4-1-2004

Towards a General Methodology for Designing Sub-Noise Measurement Procedures

Roberto Osegueda

George R. Keller
University of Texas at El Paso, keller@utep.edu

Scott A. Starks
University of Texas at El Paso, sstarks@utep.edu

R. Araiza

Dm. Bizyaev

See next page for additional authors

Comments:
UTEP-CS-04-12.
Published in *Proceedings of 10th IMEKO TC7 International Symposium*, Saint-Petersburg, Russia, June 30-July 2, 2004, Vol. 1, pp. 59-64.

10^{th} IMEKO TC7 International Symposium
June 30–July 2, 2004, Saint-Petersburg, Russia

TOWARDS A GENERAL METHODOLOGY FOR DESIGNING SUB-NOISE MEASUREMENT PROCEDURES

R. Osegueda, G. R. Keller, S. A. Starks, R. Araiza, Dm. Bizyaev, V. Kreinovich

University of Texas at El Paso, El Paso, TX 79968, USA

Abstract. In many practical situations, the measurement result z depends not only on the measured value x, but also on the parameters s describing the experiment's setting and on the values of some auxiliary quantities y; the dependence $z = f(x, s, y)$ of z on x, s, and y is usually known. In the ideal case when we know the exact value of the auxiliary parameter y, we can solve the above equation and find the desired value x. In many real-life situations, we only know y with some uncertainty, and this uncertainty leads to additional uncertainty in x.

If we are trying to reconstruct x based on a *single* measurement result, then, of course, the measurement error in y leads to the corresponding measurement error in x – and, unless we perform more accurate measurements, we cannot improve x's accuracy.

In many practical situations, however, if we have *several* measurement results corresponding to different values of t and/or y, we can reconstruct x with a much higher accuracy – because we can combine these measurement results in such a way that the influence of y drastically decreases. As a result, we get a *sub-noise* measurement accuracy, the accuracy that is much better than the accuracy with which we know y.

Keywords: error correction, general measurement methodology

1. FORMULATION OF THE PROBLEM

In many practical situations, the measurement result z depends not only on the measured value x, but also on the parameters s describing the experiment's setting and on the values of some auxiliary quantities y; the dependence $z = f(x, s, y)$ of z on x, s, and y is usually known. In the ideal case when we know the exact value of the auxiliary parameter y, we can solve the above equation and find the desired value x. In many real-life situations, we only know y with some uncertainty, and this uncertainty leads to additional uncertainty in x.

If we are trying to reconstruct x based on a *single* measurement result, then, of course, the measurement error in y leads to the corresponding measurement error in x – and, unless we perform more accurate measurements, we cannot improve x's accuracy.

In many practical situations, however, if we have *several* measurement results corresponding to different values of y, we can reconstruct x with a much higher accuracy – because we can combine these measurement results in such a way that the influence of y drastically decreases. As a result, we get a *sub-noise* measurement accuracy, the accuracy that is much better than the accuracy with which we know y.

In different areas of science and engineering, a lot of different ingenious measurement procedures have been invented that lead to such sub-noise accuracy. Our objective is to design a general methodology for designing such procedures.

We start with describing several measurement situations where procedures of this type have been applied. Then, we provide the basic mathematical foundations for the desired general methodology.

2. CASE STUDIES

2.1. Multi-spectral astronomical imaging

In *multi-spectral imaging*, we may have cosmic dust preventing us from seeing details of an image.

Let $\vec{p}$ be an arbitrary point in the image, let f denote the observation frequency, and let $I(f, \vec{p})$ denote the intensity of the object of interest at the point $\vec{p}$ at frequency f. Usually, for astronomical objects, observations at different wavelengths reflect the same structure $I(\vec{p})$, and our objective is to reveal this structure. In the first approximation, we can therefore assume that $I(f, \vec{p}) = C(f) \cdot I(\vec{p})$ for some (partially unknown) function $C(f)$.

Because of the dust, instead of observing the intensity $I(f, \vec{p})$ of the object's radiation, what we actually measure is the sum $\widetilde{I}(f, \vec{p}) = I(f, \vec{p}) + D(f, \vec{p})$, where $D(f, \vec{p})$ is the intensity of dust radiation at point $\vec{p}$ at frequency f. For many astronomical objects, the dust radiation is much more intensive than the radiation of the object; as a result, from the observed values $\widetilde{I}(f, \vec{p})$, we cannot determine the object's structure.

If we only have an observation at a single wavelength, then there is nothing that we can do to improve the quality of the image. Luckily, however, we have observations at different wavelengths, and we know how dust effect depends on the frequency. In the first approximation, we can describe this dependence by the power law $D(f, \vec{p}) = D(\vec{p}) \cdot f^{\alpha}$ for some known real value α.

In this case, for each point $\vec{p}$:

- the measured value x is the actual object's intensity $I(\vec{p})$ at this point;
- the experiment setting s is describe by the frequency f,

- the auxiliary quantities y include the amount of dust $D(\vec{p})$ at this point, and
- the dependence of the measurement result $z = \widetilde{I}(f, \vec{p})$ on x, s, and y is given by the following formula:

$$\widetilde{I}(f, \vec{p}) = C(f) \cdot I(\vec{p}) + D(\vec{p}) \cdot f^{\alpha}. \quad (1)$$

In other words, here

$$z = f(x, s, y) = C(s) \cdot x + y \cdot s^{\alpha}. \quad (2)$$

It turns out that we can combine the intensities from different wavelengths in such a way that the resulting combined image does not depend on the dust. Specifically, after observing the same object at two different wavelengths s_1 and s_2, we get two measurement results

$$z_1 = C(s_1) \cdot x + y \cdot s_1^{\alpha};\ z_2 = C(s_2) \cdot x + y \cdot s_2^{\alpha}. \quad (3)$$

Well known variable elimination techniques from linear algebra enable us to get rid of the parameter y: namely, we multiply z_1 by s_2^{α}, z_2 by s_1^{α}, and subtract the result, to get

$$z_1 \cdot s_2^{\alpha} - z_2 \cdot s_1^{\alpha} = x \cdot (C(s_1) \cdot s_2^{\alpha} - C(s_2) \cdot s_1^{\alpha}). \quad (4)$$

In other words, this simple combination enables us to eliminate the effect of dust and observe the desired structure $x = I(\vec{p})$.

Comment. Strictly speaking, we do not reconstruct the exact values of the structure, we only reconstruct it modulo a constant $C(s_1) \cdot s_2^{\alpha} - C(s_2) \cdot s_1^{\alpha}$, so while we can determine the relative intensity of different parts of the image, we cannot reconstruct the *absolute* values of these intensities unless we know the exact dependence $C(s)$.

This is a somewhat simplified description of the dust. The effect of the actual dust can be better described by assuming that there dust consists of two components $D(f, \vec{p}) = D_1(f, \vec{p}) + D_2(f, \vec{p})$ whose dependence on the observation frequency follows two different power laws: $D_1(f, \vec{p}) = D_1(\vec{p}) \cdot f^{\alpha_1}$ and $D_2(f, \vec{p}) = D_2(\vec{p}) \cdot f^{\alpha_2}$. In this case, to eliminate the effect of the dust, we must perform the observations not on two, but at least on three different wavelengths f_1, f_2, and f_3. After performing these observations and computing the appropriate linear combination $C_1 \cdot z_1 + C_2 \cdot z_2 + C_3 \cdot z_3$ of the observation results z_i, we get an expression that is proportional to the desired intensity $I(\vec{p})$ and is not affected by the dust.

This technique has been successfully used, it enables us to uncover previously unseen spiral and ring-like structures in distant galaxies [1].

2.2. Astrometry coming from VLBI

A similar situation occurs in *astrometry*, especially in astrometry coming from the Very Large Baseline Interferometry (VLBI).

The main idea behind VLBI is that we simultaneously observe a distant radiosource by two (or more) radioantennas i, j located very far from each other. The signal generated by a distant source k at some moment of time reaches both antennas. Since the path from the source to the antennas is slightly different, there is a time delay $\tau_{i,j,k}$ between these antennas. If the two antennas have precise synchronized clocks, then we can determine this delay by comparing the signals observed by the two antennas and finding the shift that makes these two signals maximally correlated.

Geometric analysis enables us to conclude that the difference d_{ij} between the lengths of the paths leading to the two antennas is equal to $d_{i,j,k} = b_{i,j} \cdot \cos(\alpha_{i,j,k})$, where $b_{i,j}$ is the distance between the two antennas (called *baseline*) and $\alpha_{i,j,k}$ is the angle between the direction towards the source and the direction connecting the two antennas. In algebraic terms, $d_{i,j,k} = \vec{b}_{i,j,k} \cdot \vec{s}_k$, where $\vec{b}_{i,j}$ is the vector connecting the two antennas i and j, $\vec{s}_k$ is the unit vector in the direction of the source k, and $\vec{b}_{i,j} \cdot \vec{s}_k$ denotes a "dot" (scalar) product of the two vectors. Thus, in the ideal case, the time delay $\tau_{i,j,k}$ between the two antennas is equal to

$$\tau_{i,j,k} = \frac{1}{c} \cdot \vec{b}_{i,j} \cdot \vec{s}_k, \quad (5)$$

where c denotes the speed of light.

In real life, the synchronization between the antennas is not perfect; for each antenna, there is an (unknown) synchronization error Δt_i – the difference between the actual and the time recorded on this antenna. As a result of these synchronization errors, the measured time delay is different from its ideal value:

$$\tau_{i,j,k} = \frac{1}{c} \cdot \vec{b}_{i,j} \cdot \vec{s}_k + \Delta t_i - \Delta t_j. \quad (6)$$

In this case:

- the desired variables x are the coordinates of the radiosource, i.e., the components of the vector $\vec{s}_k$, and
- the auxiliary quantities y are the baseline vectors $\vec{b}_{i,j}$ and the synchronization errors Δt_i.

If we knew the exact value of the baseline vectors and of the synchronization errors, then, based on the measured time delay, we would have obtained the exact value of the projection of the desired vector $\vec{s}_k$ on the baseline. By performing two measurements by two pairs of radiotelescopes (or, alternatively, the same pair at different time when, due to Earth's rotation, the orientation of the baseline changes), we would thus uniquely determine the unit vector $\vec{s}_k$ and thus, measured the exact location of the radiosource. In this ideal situation, the only source of measurement error would be the noise which translates into milliarcsecond accuracy ($\approx 0.001''$).

In reality, we only have an approximate knowledge of the baseline vector and of the synchronization errors.

As a result, the accuracy with which we can determine the location of a radiosource based on a single (or two) measurement, is several orders of magnitude lower that it could be if the signal noise was the only source of the measurement error.

It turns out, however, that we can drastically improve this accuracy if we simultaneously observe several different sources by using several different antenna pairs.

First of all, we can get rid of the synchronization errors. Specifically, if we observe two different sources k and l by using the same antennas i and j, we can simply compute the difference of the measurement results and thus get rid of the synchronization errors. Namely,

$$\Delta\tau_{i,j,k,l} = \frac{1}{c} \cdot \vec{b}_{i,j} \cdot \Delta\vec{s}_{k,l}, \tag{7}$$

where $\Delta\tau_{i,j,k,l} \stackrel{\text{def}}{=} \tau_{i,j,k} - \tau_{i,j,l}$ is the difference between the measured time delays, and $\Delta\vec{s}_{k,l} \stackrel{\text{def}}{=} \vec{s}_k - \vec{s}_l$ is the difference between the corresponding unit vectors. This technique is called *differential astrometry*.

Getting rid of the unknown baseline vectors is a little bit more complicated. For that, we need at least 4 antennas so that, in general, the baseline vectors $\vec{b}_{1,2}$, $\vec{b}_{2,3}$, and $\vec{b}_{3,4}$ are linearly independent. We also need to fix at least 4 different "basic" sources 1, 2, 3, and 4. After observing each pair of sources k and l, we get the three values $\Delta\tau_{i,j,k,l}$ that are related to the unknown baseline vectors and source locations by the formulas:

$$\Delta\tau_{1,2,k,l} = \frac{1}{c} \cdot \vec{b}_{1,2} \cdot \Delta\vec{s}_{k,l},$$

$$\Delta\tau_{2,3,k,l} = \frac{1}{c} \cdot \vec{b}_{2,3} \cdot \Delta\vec{s}_{k,l},$$

$$\Delta\tau_{3,4,k,l} = \frac{1}{c} \cdot \vec{b}_{3,4} \cdot \Delta\vec{s}_{k,l}.$$

We define the *dual basis* $\vec{B}_{i,j}$ in such a way that

$$\vec{B}_{i,j} \cdot \frac{1}{c} \cdot \vec{b}_{i,j} = 1;\ \vec{B}_{i,j} \cdot \vec{b}_{i',j'} = 0 \text{ for } (i',j') \neq (i,j).$$

Due to Kramer's rule, e.g.,

$$\vec{B}_{1,2} = \frac{c \cdot \vec{b}_{2,3} \times \vec{b}_{3,4}}{\vec{b}_{1,2} \cdot (\vec{b}_{2,3} \times \vec{b}_{3,4})}. \tag{8}$$

Then, from the above three equations, we conclude that

$$\vec{s}_{k,l} = \Delta\tau_{1,2,k,l} \cdot \vec{B}_{1,2} + \Delta\tau_{2,3,k,l} \cdot \vec{B}_{2,3} + \Delta\tau_{3,4,k,l} \cdot \vec{B}_{3,4}.$$

We have these linear expansions for $\vec{s}_{1,2}$, for $\vec{s}_{1,3}$, and for $\vec{s}_{1,4}$. Thus, we can determine the dual vectors $\vec{B}_{i,j}$ as linear combinations of $\vec{s}_{1,2}$, $\vec{s}_{1,3}$, and $\vec{s}_{1,4}$. Now, for any other source k, we have a similar expression

$$\vec{s}_{k,1} = \Delta\tau_{1,2,k,1} \cdot \vec{B}_{1,2} + \Delta\tau_{2,3,k,1} \cdot \vec{B}_{2,3} + \Delta\tau_{3,4,k,1} \cdot \vec{B}_{3,4}.$$

Since we already know how to describe the dual vectors $\vec{B}_{i,j}$ as linear combinations of $\vec{s}_{1,2}$, $\vec{s}_{1,3}$, and $\vec{s}_{1,4}$, we therefore get an explicit expression of $\vec{s}_{k,1} = \vec{s}_k - \vec{s}_1$ as a linear combination of $\vec{s}_{1,2}$, $\vec{s}_{1,3}$, and $\vec{s}_{1,4}$, with known coefficients.

In other words, we have an affine transformation between the actual and the observed values $\vec{s}_k$. Since all the vectors $\vec{s}_k$ must be unit vectors, the only possible affine transformation is rotation. Thus, we can determine all the position modulo rotation. The resulting method called *arc method* is described in detail in [2,3].

Thus, by combining the signals from several sources on several antennas, we can combine these results in such a way as to minimize the effect of the (not precisely known) antenna coordinates and clock rates.

2.3. VLBI imaging

In the astrometry section, we described how VLBI responds to point sources. For non-point sources, we can use VLBI not only to locate the source, but also to determine its image $I(\vec{p})$, i.e., to determine how the intensity of the radiosignal depends on the point $\vec{p}$ within the source.

In the ideal case of well-synchronized antennas, the phase shift $\widetilde{\varphi}_{i,j}$ between the signals observed by antennas i and j is equal to the phase $\varphi_{i,j}$ of the complex value $F(\vec{b}_{ij})$, where $F(\vec{\omega})$ is a Fourier transform of the desired image $I(\vec{p})$.

In real life, due to synchronization errors $\Delta\varphi_i$, the observed phase $\widetilde{\varphi}_{i,j}$ is different from the desired phase $\varphi_{i,j}$:

$$\widetilde{\varphi}_{i,j} = \varphi_{i,j} + \Delta\varphi_i - \Delta\varphi_j. \tag{9}$$

The synchronization errors are so huge that, based on a single measurement of the phase, we cannot say anything at all about the desired phase $\varphi_{i,j}$ of the image.

In this case:

- the desired parameters x are the phases $\varphi_{i,j}$ that correspond to the actual image;
- the auxiliary parameters y are the synchronization errors $\Delta\varphi_i$.

A known way to eliminate the effect of the auxiliary parameters is to combine the measured phases $\widetilde{\varphi}_{ij}$ between antennas i and j into a combination ("closure phase") $\widetilde{\varphi}_{ij} + \widetilde{\varphi}_{jk} + \widetilde{\varphi}_{ki}$; this combination is called a *closure phase.*

As one can see from the above formula for $\widetilde{\varphi}_{i,j}$, we have:

$$\widetilde{\varphi}_{ij} + \widetilde{\varphi}_{jk} + \widetilde{\varphi}_{ki} = \varphi_{ij} + \varphi_{jk} + \varphi_{ki}, \tag{10}$$

so the dependence on the synchronization errors disappears [5,9,10].

2.4. Image georeferencing

In *image georeferencing*, we are interested in finding the relative orientation of the two geospatial images $I_1(\vec{p})$ and $I_2(\vec{p})$, i.e., we must find the shift, the rotation angle between the images, and the scaling between them.

Overall, given two images, we must find 4 parameters: 2 parameters describing the shift, 1 parameter (angle) describing the rotation, and 1 parameter describing scaling.

This is a difficult problem. Indeed, if all we had to do is determine one single parameter – e.g., the rotation angle – then we could, in principle, determine the value of this parameter as follows: we test all possible angle and finding the rotation angle such that it we rotate the first image by this angle, we get the best match with the second image. Even if we had to try all possible angle with a step of 1 degree, it would be only 360 possible tests – which is quite doable on modern computers, even for large images.

In reality, we must determine 4 parameters. If we take 360 possible values of each parameter, then we need to test $360^4 \approx 10^9$ possible combinations of these parameters – something that is practically impossible.

It is therefore desirable to separate the problem so that we will be able to determine, e.g., rotation angle and scaling separately from determining the shift.

Many signal and image processing techniques involve using the frequency domain, i.e., involve taking the Fourier transforms $F_1(\vec{\omega})$ and $F_2(\vec{\omega})$ of the given images $I_1(\vec{p})$ and $I_2(\vec{p})$.

In Fourier domain, the shift $\vec{p} \to \vec{p} + \vec{a}$ leads to the following transformation: when $I_2(\vec{p}) = I_1(\vec{p} + \vec{a})$, then

$$F_2(\vec{\omega}) = F_1(\vec{\omega}) \cdot \exp(\mathrm{i} \cdot \vec{\omega} \cdot \vec{a}). \tag{11}$$

In order to determine the rotation angle and the scaling, we would like top be able to eliminate the effect of the shift. In other words, here, for each frequency $\vec{\omega}$:

- the desired value x is $F(\vec{\omega})$, and
- the auxiliary parameter is the shift $\vec{a}$.

In this case, the shift-independent combination is easy to describe: it is the absolute value $|F_i(\vec{\omega})|$ of the image's Fourier transform. Indeed, since

$$|\exp(\mathrm{i} \cdot \vec{\omega} \cdot \vec{a})| = 1,$$

the above relation leads to

$$|F_2(\vec{\omega})| = |F_1(\vec{\omega})|. \tag{12}$$

Thus, if we want to determine the rotation angle and the scaling between the two images, it is possible to combine the two referenced images so that the effect of possible shift between these images is minimized [4,8]: namely, we can take the absolute value of the image's Fourier transform.

2.5. Measuring strong electric current

A typical example of *measuring strong electric currents* is measuring the cable current at an aluminum plant. These current are so huge that it is difficult to measure them directly, they are measured by the magnetic fields that they generate.

If we have a single cable, then the magnetic filed generated by the current I flowing through this cable is determined by a simple formula $E = I/r$, where r is the distance between the sensor and the cable's central axis.

In real plants, in addition to the cable in which we are interested, there is often nearby an auxiliary cable that influence the measurement results. It is therefore desirable to somehow eliminate the effect of this auxiliary cable.

We can do that by considering several cables. In this case:

- the desired parameter x is the current flowing through the main cable;
- the experiment settings s are the locations of the sensors; and
- the auxiliary variables y are the location of the auxiliary cable and the current flowing through this auxiliary cable.

The dependence of the observed magnetic filed z on the values of x, s, and y is described by the standard formulas of electrodynamics. These formulas are linear in currents but non-linear in terms of the unknown location of the auxiliary cable.

It turns out it is possible to combine the measurement results at different points so as to eliminate the influence of the current in the auxiliary cable [7].

2.6. Ultrasonic non-destructive testing

In ultrasonic *non-destructive testing*, if we are only interested in the orientation of the fault, we can combine the measurement results in such a way that the effect of location minimizes [6].

3. TOWARDS A GENERAL METHODOLOGY

3.1. Formulation of the general problem

Let us describe the problem in the most general terms.

- We are interested in the parameters x. Let n_x denote the overall number of scalar quantities that form the desired x.
- The measurement results z depend not only on the values x of the desired quantities, but also on the values of the auxiliary quantities y: $z = f(x, s, y)$. Let n_y denote the overall number of scalar quantities that form y, and let n_z denote the overall number of quantities that constitute a single measurement.

We would like to determine x without knowing y precisely.

As we have seen from the above examples, we have two possible situations:

- In situations like multi-spectral astronomical imaging, the values of y are fixed and cannot be varied. We can, however, change the settings s.

- In situations like VLBI astrometry, we cannot change the settings, but we can use different values of y.

Let us describe these situations one by one.

3.2. Variable settings: analysis of the problem

In the first situation, to determine x, we must perform the measurements in several different settings.

After we performed the measurement in N_s different settings $s_1, \ldots, s_{N_s}$, we get N_s measurement results $z_1, \ldots, z_{N_s}$. Based on these results, we must be able to uniquely reconstruct the desired value x. Since we do not know y, we must select N_s in such a way that from N_s measurement results, we will be able to uniquely determine both x and y.

After N_s measurements, we have N_s equations $z_i = f(x, s_i, y)$ to determine the unknown x and y. Since the measurement result may contain several components, these equations are, in general, vector-valued. Each of the values z_i has n_z scalar components, so each of these N_s vector-valued equations can be described as n_z component scalar-valued equations. Overall, we have $N_s \cdot n_z$ scalar equations to determine n_x parameters that form x and n_y parameters that form y.

In general, a system of equation is sufficient to determine the values of all its unknowns if the number of equations is at least as large as the overall number of unknowns. We have $N_s \cdot n_z$ equations to determine $n_x + n_y$ unknowns, so we must select N_s in such a way that $N_s \cdot n_z \geq n_z + n_y$.

As a result, we arrive at the following recommendation:

3.3. Variable settings: general recommendation

In the situation with variable settings, we must perform the measurements in at least $N_s \geq (n_x + n_y)/n_z$ different settings.

3.4. Practical question: how can we actually solve the corresponding system of equations?

We showed that if the above inequality is satisfied then, in principle, we can uniquely determine the desired value x. This theoretical possibility leads us to a practical question: how can we actually determine x?

In general, the dependence $z = f(x, y)$ is non-linear, so we must solve a system of *non-linear* equations, a systems that is, in general, rather difficult to solve.

Most often, however, we know the approximate values $x^{(0)}$ and $y^{(0)}$. In this case, all we have to determine is the differences $\Delta x \stackrel{\text{def}}{=} x - x^{(0)}$ and $\Delta y \stackrel{\text{def}}{=} y - y^{(0)}$ between the actual values x and y and their known approximate values. The approximations are usually good enough, so we can *linearize* the above system of non-linear equations. Namely, we expand the dependence $f(x, y)$ in Taylor series in Δx and Δy and ignore quadratic and higher order terms in this expansion.

As a result, to determine Δx and Δy, we get a much easier-to-solve system of *linear* equations.

3.4. Variable settings: example

Let us illustrate this recommendation on the example of multi-spectral astronomical imaging. In this case, $n_x = 1$ and $n_z = 1$.

In the first approximation, we have only one auxiliary variable, i.e., $n_y = 1$. In this case, the above recommendation means that the number of different settings N_s should be at least as large as $(n_x + n_y)/n_z = (1 + 1)/1 = 2$. Indeed, as we have shown, based on measurements in two different settings, we can uniquely determine the desired value x.

In a more realistic description, we need two auxiliary parameters to describe the cosmic dust, i.e., $n_y = 2$. In this case, the above recommendation means that the number of different settings N_s should be at least as large as $(n_x + n_y)/n_z = (1 + 2)/1 = 3$. Indeed, as we have shown, in this more realistic description, based on measurements in three different settings, we can uniquely determine the desired value x.

3.5. Different values of y: analysis of the problem

In this situation, the general idea is that we measure several (N_x) objects x_i, and we measure each object under several (N_y) circumstances y_j, $j = 1, \ldots, N_y$. Based on the results $z_{i,j} = f(x_i, y_j)$ of these measurements, we must be able to uniquely determine both x_i and y_j.

For example, in the VLBI astrometry example, we observe several sources x_i by using several radiotelescopes y_j. Based on the results of these observations, we determine the coordinates of the objects.

Comment. In principle, we can also determine the baseline vectors $\vec{b}$ and the synchronization errors Δt_i: knowing $\vec{b}$ determines how tectonic plates move relative to each other; knowing Δt_i helps to synchronize the clocks.

Overall, we perform $N_x \cdot N_y$ measurements, so we end up with $N_x \cdot N_y$ vector-valued equations for determining x_i and y_j. Each of these equations has n_z scalar components, so we have $n_z \cdot N_x \cdot N_y$ scalar equations.

Based on these equations, we must determine N_x unknown vectors x_i with n_x components in each of these vectors, and N_y unknown vectors y_j with n_y components in each of these vectors. Overall, we need to determine $N_x \cdot n_x + N_y \cdot n_y$ scalar unknown.

To be able to uniquely determine all the unknowns, the number of equations must be at least as large as the overall number of unknowns. Thus, we arrive at the following recommendation:

3.6. Different values of y: recommendation

In the situation with different values of y, we must select the number of objects N_x and the number of environments N_y in such a way that:

$$n_z \cdot N_x \cdot N_y \geq N_x \cdot n_x + N_y \cdot n_y. \qquad (13)$$

To actually find the values x_i and y_j, we must, in general, solve the corresponding system of non-linear equations.

In many practical cases, we know reasonably good approximations $x_i^{(0)}$ and $y_j^{(0)}$ to x_i and y_j. In such cases, we can linearize this system.

3.7. Different values of y: good news and bad news

Good news is that when N_x and N_y are large enough, the left-hand side of the desired inequality becomes larger than its right-hand side, so this determination is always possible.

Bad news is that the above inequality holds in a generic situation. In many practical situations, we cannot uniquely determine x_i and y_j no matter how many measurements we make. For example, in astrometry, what we observe is, in essence, the angles between the directions to the sources and the directions between the antennas. If we simply rotate the positions of all the sources and all the antennas, the angles will remain the same – thus, observations will remain the same. As a result, we cannot uniquely determine the coordinates of all the sources – we can only determine them modulo rotations.

3.8. How can we describe these non-general situations? These situations can be naturally described in terms of invariance w.r.t. transformation groups

What does it mean that we cannot uniquely determine x_i and y_j no matter how many measurements we make? It means that even if we measure all the objects x for all the values y, we will not be able ti uniquely determine all the values x and y. In other words, in addition to the actual values x and y, it would also be possible to have different values $T_x(x)$ and $T_y(y)$ for objects x and auxiliary quantities y for which the measurement results would be exactly the same as for the actual values, i.e., for which

$$f(x, y) = f(T_x(x), T_y(y)). \tag{14}$$

If this invariance property holds for a pair of functions (T_x, T_y) and for another pairs (S_x, S_y), then it will also hold for their composition and for the inverse functions. In mathematical terms, it means that such pairs of transformations form a *group*.

In this case, we can only determine x modulo transformations from this group. In the astrometry example, this group was the group of all rotations. What we are claiming here is that every time we have a non-uniqueness, it is because there is some transformation group under which the function $f(x, y)$ is invariant.

ACKNOWLEDGMENTS

This work was supported in part by NASA grant NCC5-209, by AFOSR grant F49620-00-1-0365, by NSF grants EAR-0112968, EAR-0225670, and EIA-0321328, and by the Army Research Laboratories grant DATM-05-02-C-0046.

The authors are thankful to the anonymous referees for valuable suggestions.

REFERENCES

[1] D. Bizyaev, A.A. Zasov, S. Kajsin, "Combined color indexes and photometric structures of galaxies NGC 834 and NGC 1134", In: F. Favata, A.A. Kass, A. Wilson, eds., *Proc. of 33 ESLAB Symposium on Star Formation from the Small to the Large Scale*, European Space Agency, 2000.

[2] A. Dravskikh, A.M. Finkelstein, V. Kreinovich, "Astrometric and geodetic applications of VLBI 'arc method'," *Modern Astrometry, Proceedings of the IAU Colloquium No. 48*, Vienna, 1978, 143–153.

[3] A.F. Dravskikh, O.M. Kosheleva, V. Kreinovich, A.M. Finkelstein, "The method of arcs and differential astrometry," *Soviet Astronomy Letters*, **5**(3), 160–162, 1979.

[4] S. Gibson, V. Kreinovich, L. Longpré, B. Penn, S.A. Starks, "Intelligent Mining in Image Databases, With Applications to Satellite Imaging and to Web Search", In: A. Kandel, H. Bunke, M. Last (eds.), *Data Mining and Computational Intelligence*, Springer-Verlag, Berlin, 2001, pp. 309–336.

[5] V. Kreinovich, A. Bernat, O. Kosheleva, A. Finkelstein, "Interval estimates for closure phase and closure amplitude imaging in radio astronomy", *Interval Computations*, No. 2(4), 51–71, 1992.

[6] R. Osegueda, V. Kreinovich, S. Nazarian, E. Roldan, "Detection of cracks at rivet holes in thin plates using Lamb-wave scanning", In: T. Kundu (ed.), *Smart Nondestructive Evaluation and Health Monitoring of Structural and Biological Systems II*, Proc. SPIE, Vol. 5047, San Diego, CA, March 3–5, 2003, pp. 55–66.

[7] V. Kreinovich, J. Pierluissi, M. Koshelev, "A new method of measuring strong currents by their magnetic fields", *Computers & Electrical Engineering*, **23**(2), 121–128, 1997.

[8] B.S. Reddy, B.N. Chatterji, "An FFT-Based Technique for Translation, Rotation, and Scale-Invariant Image Registration," *IEEE Transactions on Image Processing*, **5**(8), 1266–1271, 1996.

[9] A.R. Thompson, J.M. Moran, G.W. Swenson Jr., *Interferometry and Synthesis in Radio Astronomy*, Wiley, New York, 2001.

[10] G.L. Verschuur, K.I. Kellerman, *Galactic and Extragalactic Radio Astronomy*, Springer-Verlag, N.Y., 1988.

[11] H. Xie, N. Hicks, G.R. Keller, H. Haitao, V. Kreinovich, "Implementation, test, and analysis of an automatic image registration algorithm based on FFT and IDL/ENVI", Computers and Geosciences, **29**(8), 1045–1055, 2003.

Authors: Roberto Osegueda, FAST Center for Structural Integrity of Aerospace Structures, University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA, email osegueda@utep.edu.

G. Randy Keller, Scott A. Starks, Roberto Araiza, Dmitry Bizyaev, and Vladik Kreinovich, NASA Pan-American Center for Earth and Environmental Studies (PACES), University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA. Corresponding author V.K., email vladik@cs.utep.edu, phone 1-915-747-6951, fax 1-915-747-5030.