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“Weird” Fuzzy Notations: 
An Algebraic Interpretation 

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Abstract 
Traditionally, fuzzy logic used non-standard notations like 
\[ m_1/x_1 + \ldots + m_n/x_n \] 
for a function that attains the value \( m_1 \) at \( x_1 \), . . . , and the value \( m_n \) at \( x_n \). In this paper, we provide an algebraic explanation for these notations. 

Mathematics Subject Classification: 03B52 

Keywords: fuzzy notations, algebraic explanation 

Formulation of the problem. In fuzzy logic, traditionally, researchers and practitioners used non-standard notations to describe functions; see, e.g., [1]. In these notations, an expression of the type 
\[ m_1/x_1 + m_2/x_2 + \ldots + m_n/x_n \] 
indicates a function that is defined on the set \{x_1, x_2, \ldots, x_n\} and that takes: 

• the value \( m_1 \) for \( x = x_1 \),
• the value \( m_2 \) for \( x = x_2 \),
• . . . , and
• the value \( m_n \) for \( x = x_n \).

To a mathematician, these non-standard notations are very confusing. 
In this paper, we provide an algebraic justification for these “weird” notations, justification that will helpfully make them somewhat less confusing.
Main idea: application of a function to a value as a “multiplication” operation. In mathematics, the division operation \( a/b \) is usually understood as the inverse to a “multiplication” operation \( ab \). Thus, to provide a reasonable interpretation for the fuzzy “division” operation, we must find the appropriate “multiplication” operation.

In the context in which the above notations are used, we have a universal set \( U \), the set \( T \) of possible values, and we have partial functions defined on this set, i.e., functions from the set \( U \) (or from its proper subset) to the set \( T \). The only operation that we have is the operation of applying a function \( f \) to the value \( x \in U \).

It is therefore reasonable to use this application operation as the multiplication operation.

Comment. This usage is in full agreement with the usual notations, in which the result of applying a function \( f \) to the value \( x \) is denoted either by \( f(x) \), or simply by \( fx \). This simplified notation is exactly the notation for a multiplication operation.

Resulting division operations: discussion. For this multiplication operation, what is the resulting division operation? For commutative multiplication operations, a division operation corresponding to a multiplication operation is defined as follows: \( a/b = c \) if and only if \( a = bc \). For non-commutative multiplication operations (and the operation \( fx \) is clearly non-commutative, since \( xf \) does not even make sense), we can distinguish between left and right divisions:

- in the left division, \( a/b = c \) if and only if \( a = bc \); and
- in the right division, \( a/b = c \) if and only if \( a = cb \).

In our case, when \( a = bc \), then \( b \) is a function, \( c \) is an element of the universal set \( U \), and \( a \) is the element of the set \( T \). Thus, the corresponding left division operation would correspond to dividing an element \( a \in T \) by a function. The only case that leads to dividing an element \( a \in T \) by a value \( x \in U \) is the right division.

Since the condition \( m = fx \) means that \( f(x) = m \), the right division means the following: \( f = m/x \) if and only if \( f(x) = m \). This interpretation cannot be taken literally, since there are many different functions for which \( f(x) = m \), and they cannot be all equal to the same object \( m/x \).

However, in the class of all the functions for which \( m = fx \), there exists the smallest one (in terms of inclusion): a function which is defined only at a single point \( x \) and whose value is equal to \( m \). It is therefore reasonable to define this smallest element as the desired “ratio” \( m/x \).

Comment. This definition is in line with the way fuzzy implication \( a \rightarrow b \) is sometimes defined (see, e.g., [1]): as the smallest possible degree \( c \) for which \( c \& a = b \), where \& is the fuzzy “and” operation (t-norm).
**Relation to function composition as multiplication.** In addition to applying a function to an object, we can also consider composition of functions. A composition is also sometimes denoted simply by $fg$ (e.g., $\log \sin(x)$ is a usual notation for $\log(\sin(x))$), so it is also natural to view it as a multiplication operation.

This multiplication operation is in line with the above definition of division: e.g., if $f = m/x$, and $g = n/m$, then formally, $gf = (n/m)(m/x) = n/x$. And indeed, here:

- $f = m/x$ means that $f(x) = m$ and $f$ is undefined for all other $x$;
- $g = n/m$ means that $g(m) = n$;
- hence $g(f(x)) = g(m) = n$ (and $g(f(y))$ is undefined for all $y \neq x$), which is exactly what $gf = n/x$ means.

**Meaning of the sum.** In our interpretation, each expression like $m_i/x_i$ means a partial function which are defined at only one point $x_i$ and has the value $m_i$ at this point. Since in mathematics, a function $f$ is defined as a set of (ordered) pairs $(x, f(x))$, the notation $m_i/x_i$ means a set consisting of a single ordered pair: $m_i/x_i = \{(x_i, m_i)\}$.

A natural “addition” operation for sets is union. It is not a standard notation for the union, but it is not as non-standard as the notations for fuzzy sets:

- a few decades ago, union was indeed routinely denoted by $+$, and
- even now, in many engineering applications, addition is used as a symbol for set union (and for the corresponding logical “or” operation).

Moreover, while the union is not any more routinely described by the plus sign $+$, the minus sign $-$, a typical sign of an operation which is inverse to $+$, is still routinely used to describe the difference between the two sets.

Also, in Boolean algebra, $+$ is often used to describe the “exclusive or” operation, which, for our one-point functions $m_i/x_i$, is equivalent to the union.

**Conclusion.** So, we will interpret the sum

$$m_1/x_1 + \ldots + m_n/x_n$$

as the union of the partial functions $\{(x_1, m_1)\}, \ldots, \{(x_n, m_n)\}$, i.e., as the set of pairs

$$\{(x_1, m_1), \ldots, (x_n, m_n)\},$$

which is a function that maps $x_1$ into $m_1$, \ldots, and maps $x_n$ into $m_n$ – exactly the meaning that we are trying to interpret.

Now, this seemingly weird expression has a reasonable algebraic explanation.

**ACKNOWLEDGEMENTS.** This work was supported in part by NSF grant HRD-0734825, by Grant 1 T36 GM078000-01 from the National Institutes
of Health, and by Grant 5015 “Application of fuzzy logic with operators in the knowledge based systems” from the Science and Technology Centre in Ukraine (STCU), funded by European Union.

References