4-2017

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Comments:
Technical Report: UTEP-CS-17-34
To appear in *Journal of Uncertain Systems*

A Symmetry-Based Explanation for an Empirical Model of Fatigue Damage of Composite Materials

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Abstract

In this paper, we provide a symmetry-based explanation for an empirical formula that describes fatigue damage of composite materials.

1 Formulation of the Problem

Need for composite materials. In many practical applications, it is desirable to use a material with certain properties like density, strength, cost, chemical properties, etc. Often, when there is no natural material with these properties, we can get the desired properties if we combine several different materials – e.g., layer-by-layer, or by including threads of one material inside another one. Typical engineered composite materials include mortars, concrete, reinforced plastics, metal and ceramic composites, etc. The resulting composite materials are actively used for buildings, bridges, boat hulls, spacecraft, aircraft, etc.

Need to predict fatigue damage. One of the main applications of composite materials is in structures that undergo repeated stress cycles. For example, an airplane is stressed during every flight. While the composite materials are often more resistant to stress than the traditional materials, they still take some damage and eventually fail. It is therefore important to be able to accurately predict the failure time – so that we can stop using the corresponding device when it is close to failure.

The need for the accurate prediction is easy to explain:

- If our prediction $\widetilde{T}$ for the failure time is an overestimate, i.e., if it is larger than the actual failure time, then we will continue using a device when it is bound to fail – and thus, risk a catastrophic failure.
- On the other hand, if we underestimate the failure time, then we stop using the device while it is still capable of safe functioning – and thus,

waste our resources on building, e.g., a new airplane while the old one could still safely function for several years.

An empirical formula for fatigue damage. The degradation of a composite material can be traced by the decrease of its Young's modulus E: it starts with some initial value E_0 and then decreases until it reaches a certain level $E_f < E_0$ at which the material fails.

Let us denote by N the number of cycles after which the material fails. For each value $n \leq N$, let $E(n)$ denote the value of the Young's modulus after n cycles. Then, $E(0) = E_0$ and $E(N) = E_f$.

To describe the degradation of the material, it is therefore desirable to know how the value $E(n)$ changes with n. It turns out that the following empirical formula accurately describes this change:

$$D(n) \stackrel{\text{def}}{=} \frac{E_0 - E(n)}{E_0 - E_f} = 1 - \left(1 - \left(\frac{n}{N}\right)^B\right)^A, \tag{1}$$

for some values A and B. This formula was first proposed in [3] and remains the most accurate formula for describing the fatigue damage (see, e.g., [2]).

Remaining problem. For practical applications, it is always better to use formulas that are not only empirically valid but which also have a good theoretical explanation: indeed, the presence of such an empirical explanation makes the corresponding formulas more reliable. It is therefore desirable to come up with a theoretical explanation for the empirical formula (1).

What we do in this paper. In this paper, we provide such a theoretical explanation – based on the general symmetry ideas.

2 General Idea: Physical Formulas Should Not Depend on the Choice of the Measuring Unit

Data processing algorithms deal with the numerical values of the physical quantities. However, the numerical values depend on the choice of a measuring unit. For example, depending on the choice of a measuring unit, we can describe the height of the same person as 1.7 m or 170 cm. It is therefore reasonable to require that the physical formulas not depend on the choice of a measuring unit.

Of course, if we change the units in which we measure one of the quantities, then we may need to adjust units of related quantities. For example, if we replace meters with centimeters, then for the formula $v = d/t$ (that describes velocity v as a ratio of distance d and time t) to remain valid we need to replace meters per second with centimeters per second when measuring velocity. Once the appropriate adjustments are made, we expect the formulas to remain the same.

3 Analysis of the Problem and the Resulting Derivation of the Empirical Formula

Physical description of the problem. It takes some effort – i.e., some energy – to break the material. The fact that the material breaks after a certain number of cycles means that with each cycle, the energy is "pumped" into the system.

Notations. Let P_0 denote the amount of energy needed to break the material in its original state.

Usually, the cycles are periodic, with a period Δt. Thus, the time t during which we perform n cycles is equal to $t = n \cdot \Delta t$. Let us denote the overall amount of energy pumped into the system by time t by $P(t)$.

The stability of a system at each moment of time can be characterized by the remaining amount of energy needed to break the system. Let us denote this remaining amount by R. At each moment of time t, we thus have

$$R = P_0 - P(t). \tag{2}$$

In particular, in the beginning, at moment $t = 0$, when no energy has yet been pumped into the system, we have $P(t) = 0$ and thus, $R = P_0$.

The amount of energy is difficult to measure directly without destroying the sample, but we can gauge the changing state of the system by tracing how its Young's modulus changes: the closer the Young's modulus E gets to the value E_f corresponding to the failure state, the close the state of the composite material to the failure state, and thus, the fewer energy we need to pump to break the material. From this viewpoint, the remaining amount of energy depends on the difference $E - E_f$:

$$R = R(E - E_f).$$

In these terms, the formula (2) takes the form

$$R(E - E_f) = P_0 - P(t). \tag{3}$$

To finalize our analysis, we thus need to find the corresponding two dependencies:

- the dependence $P(t)$ that describes how the amount $P(t)$ of pumped energy depends on time t, and
- the dependence $R(E - E_f)$ that describes how the remaining amount of energy R depends on the difference $E - E_f$ between the current value E of the Young's modulus and the value E_f corresponding to the failure state.

Let us use scale invariance. To find both dependencies, let us use scale invariance, i.e., the natural requirement that physical dependencies should not change if we simply change the measuring unit.

Let us first apply this requirement to finding the dependence $P(t)$. The change in time unit means that for some $\lambda > 0$, we replace the original unit of time by a new unit which is λ times larger. After this change, all numerical values of time get divided by λ, i.e., instead of each original value t, we get the new numerical value $t' = \dfrac{t}{\lambda}$. In these terms, the above requirement means that after an appropriate change of the unit of energy $P \to P' = \dfrac{P}{\mu}$ for some μ (depending on λ), the dependence between the re-scaled energy A' and the re-scaled time t' should be described by exactly the same formula as the dependence of A on t.

In terms of t', the value t is equal to $t = \lambda \cdot t'$. Thus, for each t', the amount of energy as described in the original units is equal to $P(\lambda \cdot t')$. In the new energy units, this value has the form $P'(t') = \dfrac{P(\lambda \cdot t')}{\mu(\lambda)}$. Thus, the requirement that this dependence is described by the same function $P(t)$ as in the original units means that $P(t') = \dfrac{P(\lambda \cdot t')}{\mu(\lambda)}$, i.e., equivalently.

$$P(\lambda \cdot t') = \mu(\lambda) \cdot P(t') \tag{4}$$

for all t' and $\lambda > 0$.

The amount of energy pumped into the system increases with time. Thus, the function $P(t)$ should be increasing. It is known – see, e.g., [1] – that all monotonic solutions of the equation (4) have the form

$$P(t) = c \cdot t^B \tag{5}$$

for some constants c and B.

This can be easily proven in the case when we additionally assume that the dependence $P(t)$ is differentiable. In this case, due to the formula (4), the function $\mu(\lambda)$ is proportional to a differentiable function $P(\lambda \cdot t)$ and is, thus, differentiable too. Differentiating both sides of the equation (4) with respect to λ and taking $\lambda = 1$, we conclude that $t \cdot \dfrac{dP}{dt} = B \cdot P(t)$, where we denoted $B \stackrel{\text{def}}{=} \dfrac{d\mu}{d\lambda}_{|\lambda=1}$. Separating the variables A and t, we get $\dfrac{dA}{A} = B \cdot \dfrac{dt}{t}$. Integrating both parts, we get $\ln(A) = B \cdot \ln(t) + C$, thus $A = \exp(\ln(A)) = c \cdot t^B$, where $c \stackrel{\text{def}}{=} \exp(C)$.

Similarly, from the requirement that the dependence $R(E - E_f)$ be scale-invariant, we conclude that

$$R(E - E_f) = c' \cdot (E - E_f)^D \tag{6}$$

for some values c' and D.

Deriving the empirical formula (1). Let us use the resulting expressions (5) and (6) to derive the desired empirical formula (1). Indeed, substituting the formulas (5) and (6) into the formula (3), we conclude that

$$c' \cdot (E - E_f)^D = P_0 - c \cdot t^B.$$

Substituting $t = n \cdot \Delta t$ into this formula, we get

$$c' \cdot (E(n) - E_f)^D = P_0 - c'' \cdot n^B,$$

where we denoted $c'' \stackrel{\text{def}}{=} c \cdot (\Delta t)^B$. Dividing both sides of this formula by P_0, we get a simplified expression

$$C' \cdot (E(n) - E_f)^D = 1 - C'' \cdot n^B, \tag{7}$$

where we denoted $C' \stackrel{\text{def}}{=} \dfrac{c'}{P_0}$ and $C'' \stackrel{\text{def}}{=} \dfrac{c''}{P_0}$.

By definition, the number of cycles at which the material fails is equal to N. For this value N, we have $E(N) = E_f$, thus $1 = C'' \cdot N^B$ and $C'' = \dfrac{1}{N^B}$. Substituting this expression for C'' into the formula (7), we get

$$C' \cdot (E(n) - E_f)^D = 1 - \left(\frac{n}{N}\right)^B. \tag{8}$$

For $n = 0$, the Young's module E takes its initial value E_0, so we have

$$C' \cdot (E_0 - E_f)^D = 1$$

and thus, $C' = \dfrac{1}{E_0 - E_f}$. Substituting this expression for C' into the formula (8), we conclude that

$$\left(\frac{E(n) - E_f}{E_0 - E_f}\right)^D = 1 - \left(\frac{n}{N}\right)^B. \tag{9}$$

Raising both sides of this formula to the power $A \stackrel{\text{def}}{=} 1D$, we conclude that

$$\frac{E(n) - E_f}{E_0 - E_f} = \left(1 - \left(\frac{n}{N}\right)^B\right)^A. \tag{10}$$

Thus, for

$$D(n) = \frac{E_0 - E(n)}{E_0 - E_f} = 1 - \frac{E(n) - E_f}{E_0 - E_f},$$

we get the desired formula (1):

$$D(n) = 1 - \left(1 - \left(\frac{n}{N}\right)^B\right)^A.$$

The empirical formula has been justified.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

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