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To my

family.

ASYMPTOTIC APPROXIMATION OF THE FREE BOUNDARY FOR THE AMERICAN PUT NEAR EXPIRY

by

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THESIS

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Abstract

An introduction to boundary value problems for the heat operator will focus on the Dirichlet and Neumann problems in bounded domains. Following an overview of the basic financial principles of derivative securities, a derivation of the Black-Scholes formulae is given. The valuation of commodity options is then discussed. Partial differential equations as free boundary problems are then applied to valuation of the American Put option. A method of solution to this problem close to expiry is investigated.

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Chapter 1

Introduction

The value of any asset is ultimately determined by the market. The highest price that an investor is willing to pay for a particular thing is, by definition, its value to the investor. The decision making process that sets this value in the mind of the investor is influenced by a complex mix of factors. An analysis of supply and demand, experience, gut feel, logic etc... collectively set the "mood" of a body of investors which establish the market price for an asset. This mood can change quickly, and can be influenced by a wide range of factors. Investor confidence can be up or down, and news arriving from the other side of the world can affect prices locally. One needs only to look at a brief history of a stock price to see the effect of this phenomenon; short-term up and down spikes with longer term increasing and decreasing trends.

With our understanding of the drivers of this behavior, it seems like an unlikely candidate for mathematical analysis. Indeed, if the market could be described accurately by a purely deterministic model then the surest path to riches would be the study of mathematics. As this is apparently not the case, the best we can hope for is to develop a probabilistic model. This model cannot tell us with certainty the path that an asset value will take, but given a list of qualifying assumptions, it can tell us that some paths are more probable than others.

The graph of the history of a stock price bears a striking resemblance to the path followed by a very small particle suspended on the surface of a liquid. The nature of this apparently random movement of a particle due to myriad unseen forces has perhaps been the subject of human fascination since before recorded history, but the first written description of the phenomenon is credited to the Roman poet Lucretius, circa 60 B.C. The first mathematical treatment was published by the Scottish mathematician Robert Brown in 1827. The phenomenon is known as Brownian motion in his honor.

In order to understand the financial setting in which we are working, some basic discussion of options is necessary. An option is a derivative security; it derives its value from the value of an asset, called the underlying asset, or simply the underlying. The asset can be anything that is traded, including shares of stock. There are two basic types of options; the call and the put. A call option is a contract which gives the holder (investor) the right to buy a certain quantity of the underlying for a certain price (the strike price) at a specified expiration date (expiry). A put is identical, but gives the holder the right to sell. It is basically a bet on how the value of the underlying will move during the option's lifespan. If one holds a put option to sell a stock for \$20, for example, and the price of the stock at expiration is \$15, the holder can buy the stock for \$15 and sell it for \$20. If the price of the stock is \$21 at expiration, the holder of the option will let the option expire without exercising it - thus the name "option". Since the option provides the investor the possibility to make a profit, it has value - it costs money to enter into an option contract.

The option contract described above is the simplest type of option - a vanilla European option. It can only be exercised on expiration. Options which can be exercised before expiration are called American options. The names describe contract specifications only, having nothing to do with geographic regions. There are many other types of option contracts. Complex options contracts are called exotic options.

The French mathematician Louis Bachelier wrote the pioneering thesis linking the study of Brownian motion to the value of options in 1900. His thesis, quite unappreciated at the time, is considered by many to mark the birth of the field of financial mathematics. In the present work, we begin with the analysis performed by Fischer Black and Myron Scholes, first published in 1973. Subsequent work on this theme by Scholes and Robert Merton (Black passed away in 1995) won the Nobel Prize in Economics in 1997.

The famous Black-Scholes analysis leads to a partial differential equation (PDE) which gives the value of the investment (option). This PDE is then transformed into the heat equation, for which methods of solution have been extensively studied, perhaps most notably by Joseph Fourier, who published his classic work in 1822.

Fourier studied the diffusion of heat in conducting materials under various conditions. The basic physical model is a bar made of some material through which heat can flow. At the start of the process, using t for time, $t = 0$ with each location on the bar initially at some (different) specific temperature. If we consider the bar to be lying on the x-axis, the temperature of the bar can be considered to be a function of x, with each x value representing the location at which we are measuring the temperature, and time t, with each t value being the time at which we are measuring the temperature, or $f(x, t)$. The temperature at the start of the heat diffusion process is $f(x, 0)$. Then we start the time running and see how the temperature of the bar changes. Heat will flow from the hotter places to the cooler areas of the bar at some rate which depends on the material. At any time t, and any place x, we want to be able to tell what the temperature of the bar is. This temperature is a function of the time t and the location x, given the initial temperature pattern of the bar at $t = 0$. This temperature function is exactly the solution to the heat equation.

By converting the Black-Scholes PDE to the heat equation we can use the theoretical background and methods of solution developed for the heat equation to solve the Black-Scholes PDE and find the value of an option. We also have proofs that unique solutions to these problems always exist.

In keeping with the context of the mathematical analysis of options pricing, we need to add an additional constraint to the above model of heat diffusion in a bar. In the Dirichlet problem, in addition to knowing the initial temperature pattern of the bar at $t = 0$, we also have a function for the temperature at the end of the bar. The Neumann problem deals with how fast heat is flowing into or out of the end of the bar. These are known as boundary value problems and there is also a wealth of theoretical background we can call on to solve these problems.

Of particular interest in this study is the situation where we know the conditions at the boundary, but we do not know exactly where the boundary is. This is known as a free boundary problem, and this type of problem models various physical settings. One example is the Stefan problem, which models a system of warming water and ice, where the boundary between the ice and liquid water is the free boundary. This problem has been extensively studied, but unfortunately does not model option pricing.

Remembering that the market determines the price of an asset, it can be asked what value this analysis brings to the pricing of options. We discuss the many assumptions which must be made in order to construct the mathematical model of options pricing presented here, and these assumptions do conflict with reality to some degree. The mathematical modeling of a system as complex as the options market requires many simplifications as well. As such, the material discussed here on options pricing is essentially an example of the mathematical modeling of a highly complex system which at first glance seems to be without opportunity to impose any sort of mathematical order. We assert that the first impression in this case is incorrect, and that some beautiful mathematics can be discovered in this endeavor.

Since our goal is to convert the Black-Scholes equation to the heat equation and then use established methods of solution in order to calculate the value of an option, we begin with some background on the heat equation. The Dirichlet problem is discussed in some detail and references are given for related results pertaining to the Neumann problem. We establish that by using the heat equation we can obtain a unique solution to the problem of finding the option value at any time before expiry. We then provide some background information on derivative securities, developing the context of the problem. Derivative securities is a vast and complex field, and the discussion provided here is by no means extensive. The goal is to provide the minimum information required to understand the financial setting for the mathematical modeling of valuation of options on stocks and commodities. Of particular interest in this discussion is the concept of arbitrage, or risk-free profit. This principle is key to several essential arguments made during the development of options pricing models.

With a basic understanding of the financial setting, we move to the Black-Scholes analysis. A derivation of the famous Black-Scholes PDE is presented, and the manipulations required to perform the conversion to the heat equation are shown. We then arrive at the Black-Scholes formula for the valuation of a stock option. Following the same analysis, we derive the formula for valuation of commodity options. A discussion of American options follows, with an arbitrage-based argument illustrating why American option values must behave differently from their European counterparts. Valuation of American options leads to a discussion of free boundary problems and the Stefan problem is presented. Finally, we discuss a free boundary problem for the value of an American put option and a method of solution close to the expiry.

Chapter 2

The Heat Equation

The heat equation

$$\partial_t u - \Delta u = 0$$

models heat flow in various physical settings. By appropriate manipulations the heat equation can also be used to model the value of certain financial instruments. This application has been the focus of intense interest since the Nobel Prize-winning work of Black and Scholes in the early 1970's. Since the financial applications in which we are interested involve solutions of the heat equation in a bounded region of space $\Omega \subset \mathbb{R}^n$ over a time interval $0 \leq t \leq \infty$, it is appropriate to specify both initial conditions $u(x, 0)$ ($x \in \Omega$) and boundary conditions on $\partial\Omega \times [0, T]$. If the boundary condition takes the form $u = g$, it is known as a Dirichlet problem. If we specify $\partial_v u = g$ at the boundary, it is known as a Neumann problem.

2.0.1 Existence and Uniqueness of Solutions

Notation

For $x \in \mathbb{R}^n$ and the multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

For derivatives on $\mathbb{R}^n$,

$$\partial_j = \frac{\partial}{\partial x_j}$$

and higher order derivatives are expressed by

$$\partial^\alpha = \prod_1^n \left(\frac{\partial}{\partial x_j}\right) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

Definition

The Schwartz Class $S = S(\mathbb{R}^n)$ is the space of all C^∞ functions on $\mathbb{R}^n$ such that $u \in S$ if and only if $u \in C^\infty$ and for all mulit-indices α and β,

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta u(x) \right| < \infty.$$

Definition

If $f \in L^1(\mathbb{R}^n)$, its Fourier Transform $\widehat{f}$ is the bounded function on $\mathbb{R}^n$ defined by

$$\widehat{f}(\xi) = \int e^{-2\pi i x \xi} f(x) dx.$$

The Maximum Principle

Let Ω be a bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. Suppose u is a real valued continuous function on $\overline{\Omega} \times [0, T]$ that satisfies $\partial_t u - \Delta u = 0$ on $\Omega \times (0, T)$ (and hence is C^∞ there). Then u assumes its maximum either on $\Omega \times \{0\}$ or on $\partial\Omega \times [0, T]$.

Replacing u by $-u$, we see that the minimum is also achieved. Thus, as a corollary, there is at most one continuous function u on $\overline{\Omega} \times [0, T]$ which agrees with a given continuous function on $\Omega \times \{0\}$ and on $\partial\Omega \times [0, T]$ and satisfies $\partial_t u - \Delta u = 0$ on $\Omega \times (0, T)$.
Our notation and statement of theorems follows that of [3].

Existence of solution to the Dirichlet problem

Consider the Dirichlet problem

$$\partial_t u - \Delta u = 0$$

with initial condition

$$u(x, 0) = f(x).$$

and boundary condition

$$u(x, t) = g(t) \text{ for } x \in \partial\Omega$$

Assuming that f is in the Schwartz class S, and denoting the Fourier transform of $u(x, t)$ with respect to x by $\widehat{u}(\xi, t)$, we have

$$\partial_t \widehat{u}(\xi, t) + 4\pi |\xi|^2 \widehat{u}(\xi, t) = 0$$

$$\widehat{u}(\xi, t) = \widehat{f}(\xi)$$

This is an initial value problem for an ordinary differential equation in t, with solution

$$\widehat{u}(\xi, t) = \widehat{f}(\xi) e^{-4\pi^2 |\xi|^2 t}$$

$$(t > 0)$$

Thus $u(x, t) = f * K_t(x)$ where $*$ indicates convolution and $\widehat{K}_t(\xi) = e^{-4\pi^2 |\xi|^2 t}$.

Making use of the result

$$f(x) = e^{-\pi a|x|^2}, (a > 0) \Rightarrow \widehat{f}(\xi) = a^{-\frac{n}{2}} e^{-\frac{\pi|\xi|^2}{a}},$$

we have

$$K_t(x) \equiv K(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

$$(t > 0)$$

The function K on $\mathbb{R}^n \times (0, \infty)$ is called the Gaussian Kernel [3]. With this, we can state the following:

Theorem

Suppose $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then $u(x,t) = f * K_t(x)$ satisfies $\partial_t u - \Delta u = 0$ on $\mathbb{R} \times (0, \infty)$. If f is bounded and continuous, then u is continuous on $\mathbb{R} \times [0, \infty)$ and $u(x, 0) = f(x)$. If $p < \infty$, $u(\cdot, t) \to f$ as $t \to 0$ in the L^p norm [3].

In one spatial variable we have [7]

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} u_0(s)\, e^{\frac{-(x-s)^2}{4t}} ds.$$

Chapter 3

Introduction to Derivative Securities

3.0.2 Options

An option is a type of investment known as a derivative security. An option derives its value from the value of some underlying security. The underlying asset can be practically anything that is traded, including stocks, commodities, etc... The global derivative securities market is huge, with an estimated value (2004) of over $220 trillion [5, Hull].

An option is a contract giving the holder the right to buy or sell a specified quantity of the underlying asset at a specified time in the future. A "call" option gives the holder the right to buy, while a "put" option gives the holder the right to sell. The future date specified is known as the expiration date or expiry, and the agreed-upon price is known as the strike price. Options which can only be exercised at expiry are known as European options. American options can be exercised at expiry or any time before. Exotic options derive their value based on the history of the underlying asset price rather than solely on the price at expiry.

Since an option provides the investor with the opportunity to make a profit, it has a value. This value is determined by supply and demand in the market. Entering into an option contract costs money, but options are attractive to investors because they require less up-front investment than purchasing the underlying assets. Also, options can never cost the option holder more than the price of the option itself, no matter how the asset price moves.

3.0.3 Futures

An option may have a futures contract as its underlying asset. A futures contract specifies the price of a quantity of some commodity for delivery at a future date. The futures price is the price at which a commodity can be purchased for delivery at the specified future date. The spot price is the price at which it is traded today. Futures contracts do not give the holder the option of allowing the contract to expire without exercising it.

The holder of a contract to buy is said to have a long position, while a contract to sell implies a short position. Futures contracts are settled daily, with the investors with long and short positions paying each other (indirectly) the difference in the daily movement of the futures price. If the futures price of the commodity goes up, for instance, investors with long positions will pay investors with short positions. These transactions are handled through the exchanges which organize futures trading. The US Commodity Futures Trading Commission is the governmental regulatory agency that oversees these activities, although many exchanges engage in self-regulation.

Since the futures contract is settled daily, its value is reset to zero every day. The resetting process is called "marking to market". This fact is critical in valuing options on futures contracts. It costs nothing to enter into a futures contract, but money must be deposited into a margin account to facilitate the daily settlement process.

On the delivery date of a futures contract the futures price must equal the spot price. The explanation of why this is so rests on the important concept of arbitrage, or risk-free profit. Arbitrage plays a key role in the theory of mathematical derivative pricing models. In the case of futures, arbitrage works to assure that on the delivery date the spot and futures prices must be equal. Say that the futures price of some commodity on the delivery date is higher than the spot price. An arbitrageur would enter into a short position in a futures contract, buy the commodity at the spot price and deliver, assuring a risk-free profit equal to the difference between the spot and futures prices. Since this profit is risk-free, every investor would seek to take advantage of such an opportunity. The resulting increase in demand for the commodity would drive the spot price up, thus eliminating the opportunity very quickly. So quickly, in fact, that many mathematical models consider these arbitrage opportunities to be non-existent. The background information on derivative securities presented here can be found in [5].

Chapter 4

Mathematical Models of Options Pricing

4.0.4 A Simple Asset Pricing Model

Most models of option pricing are based on a simple model of asset price variation. One of the basic assumptions is that of the efficient market hypothesis, which states that:

1. The present price of an asset reflects all of the historical information available and no further information is contained in it.

2. The market responds immediately to any new information.

Thus, the movement of asset prices is a Markov process. For modeling purposes the return on investment, or the change in asset price dS divided by the original value S, is a better indicator of investment performance than the absolute change in asset value. We can express this as

$$\frac{dS}{S}$$

One common approach is to decompose the return into predictable and random components. The predictable component is the anticipated return if the asset value were invested in a practically risk-free instrument, such as US government treasury bills. This can be expressed as

$$\mu dt$$

A second component reflects the random contribution of market effects, expressed as

$$\sigma dX$$

Where dX represents a random sample drawn from a normal distribution with mean zero and variance dt. σ is the volatility or standard deviation of the returns. Together, these components yield the stochastic differential equation

$$\frac{dS}{S} = \sigma dX + \mu dt \tag{4.1}$$

The term dX is known as a Wiener process. Equation (4.1) is an example of a random walk. While it cannot be solved deterministically, it does yield significant information about the behavior of the asset price S in a probabilistic sense. Since assets are priced in discrete time steps, equation (4.1) yields a time series which can be used to model the movement of an asset price. But over practical timescales this approach leads to unmanageably large amounts of data. A more practical approach is to set up the model in the continuous time limit $dt \to 0$ and solve the resulting differential equations. To accomplish this we use Ito's lemma and the result

$$dX^2 \to dt \text{ as } dt \to 0 \tag{4.2}$$

4.0.5 Elimination of Randomness

Consider a function of the random variable S and time: $f(S,t)$

Expanding $f(S+dS, t+dt)$ about (S,t) yields:

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial^2 S}dS^2 + ... \tag{4.3}$$

Using (4.1), and (4.3) we obtain

$$df = \sigma S\frac{\partial f}{\partial S}dX + \left(\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial^2 S} + \frac{\partial f}{\partial t}\right)dt \tag{4.4}$$

Both random walks (4.1) and (4.4) depend on the single random variable dX. This allows the construction of a variable g whose variation dg is deterministic during a small time step dt:

Let Δ be a number and let

$$g = f - \Delta S$$

where Δ is constant during the time interval dt. Thus, we have

$$\begin{aligned} dg = df - \Delta dS &= \sigma S\frac{\partial f}{\partial S}dX + \left(\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt - \Delta\left(\sigma S dX + \mu S dt\right) \\ &= \sigma S\left(\frac{\partial f}{\partial S} - \Delta\right)dX + \left(\mu S\left(\frac{\partial f}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt \end{aligned}$$

By choosing $\Delta = \frac{\partial f}{\partial S}$ we can eliminate the dX terms, yielding a deterministic result for g. Our development of this model follows that of [7].

4.0.6 The Black-Scholes Model

The assumptions made for the Black-Scholes analysis are:

1. The asset price S follows the lognormal random walk of equation (4.1).
2. The risk-free interest rate r and the volatility σ are known functions of time over the option lifespan.
3. There are no transaction costs.
4. The underlying asset pays no dividends.
5. Arbitrage opportunities do not exist.
6. Trading in the underlying can occur continuously.
7. Assets are infinitely divisible and "short selling" (selling assets which are not owned) is possible. (Short selling involves borrowing assets for sale and replacing them at a later date.)

Given these assumptions, we can give the value of an option as a function of the current price of the underlying and time, $V(S,t)$. We can thus write

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt$$

for the random walk followed by V. A portfolio Π, consisting of one option and some quantity $-\Delta$ of the underlying asset will have value

$$\Pi = V - \Delta S \tag{4.5}$$

The change in value of this portfolio in one time step dt is

$$d\Pi = dV - \Delta dS$$

So Π follows the random walk

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt$$

We can eliminate the random component by choosing

$$\Delta = \frac{\partial V}{\partial S} \tag{4.6}$$

We then have

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

The return on an amount Π invested at the risk-free rate r would be $r\Pi dt$ over the time step dt. If the right side of the above equation were larger than this amount, an arbitrageur could sell the option, borrow Π, invest in it the portfolio and make a guaranteed riskless profit. If it were less than this amount, the arbitrageur could buy the option, short (sell) the portfolio and invest the proceeds in the bank, also guaranteeing a riskless profit. The existence of arbitrageurs who seek to take advantage of such situations combined with the law of supply and demand make it practically impossible for such situations to exist for any significant amount of time. Thus we can conclude that

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{4.7}$$

Substituting (4.5) and (4.6) into (4.7), we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{4.8}$$

This is the celebrated Black-Scholes partial differential equation. Under the assumptions stated at the beginning of this section, the value of any derivative security whose price depends only on the current value of the underlying asset S and the time to expiry t must satisfy the Black-Scholes equation. This derivation is essentially that of [7].

4.0.7 The Black-Scholes Formula

The Black-Scholes equation and boundary conditions for a European call option, that is, an option which can only be exercised at the expiration date T and gives the holder the right to buy a quantity of the underlying asset at a predetermined strike price E, are

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{4.9}$$

with

$$C(0,t) = 0, \quad C(S,t) \approx S \text{ as } S \to \infty, \quad C(S,T) = max(S-E,0).$$

Setting

$$S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad C = E\upsilon(x,\tau).$$

we obtain

$$\frac{\partial \upsilon}{\partial \tau} = \frac{\partial^2 \upsilon}{\partial x^2} + (k-1)\frac{\partial \upsilon}{\partial x} - k\upsilon$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$, with initial condition $\upsilon(x,0) = max(e^x - 1, 0)$.

By using the change of variables

$$\upsilon = e^{\alpha x + \beta \tau} u\left(x, \tau\right)$$

and differentiating, we obtain

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \left(k-1\right)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

The u and $\frac{\partial u}{\partial x}$ terms can be eliminated by choosing

$$\beta = \alpha^2 + \left(k-1\right)\alpha - k$$

$$0 = 2\alpha + \left(k-1\right)$$

which yield

$$\alpha = -\frac{1}{2}\left(k-1\right), \beta = -\frac{1}{4}\left(k+1\right)^2.$$

Then

$$\upsilon = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2 \tau} u\left(x, \tau\right)$$

with

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad \tau > 0$$

and

$$u\left(x, 0\right) = u_0\left(x\right) = max\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right). \tag{4.10}$$

As previously established, the solution to this is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s)\, e^{\frac{-(x-s)^2}{4\tau}} ds.$$

where $u_0(x)$ is given by (4.10). To evaluate the integral we make the change of variable $x' = \frac{(s-x)}{\sqrt{2\tau}}$, obtaining

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0\left(x'\sqrt{2\tau} + x\right) e^{-\frac{1}{2}x'^2} dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)\left(x+x'\sqrt{2\tau}\right)} e^{-\frac{1}{2}x'^2} dx'$$

$$-\frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)\left(x+x'\sqrt{2\tau}\right)} e^{-\frac{1}{2}x'^2} dx'$$

$$= I_1 - I_2$$

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)\left(x+x'\sqrt{2\tau}\right)-\frac{1}{2}x'^2} dx'$$

$$= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau} e^{-\frac{1}{2}\left(x'-\frac{1}{2}(k+1)\sqrt{2\tau}\right)^2} dx'$$

$$= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}-\frac{1}{2}(k+1)\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\rho^2} d\rho$$

$$= e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau} N(d_1),$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$N(d_1) = \frac{1}{\sqrt{2\tau}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}s^2} ds$$

is the cumulative distribution function for the normal distribution. I_2 is calculated similarly, with $(k-1)$ replacing $(k+1)$.

Thus, we have

$$\upsilon(x,\tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau).$$

Substituting $x = log\left(\frac{S}{E}\right), \tau = \frac{1}{2}\sigma^2(T-t)$ and $C = E\upsilon(x,t)$, we have

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)N(d_2)}$$

where

$$d_1 = \frac{log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = \frac{log\left(\frac{S}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}.$$

We have reached the Black-Scholes formula for the value of a European call option. The method shown here closely follows that given in [7].

The formula for the European put follows similarly, using the put-call parity relationship. Put-call parity can be understood by the following argument. Consider a portfolio which is long one asset, long one put and short one call. Assume further that the options in the portfolio have the same expiration date T and the same strike price E. If we use Π for the value of the portfolio, we have

$$\Pi = S + P - C.$$

Note that whether S is greater or less than E at expiry, the payoff is exactly E. By an arbitrage argument we have that the value of a portfolio that guarantees a payoff of E at time T must be $Ee^{-r(T-t)}$. Thus we have the put-call parity formula:

$$C - P = S - Ee^{-r(T-t)}.$$

For the european put, this yields

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1).$$

This derivation of the put-call parity formula is essentially that of [7].

Chapter 5

Valuation of Derivative Securities

5.0.8 Commodity Options Pricing

Take the futures price as F, and let the spot price as a function of time be given by $S(t)$. If an investor enters into a short position in a futures contract, he must deliver the specified quantity of the commodity for the amount F at a specified future date T. The investor can borrow the amount $S(t)$ at the beginning of the contract, buy the commodity and use the proceeds F to pay off the loan. With risk-free rate r, the loan will cost $S(t)\, e^{r(T-t)}$. Thus, by the arbitrage argument given earlier,

$$F = S(t)\, e^{r(T-t)}.$$

The value of an option with a futures contract as the underlying must then be a function of F and t: $V(F,t)$. Using the same steps followed in the Black-Scholes analysis, we consider the portfolio $\Pi = V - \Delta F$. Since the volatility of F is also σ, we have that $dF^2 \to \sigma^2 F^2 dt$ and obtain by the same method used previously

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2}\right) dt + \frac{\partial V}{\partial F} dF.$$

Following the same steps as for the stock option and using $\Delta = \frac{\partial V}{\partial F}$ to make the portfolio risk-free we obtain

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} \right) dt.$$

Since the cost of the portfolio is just V (it costs nothing to enter into a futures contract), by the same arbitrage argument used in the Black-Scholes analysis, we must have

$$d\Pi = rV dt$$

These two equations for $d\Pi$ lead to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0.$$

Reducing this equation to the heat equation and solving as before, we find that the value of a European call, for example, is

$$C(F, t) = e^{-r(T-t)} \left(FN(d_1) - EN(d_2) \right).$$

This analysis of commodity options pricing and further discussion can be found in [2].

5.0.9 American Options

Since American options can be exercised at expiry or any time before, they can potentially have greater value than European options. A simple example shows how this can be true. In the case of the put option, for example, there exists the possibility that at some time during the lifespan of the option the price of the underlying asset, S, will fall to zero, such as in the case of bankruptcy. If this occurs, the American option clearly has higher value than the European option, where early exercise is not allowed. This argument illustrates that allowing the possibility of early exercise must make a difference in the value of an option at certain values of the underlying.

Since the Black-Scholes formula derived earlier for the European option gives its value for all values of S, it is apparent that this formula cannot always give the correct value for American options. If it could, then the value of European and American options would always be the same, and we have shown that this is not the case. Furthermore, there must be values of S for which early exercise is optimal for the investor. If this were not the case then the best strategy for the investor would be to hold the option until expiry no matter how the asset price changed. But then the option would have the same value as its European counterpart, again in contradiction with the conclusion of the above arbitrage argument [7].

So for American options, at any time before expiry there exists a threshold value of the underlying. In order to maximize the profit to the investor the decision whether to hold or exercise the option at a specific time depends on whether the value of the underlying is above or below this threshold value. The curve formed by these threshold values forms a boundary between the region where it is optimal to hold the option and the region where it is optimal to exercise the option. Since we do not know where this boundary is we refer to it as a free boundary, and free boundary problems are an important area in the study of PDE's.

Free Boundary Problems

Some free boundary problems occurring in physical settings have been extensively studied. One such problem is the Stefan model of melting ice. This model arises from the problem of determining the solid/liquid boundary in a system consisting of a bar of ice which is held to a temperature above its melting point at one end. The ice will melt there and the liquid water will increase in temperature in accordance with the heat equation. The heat from the water will flow into the ice, and the temperature of the ice will increase, also in accordance with the heat equation, until it reaches its melting point. At that temperature heat will be used to bring about the solid/liquid phase transformation and the temperature will remain constant until the ice has melted. Say the ice occupies the interval $a \leq x < \infty$ and is initially at $0°C$ except at $x = a$, where it is maintained at some temperature $T > 0$. Considering the region where the water is liquid to be $a \leq x \leq s(t)$ and denoting the temperature by u, we state the Stefan problem as:

$$\alpha^2 u_{xx} - u_t = 0 \ \text{ for}\, a \leq x \leq s(t), t > 0,$$

$$u(a, t) = T \ \text{ for}\, t > 0,$$

$$u(s(t), t) = 0 \ \text{ for}\, t > 0,$$

where α is some non-zero constant.

We do not know the free boundary $s(t)$ a priori, but we know that at the boundary

$$\frac{ds(t)}{dt} = -k u_x(s(t), t) \ \text{ for}\, t > 0,$$

where k is some positive constant. The Stefan problem can be solved by reduction to an integral equation. This statement of the Stefan problem, as well as existence, uniqueness and asymptotic behavior of solutions are discussed in [4].

Turning now to a derivation of the boundary conditions for the American put option, we see that the Stefan model is not appropriate for the valuation of the American put.

The Free Boundary Problem for the American Put Option

Denoting the option value as $P(S, t)$, we have established by an arbitrage argument that we must have $P(S, t) \geq max(E - S, 0)$. Also, at each time t there is a particular value of S below which the option should be exercised and above which the option should be held. This is the boundary value S_f which is not known *a priori*. Since S is a function of t, we refer to $S_f(t)$ as the **free boundary**. If $0 \leq S \leq S_f(t)$, then $P = E - S$ and the option should be exercised. If $S_f(t) \leq S \leq \infty$, early exercise is not the optimal strategy for the investor and the option value must satisfy the Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0.$$

The boundary conditions at $S = S_f(t)$ are

$$P(S_f(t), t) = max(E - S_f(t), 0) \text{ and } \frac{\partial P}{\partial S}(S_f(t), t) = -1$$

The first of these follows from the previous arbitrage argument. The continuity constraint for $P(S(t), t)$ follows from the observation that if S reached the value at which a discontinuity in P lasting longer than an infinitesimal time occurred, an arbitrage opportunity would again arise. We can see that the slope of the option, $\frac{\partial P}{\partial S}$, at $S = S_f(t)$ must be equal to the slope of the payoff function $max(E - S_f(t), 0)$ by noting that if $\frac{\partial P}{\partial S} < -1$, as S increases from $S_f(t)$ the value of P will fall below the payoff function. If $\frac{\partial P}{\partial S} > -1$, the value of P will fall below the payoff function as S decreases. In either case the condition $P(S, t) \geq max(E - S, 0)$ is violated. This analysis is essentially that of [7].

Chapter 6

The Optimal Exercise Boundary for an American Option

6.0.10 Valuation of the American Put

The boundary value problem for the value of an American put, $P(S,t)$ with exercise boundary $S = S_f(t)$ can be stated as

$$-P_t = \frac{\sigma^2 S^2}{2} P_{SS} + rSP_S - rP, \quad 0 < t < T, \quad S > S_f(t)$$

$$S_f(T) = E$$

$$P(S,T) = 0, \quad S \geq E$$

$$P[S_f(t),t] = E - S_f(t), \quad P_S[S_f(t),t] = -1, \quad 0 \leq t \leq T \tag{6.1}$$

for $S \leq S_f(t)$ and $P(S,t) = E - S$ [6].

Rewriting, using

$$\rho = \frac{2r}{\sigma^2}$$

$$\hat{\rho} = \rho - 1$$

$$S = Ee^x$$

$$t = T - \frac{2\tau}{\sigma^2}$$

$$P(S,t) = Ee^{-\rho\tau}[1 + p(x,\tau)] - S$$

$$b\left(\tau\right) = log\left[\frac{S_f(t)}{E}\right],$$

the equation (6.1) becomes the following for the price $p\left(x,\tau\right)$ and exercise boundary $x = b\left(\tau\right)$:

$$p_\tau = p_{xx} + \hat{\rho} p_x, \quad x > b\left(\tau\right), \quad 0 < \tau < \frac{\sigma^2 T}{2} \tag{6.2}$$

$$b(0) = 0$$

$$p(x,0) = e^x - 1, \quad x \geq 0 \tag{6.3}$$

$$p\left[b(\tau),\tau\right] = e^{\rho\tau} - 1, \quad 0 \leq \tau \leq \frac{\sigma^2 T}{2} \tag{6.4}$$

$$p_x\left[b(\tau),\tau\right] = 0, \quad 0 \leq \tau \leq \frac{\sigma^2 T}{2}. \tag{6.5}$$

6.0.11 The Integral Equation for $b(\tau)$

The problem stated as (6.2) - (6.5) above can be converted into an integral equation for $b(\tau)$ by using the Green's function. This solution $G(x,\tau;\xi,s)$ is given by

$$G(x,\tau;\xi,s) = \frac{1}{\sqrt{4\pi(\tau - s)}} e^{-\frac{[x-\xi+\hat{\rho}(\tau-s)]^2}{4(\tau-s)}}, s < \tau. \tag{6.6}$$

We will apply Green's theorem to $G(x,\tau;\xi,s)$ and $p(\xi,s)$ in the region

$$\xi > b(s), \quad 0 < s < \tau.$$

We start by computing the following derivatives:

$$G_\xi(x,\tau;\xi,s) = G(x,\tau;\xi,s)\frac{1}{2}\left(\frac{x-\xi+\hat{\rho}(\tau-s)}{(\tau-s)}\right)$$

$$G_{\xi\xi}(x,\tau;\xi,s) = G(x,\tau;\xi,s)\frac{1}{4}\left(\frac{x-\xi+\hat{\rho}(\tau-s)}{(\tau-s)}\right)^2 - \frac{1}{2}G(x,\tau;\xi,s)\frac{1}{(\tau-s)}.$$

$$G_s(x,\tau;\xi,s) = \frac{G(x,\tau;\xi,s)}{2(\tau-s)}\left[1+\hat{\rho}[x-\xi+\hat{\rho}(\tau-s)] - \frac{1}{2(\tau-s)}[x-\xi+\hat{\rho}(\tau-s)]^2\right].$$

Algebraic manipulations yield the equality

$$G_{\xi\xi} - \hat{\rho}G_\xi + G_s = 0.$$

Which leads to

$$\iint_R \left[\frac{\partial}{\partial\xi}\left(Gp_\xi - G_\xi p\right) + \hat{\rho}\frac{\partial}{\partial\xi}\left(Gp\right) - \frac{\partial}{\partial s}\left(Gp\right)\right] = 0.$$

This integral can be written as

$$\iint_R \frac{\partial}{\partial\xi}\left(Gp_\xi - G_\xi p\right) + \hat{\rho}\iint_R \frac{\partial}{\partial\xi}\left(Gp\right) - \iint_R \frac{\partial}{\partial s}\left(Gp\right) = 0 = I_1 + I_2 - I_3,$$

where

$$I_1 = \int_0^\tau \int_{b(s)}^\infty \frac{\partial}{\partial \xi} \left(G p_\xi - G_\xi p \right) d\xi ds$$

$$= \int_0^\tau \left(G p_\xi(x,\tau;\infty,s) - G p_\xi(x,\tau;b(s),s) + G_\xi p(x,\tau;b(s),s) - G_\xi p(x,\tau;\infty,s) \right) ds$$

$$= \int_0^\tau \left(G_\xi(x,\tau;b(s),s) p(b(s),s) - G p_\xi(x,\tau;b(s),s) p_\xi(b(s),s) \right) ds.$$

Using $p\left[b(\tau),\tau\right] = e^{\rho\tau} - 1$ and $p_x\left[b(\tau),\tau\right] = 0$, one obtains

$$I_1 = \int_0^\tau G(x,\tau;\xi,s) \frac{x - b(s) + \hat{\rho}(\tau - s)}{2(\tau - s)} \left(e^{\rho s} - 1 \right) ds.$$

We now move to I_2:

$$I_2 = \hat{\rho} \int_0^\tau \int_{b(s)}^\infty \frac{\partial}{\partial \xi} \left(G p \right) d\xi ds$$

$$= \hat{\rho} \int_0^\tau -G(x,\tau;b(s),s) p(b(s),s) ds$$

$$= -\hat{\rho} \int_0^\tau G(x,\tau;b(s),s)(e^{\rho s} - 1) ds.$$

Finally,

$$I_3 = \int_0^{b(\tau)} \int_0^{b^{-1}(\xi)} \frac{\partial}{\partial s} \left(G p \right) ds d\xi + \int_{b(\tau)}^\infty \int_0^\tau \frac{\partial}{\partial s} \left(G p \right) ds d\xi$$

$$= \int_0^{b(\tau)} G(x,\tau;\xi,b^{-1}(\xi))p(\xi,b^{-1}(\xi)) - G(x,\tau;\xi,0)p(\xi,0)d\xi$$
$$+ \int_{b(\tau)}^{\infty} G(x,\tau;\xi,\tau)p(\xi,\tau) - G(x,\tau;\xi,0)p(\xi,0)d\xi$$

Now, letting $\xi = b(r)$ with $\xi = 0$, one concludes

$$I_3 = \int_0^{\tau} G(x,\tau;b(r),r)p(b(r),r)dr - \int_0^{b(\tau)} G(x,\tau;\xi,0)(e^{\xi}-1) + p(x,\tau)$$
$$- \int_{b(\tau)}^{\infty} G(x,\tau;\xi,0)(e^{\xi}-1)d\xi$$

Thus,

$$p(x,\tau) = -\int_0^{\tau} \left[\frac{x-b(s)}{2(\tau-s)} + \frac{1}{2}\hat{\rho} + b'(s)\right](e^{\rho s}-1)G[x,\tau;b(s),s]ds$$
$$+ \int_0^{\infty} (e^{\xi}-1)G[x,\tau;\xi,0]d\xi \qquad (6.7)$$

This is a representation of p in terms of the boundary function $b(\tau)$. Using (6.7) in the boundary condition (6.4), we obtain

$$e^{\rho\tau} - 1 = -\int_0^{\tau} \left[\frac{b(\tau)-b(s)}{2(\tau-s)} + \frac{3}{2}\hat{\rho} + b'(s)\right](e^{\rho s}-1)G[b(\tau),\tau;b(s),s]ds$$
$$+ \int_0^{\infty} (e^{\xi}-1)G[b(\tau),\tau;\xi,0]d\xi. \qquad (6.8)$$

Equation (6.8) is an integral equation for $b(\tau)$. Rewriting, using $I(\tau)$ for the first integral and (6.6) for G, we obtain

$$e^{\rho\tau} - 1 + I(\tau) = \frac{1}{\sqrt{2\pi}} \left[e^{b(\tau)+(1-\hat{\rho})\tau} \int_{-\frac{b(\tau)+(3-\rho)\tau}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} dz - \int_{-\frac{b(\tau)-\hat{\rho}\tau}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} dz \right] \tag{6.9}$$

The above is a detailed presentation of the procedure presented in [6].

6.0.12 Asymptotic Solution of the Integral Equation

We begin by showing that $I(\tau) \approx \frac{\rho\tau}{2}$ for $\tau \ll 1$. Dividing the integral into two parts

$$I = \int_0^\delta \left[-\frac{b(\tau)-b(s)}{2(\tau-s)} + \frac{1}{2}\hat{\rho} + b'(s) \right] (e^{\rho s} - 1) G[b(\tau), \tau; b(s), s] ds$$
$$+ \int_\delta^\tau \left[-\frac{b(\tau)-b(s)}{2(\tau-s)} + \frac{1}{2}\hat{\rho} + b'(s) \right] (e^{\rho s} - 1) G[b(\tau), \tau; b(s), s] ds \equiv I_1 + I_2. \tag{6.10}$$

Bounding $|I_1|$:

$$|I_1| = \left| \int_0^\delta \left[-\frac{b(\tau)-b(s)}{2(\tau-s)} + \frac{1}{2}\hat{\rho} + b'(s) \right] (e^{\rho s} - 1) G[b(\tau), \tau; b(s), s] ds \right|$$
$$\leq \int_0^\delta \rho s \left| \frac{b(\tau)-b(s)}{2(\tau-s)} \right| \frac{\tau^{\gamma(\tau,\delta)}}{\sqrt{4\pi(\tau-s)}} ds + \int_0^\tau \rho s \left| \frac{1}{2}\hat{\rho} \frac{\tau^{\gamma(\tau,\delta)}}{\sqrt{4\pi(\tau-s)}} \right| ds$$
$$+ \int_0^\tau \rho s \, |b'(s)| \, \frac{\tau^{\gamma(\tau,\delta)}}{\sqrt{4\pi(\tau-s)}} ds \tag{6.11}$$

where

$$\gamma(\tau,\delta) = \frac{1}{2}\left(\hat{\rho}\frac{\tau-\delta}{b(\tau)}\right)^2 > 0$$

Using the result that $b(\tau) \approx -\sqrt{2\tau log\tau^{-1}}$ in (6.11) yields $|I_1| = o(\tau)$. Turning now to I_2, for $0 < \delta < \tau$ we can write the integral as

$$\int_\delta^\tau f\left(s;\tau\right) e^{-\lambda^2\phi(s;\tau)} ds$$

where $\lambda^2 = \frac{\alpha^2(\tau)}{4}$ and

$$\phi(s;\tau) = \left[\frac{b(\tau)-b(s)+\hat{\rho}(\tau-s)}{\alpha(\tau)(\tau-s)^{\frac{1}{2}}}\right]^2.$$

Evaluating I_2 asymptotically for $\alpha \gg 1$:

$$I_2 = \int_\delta^\tau \left[-\frac{b(\tau)-b(s)}{2(\tau-s)} + \frac{1}{2}\hat{\rho} + b'(s)\right](e^{\rho s}-1)\frac{1}{\sqrt{4\pi(\tau-s)}} e^{-\left[\frac{b(\tau)-b(s)+\hat{\rho}(\tau-s)}{\alpha(\tau)(\tau-s)^{\frac{1}{2}}}\right]^2 \frac{\alpha^2(\tau)}{4}} ds$$

To evaluate this integral it is sufficient to consider δ close to τ [1]. Using the Taylor expansion for $e^{\rho s} - 1$ and

$$\delta < s < \xi < \tau \Rightarrow b'(\xi) \approx b'(\tau)$$

$$-\frac{b(\tau)-b(s)}{2(\tau-s)} = -\frac{b'(\xi)}{2}$$

We obtain

$$I_2 \approx \frac{1}{\sqrt{4\pi}}\left[\frac{1}{2}b'(\tau)+\frac{1}{2}\hat{\rho}\right]\rho\tau\int_\delta^\tau \frac{1}{\sqrt{\tau-s}}e^{-\lambda^2|\phi'(\tau;\tau)|(\tau-s)}ds$$

With the change of variable

$$\lambda^2(\tau-s)\,|\phi'(\tau;\tau)| = z^2$$

$$\approx \frac{1}{\sqrt{4\pi}}\left[\frac{1}{2}b'(\tau)+\frac{1}{2}\hat{\rho}\right]\rho\tau\frac{2}{\lambda\sqrt{\phi'(\tau;\tau)}}\int_0^{\lambda\sqrt{|\phi'(\tau;\tau)|(\tau-\delta)}} e^{-z^2}dz$$

Since $\lambda^2\,|\phi'(\tau;\tau)| = \frac{(b'(\tau)+\hat{\rho})^2}{4}$, the upper limit tends to infinity as t tends to 0. Thus, the asymptotic behavior of I_2 is

$$I_2 \approx \frac{\sqrt{\pi}}{2}\frac{f(\tau;\tau)}{\left|\lambda\sqrt{\phi(\tau;\tau)}\right|} \approx \frac{\rho\tau}{2}\text{ for }\tau \ll 1.$$

Thus $I(\tau) = I_1 + I_2 = o(\tau) + \frac{\rho\tau}{2}$ for $\tau \ll 1$. The left side of (6.9) is asymptotically $\rho\tau + \frac{\rho\tau}{2} = \frac{3\rho\tau}{2}$. Making the assumption $-\tau^{-\frac{1}{2}}b(\tau) \gg 1$ for $\tau \ll 1$, we use the asymptotic value of the error function:

$$\int_y^\infty e^{-\frac{z^2}{2}}dz = e^{-\frac{y^2}{2}}\left[y^{-1}+O(y^{-3})\right]\text{ for y } \gg 1$$

to evaluate the right side of (6.9), which becomes

$$\frac{3\rho\tau}{2} \approx \frac{2\tau}{b^2(\tau)}\sqrt{\frac{\tau}{\pi}}e^{-\frac{b^2(\tau)}{4\tau}} \text{ for } \tau \ll 1. \tag{6.12}$$

We can rewrite this equation for $b(\tau)$ in terms of $\alpha(\tau)$:

$$b(\tau) = -2\tau^{\frac{1}{2}}\alpha(\tau) \tag{6.13}$$

Using (6.13) for $b(\tau)$ in (6.12), $\rho = \frac{2r}{\sigma^2}$, and $\tau = \frac{1}{2}\sigma^2(T-t)$ we obtain the result for the boundary, in parametric form with parameter α:

$$\frac{\sigma^2}{2}(T-t) \approx \left[\frac{\sigma^2 e^{-\alpha^2}}{6\pi^{\frac{1}{2}}r\alpha^2}\right]^2 \tag{6.14}$$

In order to obtain $S_f(t)$ explicitly, we solve (6.14) for α in terms of $\frac{\sigma^2}{2}(T-t)$, obtaining

$$\frac{e^{-\alpha^2}}{\alpha^2} \approx \frac{6r}{\sigma^2}\sqrt{\pi\tau}$$

Taking logarithms and substituting three times yields

$$\alpha(\tau) = \left\{log\frac{\sigma^2}{6r\sqrt{\pi\tau}} - log\left[log\frac{\sigma^2}{6r\sqrt{\pi\tau}} - log\left(log\frac{\sigma^2}{6r\sqrt{\pi\tau}} - log\alpha^2\right)\right]\right\}^{\frac{1}{2}}. \tag{6.15}$$

Next, we express $S_f(t)$ in terms of $b(\tau)$:

$$S_f(t) = Ee^{b(\tau)}.$$

Using (6.13) in this expression, we obtain

$$S_f(t) \approx E\left[1 - 2\left[\frac{\sigma^2\,(T-t)}{2}\right]\alpha\right]$$

for $\frac{1}{2}\sigma^2\,(T-t) \ll 1$.

Using the first term from (6.15), we obtain

$$S_f(t) \approx E\left[1 - \left[2\sigma^2\,(T-t)\,log\frac{\sigma^2}{6\pi^{\frac{1}{2}}r\left[\frac{1}{2}\sigma^2(T-t)\right]^{\frac{1}{2}}}\right]^{\frac{1}{2}}\right]$$

Here we have worked out the details using the method of [6]. Our result and further analysis of the boundary near expiry can be found in [6].

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Curriculum Vitae

Walter Bales was born and raised in El Paso, Texas. After a twenty-year career in engineering and management in the electronics industry on the US - Mexico border, Walt felt the desire for a change. In 2004 he decided to pursue an academic career in mathematics. He recieved his B.S. in Mathematics from the University of Texas at El Paso in 2007, graduating Magna Cum Laude. As an undergraduate student he taught Calculus labs for five semesters, recieving an award for outstanding teaching. He was also active in tutoring many other students in math and science. He continued graduate studies at UTEP with a focus on financial mathematics while teaching math courses at both UTEP and El Paso Community College. Walt completed the degree requirements for the M.S. in Mathematics in May 2009.