)}{\sqrt{r_0^2 + z^2}} + \frac{z^2}{(r_0^2 + z^2)^{3/2}} \right), \quad (2)$$

i.e.,

$$u(z) = \frac{F}{2\pi} \cdot \frac{1+\nu}{E} \cdot U(z), \quad (3)$$

where we denoted

$$U(z) \stackrel{\text{def}}{=} \frac{2(1-\nu)}{\sqrt{r_0^2 + z^2}} + \frac{z^2}{(r_0^2 + z^2)^{3/2}}. \quad (4)$$

What we measure is the deflection at the surface, when $z = 0$. Substituting $z = 0$ into the formula (2), we get the following formula for the measured deflection $\widetilde{u}$:

$$\widetilde{u} = \frac{F}{2\pi} \cdot \frac{1+\nu}{E} \cdot \frac{2(1-\nu)}{r_0}. \quad (5)$$

From the homogeneous situation to the situation when the modulus may change with depth. The formula (2) describes the case when the elasticity modulus E is the same for all the depths. We would like to use this formula to predict what will happen in the more realistic setting, when the elasticity modulus changes with depth, i.e., when $E = E(z)$.

To come up with such a formula, let us recall that as the force is applied, the deflection starts at the top, at the point where the force is applied, and propagates down, to larger depths z.

In general, a layer between depths z and $z + \Delta z$ has a thickness Δz. After the application of the force, the top boundary of this layer, at depth z, is shifted by the deflection $u(z)$, so it now at depth $z+u(z)$. Similarly, the lower boundary of this layer, at depth $z + \Delta z$, is shifted by the deflection $u(z + \Delta z)$, so it now at depth $z + \Delta z + u(z + \Delta z)$. The new thickness of the layer can be computed as the difference between the new depths of its lower and upper boundaries, i.e., as the value

$$(z + \Delta z + u(z + \Delta z)) - (z + u(z)) = \Delta z + (u(z + \Delta z) - u(z)).$$

If this layer is absolutely stiff (incompressible), then its thickness cannot change, so we have $\Delta z+(u(z+\Delta z)-u(z)) = \Delta z$ and thus, $u(z+\Delta z)-u(z) = 0$. If this layer is absolutely soft, then it will fully absorb all the compression and the bottom boundary will not change at all.

Both conclusions are based only on the elasticity of this particular layer. In general, it is reasonable to assume that the relative deflection $u(z + \Delta z) - u(z)$ of a narrow layer depends only on the elasticity of this layer. For thin layers, the difference is approximately equal to $u'(z) \cdot \Delta z$, where $u'(z)$ is the derivative of the function $u(z)$ that describes how deflection u changes with depth z. So, we conclude that the derivative $u'(z)$ should depend only on the local value $E(z)$.

To find the dependence of $u'(z)$ on $E(z)$, let us take into account that, by differentiating both sides of the formula (3), we conclude that, for every value E and for every depth z, we have

$$u'(z) = \frac{F}{2\pi} \cdot \frac{1+\nu}{E} \cdot U'(z). \quad (6)$$

Thus, the desired dependence takes the form

$$u'(z) = \frac{F}{2\pi} \cdot \frac{1+\nu}{E(z)} \cdot U'(z). \quad (7)$$

By integrating this derivative and taking into account that the deflection disappears at infinity $u(\infty) = 0$, we can get the desired formula for the deflection

$u(z)$:

$$u(z) = -(u(\infty) - u(z)) = -\int_z^\infty u'(t)\,dt. \tag{8}$$

Substituting the expression (7) into this formula, and taking the constants not depending on z outside the integral, we conclude that

$$u(z) = -\frac{F}{2\pi} \cdot (1+\nu) \cdot \int_z^\infty \frac{1}{E(t)} \cdot U'(t)\,dt. \tag{9}$$

What we observe is the value $\widetilde{u} = u(0)$ corresponding to $z = 0$. For this value, the formula (9) takes the form

$$\widetilde{u} = -\frac{F}{2\pi} \cdot (1+\nu) \cdot \int_0^\infty \frac{1}{E(t)} \cdot U'(t)\,dt. \tag{10}$$

Let us apply the resulting formula to the case of the two-layer pavement. Let us use this formula to analyze the pavement situation when we have a two-layer pavement: the first layer starts at the surface and ends at the depth z_0, and then the second layer starts. We know the overall stiffness E and we know the stiffness E_2 of the second layer. Our goal is to estimate the stiffness E_1 of the top layer,

In this case, $E(t) = E_1$ for $0 \le t \le z_0$ and $E(t) = E_2$ for $t \ge z_0$. Substituting this expression into the formula (9), we conclude that

$$\widetilde{u} = \frac{F}{2\pi} \cdot (1+\nu) \cdot \left(-\int_0^{z_0} \frac{1}{E_1} \cdot U'(t)\,dt - \int_{z_0}^\infty \frac{1}{E_2} \cdot U'(t)\,dt \right). \tag{11}$$

Let us estimate these two integrals.

In the first integral, we can take the constant factor $\dfrac{1}{E_1}$ outside the integral, then we get

$$-\int_0^{z_0} \frac{1}{E_1} \cdot U'(t)\,dt = -\frac{1}{E_1} \cdot \int_0^{z_0} U'(t)\,dt = \frac{1}{E_1} \cdot (U(0) - U(z_0)). \tag{12}$$

Similarly, in the second integral, we can take the constant factor $\dfrac{1}{E_2}$ outside the integral, then we get

$$-\int_{z_0}^\infty \frac{1}{E_2} \cdot U'(t)\,dt = -\frac{1}{E_2} \cdot \int_{z_0}^\infty U'(t)\,dt = \frac{1}{E_2} \cdot U(z_0). \tag{13}$$

Substituting the expressions (12) and (13) into the formula (11), we conclude that

$$\widetilde{u} = \frac{F}{2\pi} \cdot (1+\nu) \cdot \left(\frac{1}{E_1} \cdot (U(0) - U(z_0)) + \frac{1}{E_2} \cdot U(z_0) \right). \tag{14}$$

We know that $U(0) = \dfrac{2-2\nu}{r_0}$. Here, $z_0 \gg r_0$, so $r_0^2 + z^2 \approx z^2$, and the formula for $U(z_0)$ takes the form so the above formula takes the form

$$U(z_0) = \frac{3-2\nu}{z_0}. \tag{15}$$

Thus, we conclude that

$$\widetilde{u} = \frac{F}{2\pi} \cdot (1+\nu) \cdot \left(\frac{1}{E_1} \cdot \left(\frac{2-2\nu}{r_0} - \frac{3-2\nu}{z_0} \right) + \frac{1}{E_2} \cdot \frac{3-2\nu}{z_0} \right). \tag{16}$$

Now we are ready to solve the first problem: estimating the stiffness of the upper layer based on the overall stiffness and the stiffness of the lower layer. We know the formula (5) that determines how the observed deflection $\widetilde{u}$ depends on the overall stiffness E. By equating the expressions (5) and (16) for the deflection $\widetilde{u}$, we get

$$\frac{F}{2\pi} \cdot \frac{1+\nu}{E} \cdot \frac{2(1-\nu)}{r_0} =$$

$$\frac{F}{2\pi} \cdot (1+\nu) \cdot \left(\frac{1}{E_1} \cdot \left(\frac{2-2\nu}{r_0} - \frac{3-2\nu}{z_0} \right) + \frac{1}{E_2} \cdot \frac{3-2\nu}{z_0} \right). \tag{17}$$

Dividing both sides of this equality by $\dfrac{F}{2\pi} \cdot (1+\nu)$, we conclude that:

$$\frac{1}{E} \cdot \frac{2(1-\nu)}{r_0} = \frac{1}{E_1} \cdot \left(\frac{2-2\nu}{r_0} - \frac{3-2\nu}{z_0} \right) + \frac{1}{E_2} \cdot \frac{3-2\nu}{z_0}. \tag{18}$$

Now we are ready to describe the solution to our first problem. We assume that we know the overall stiffness E and the stiffness E_2 of the lower layer, and we would like to know the stiffness E_1 of the upper layer. To find E_1, we move the terms containing E_2 to the right-hand side of the equation (18):

$$\frac{1}{E_1} \cdot \left(\frac{2-2\nu}{r_0} - \frac{3-2\nu}{z_0} \right) = \frac{1}{E} \cdot \frac{2(1-\nu)}{r_0} - \frac{1}{E_2} \cdot \frac{3-2\nu}{z_0}. \tag{19}$$

By dividing both sides of the resulting equation by the coefficient at $1/E_1$, we can find $1/E_1$:

$$\frac{1}{E_1} = \frac{\dfrac{1}{E} \cdot \dfrac{2(1-\nu)}{r_0} - \dfrac{1}{E_2} \cdot \dfrac{3-2\nu}{z_0}}{\dfrac{2-2\nu}{r_0} - \dfrac{3-2\nu}{z_0}}. \tag{20}$$

By inverting this expression, we arrive at the following formula.

3 Solution to the First Problem

Solution to the first problem: resulting formula. If we know the overall stiffness E, the stiffness E_2 of the lower layer, and the thickness z_0 of the upper layer, then we can compute the stiffness E_1 of the upper layer as follows:

$$E_1 = \frac{\dfrac{2-2\nu}{r_0} - \dfrac{3-2\nu}{z_0}}{\dfrac{1}{E}\cdot\dfrac{2(1-\nu)}{r_0} - \dfrac{1}{E_2}\cdot\dfrac{3-2\nu}{z_0}}, \quad (21)$$

where z_0 is the effective radius of the LWD device which is used to measure the stiffness.

It is possible to make the resulting formula easier (and faster) to compute. We can make this expression faster to compute if we:

- explicitly subtract the fractions in the numerator and in the denominator, and then
- we explicitly divide the resulting fractions.

In the numerator, we get:

$$\frac{2-2\nu}{r_0} - \frac{3-2\nu}{z_0} = \frac{(2-2\nu)\cdot z_0 - (3-2\nu)\cdot r_0}{r_0\cdot z_0}. \quad (22)$$

In the denominator, we get:

$$\frac{1}{E}\cdot\frac{2(1-\nu)}{r_0} - \frac{1}{E_2}\frac{3-2\nu}{z_0} = \frac{2(1-\nu)\cdot z_0\cdot E_2 - (3-2\nu)\cdot r_0\cdot E}{z_0\cdot r_0\cdot E\cdot E_2}. \quad (23)$$

By dividing these two fractions, we conclude that

$$E_1 = \frac{((2-2\nu)\cdot z_0 - (3-2\nu)\cdot r_0)\cdot E\cdot E_2}{2(1-\nu)\cdot z_0\cdot E_2 - (3-2\nu)\cdot r_0\cdot E}. \quad (24)$$

This expression is faster to compute, since it only has one division – and in the computer, out of all arithmetic operations, division takes the longest time to perform. This, by the way, is easy to explain:

- multiplication, crudely speaking, is several additions, so multiplication takes longer than addition, and
- all known algorithms for division include several multiplications, so division takes longer than multiplication.

Numerical example. As typical values, let us take $\nu = 0.4$, $r_0 = 1$ in, $z_0 = 12$ in, and $E_2 = 0.1\cdot E_1$. In this case, from the formula (18), we get $2-2\nu = 1.2$, $3-2\nu = 2.2$, thus:

$$\frac{1}{E}\cdot\frac{1.2}{1} = \frac{1}{E_1}\cdot\left(\frac{1.2}{1} - \frac{2.2}{12}\right) + \frac{10}{E_1}\cdot\frac{2.2}{12},$$

i.e.,

$$\frac{1.2}{E} = \frac{1}{E_1} \cdot (0.166\ldots + 1.83\ldots) = \frac{2}{E_1},$$

and so,

$$E = \frac{1.2}{2} \cdot E_1 = 0.6 \cdot E_1.$$

So, in this cases, the pavement stiffness E is about 60% of the stiffness E_1 of the base.

4 Analysis of the Second Problem

Towards solving the second problem: when the overall stiffness is not sufficient, how much extra base do we need? Let us now use the formula (18) to decide how much extra base we need if the measured overall stiffness E is still smaller than the desired pavement stiffness E_0. The overall depth $z > z_0$ of the base can be found from the requirement that when we have this depth, the resulting overall stiffness will be equal to E_0, i.e., we will have

$$\frac{1}{E_0} \cdot \frac{2(1-\nu)}{r_0} = \frac{1}{E_1} \cdot \left(\frac{2-2\nu}{r_0} - \frac{3-2\nu}{z} \right) + \frac{1}{E_2} \cdot \frac{3-2\nu}{z}. \tag{25}$$

To find the desired value z, we move all the terms not containing z to the left-hand side. Thus, we get:

$$(3-2\nu) \cdot \left(\frac{1}{E_2} - \frac{1}{E_1} \right) \cdot \frac{1}{z} = \left(\frac{1}{E_0} - \frac{1}{E_1} \right) \cdot \frac{2-2\nu}{r_0}. \tag{26}$$

By subtracting the fractions, we get:

$$(3-2\nu) \cdot \frac{E_1 - E_2}{E_1 \cdot E_2} \cdot \frac{1}{z} = \frac{E_1 - E_0}{E_0 \cdot E_1} \cdot \frac{2-2\nu}{r_0}, \tag{27}$$

thus:

$$z = \frac{3-2\nu}{2-2\nu} \cdot \frac{E_1 - E_2}{E_1 - E_0} \cdot r_0. \tag{28}$$

Plugging in the formula (24) into this expression, we conclude that

$$E_1 - E_2 = \frac{N_{12}}{D},$$

where

$$N_{12} \stackrel{\text{def}}{=} ((2-2\nu) \cdot z_0 - (3-2\nu) \cdot r_0) \cdot E \cdot E_2 -$$
$$2(1-\nu) \cdot z_0 \cdot E_2^2 - (3-2\nu) \cdot r_0 \cdot E \cdot E_2$$

and

$$D \stackrel{\text{def}}{=} 2(1-\nu) \cdot z_0 \cdot E_2 - (3-2\nu) \cdot r_0 \cdot E.$$

The numerator can be simplified to

$$N_{12} = (2-2\nu)\cdot z_0\cdot E_2\cdot (E-E_2),$$

so

$$E_1 - E_2 = \frac{(2-2\nu)\cdot z_0\cdot E_2\cdot (E-E_2)}{2(1-\nu)\cdot z_0\cdot E_2 - (3-2\nu)\cdot r_0\cdot E}. \tag{29}$$

Similarly,

$$E_1 - E_0 = \frac{N_{10}}{D},$$

where

$$N_{10} = ((2-2\nu)\cdot z_0 - (3-2\nu)\cdot r_0)\cdot E\cdot E_2 - \\ 2(1-\nu)\cdot z_0\cdot E_2\cdot E_0 - (3-2\nu)\cdot r_0\cdot E\cdot E_0.$$

Here, the numerator can be simplified into

$$N_{10} = (3-2\nu)\cdot r_0\cdot E\cdot (E-E_2) - (2-2\nu)\cdot z_0\cdot E_2\cdot (E_0-E),$$

so

$$E_1 - E_0 = \frac{(3-2\nu)\cdot r_0\cdot E\cdot (E-E_2) - (2-2\nu)\cdot z_0\cdot E_2\cdot (E_0-E)}{2(1-\nu)\cdot z_0\cdot E_2 - (3-2\nu)\cdot r_0\cdot E}. \tag{30}$$

These two differences have the same denominator D, so their ratio is simply equal to the ratio of their numerators:

$$\frac{E_1-E_2}{E_1-E_0} = \frac{(2-2\nu)\cdot z_0\cdot E_2\cdot (E-E_2)}{(3-2\nu)\cdot r_0\cdot E\cdot (E-E_2) - (2-2\nu)\cdot z_0\cdot E_2\cdot (E_0-E)}. \tag{31}$$

Substituting this ratio into the formula (28) and canceling the terms $2-2\nu$ in the numerator and in the denominator, we arrive at the following formula.

5 Solution to the Second Problem

Let us assume that we know the overall stiffness E which is smaller than the desired overall stiffness E_0. Let us also assume that we know the stiffness E_2 of the lower layer, and the thickness z_0 of the upper layer. In this case, in order to reach the desired stiffness, we need to increase the thickness z_0 of the upper layer to the new value

$$z = \frac{(3-2\nu)\cdot z_0\cdot E_2\cdot (E-E_2)}{(3-2\nu)\cdot r_0\cdot E\cdot (E-E_2) - (2-2\nu)\cdot z_0\cdot E_2\cdot (E_0-E)}\cdot r_0. \tag{32}$$

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

The authors are thankful to the researchers and students from the Center for Transportation Infrastructure Systems (El Paso, Texas) for valuable discussions.

References

[1] M. A. Mooney and D. Adam, "Vibartory roller integrated measurement of earthwork compaction: an overview", *Proceedings of the International Symposium on Field Measurements in Geomechanics FMGM'2007*, Boston, Massachusetts, September 24–27, 2007.

[2] M. A. Mooney and N. W. Facas, *Extraction of Layer Properties from Intelligence Compaction Data*, Highway Innovation Deserving Exploratory Analysis Programs (IDEA) Report 145, Transportation Research Board, Washington, DC, 2013.

[3] M. A. Mooney, P. B. Gorman, E. Farouk, J. N. Gonzalez, and A. S. Akanda, *Exploring Vibration-Based Intelligent Soft Compaction*, Oklahoma Department of Transportation, Projet No. 2146, Final Report, 2003.

[4] M. A. Mooney, P. B. Gorman, and J. N. Gonzalez, "Vibration-based health monitoring during eathwork construction", *Journal of Structural Health Monitoring*, 2005, Vol. 2, No. 4, pp. 137–152.

[5] M. A. Mooney, R. V. Rinehart, N. W. Facas, O. M. Musimbi, and D. J. White, *Intelligent Soil Compaction Systems*, National Cooperative Highway Research Program (NCHRP) Report 676, Transportation Research Board, Washington, DC, 2010.