ON GEOMETRY OF RADIO ANTENNA PLACEMENTS

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Introduction to the problem. It is known that if we use an instrument of linear size D to make observations on a wavelength λ, then the smallest size of the observable details is $\approx \lambda/D$. Thus, to get a more detailed picture, we must increase the size D of the instrument. This problem is especially important for observations on the largest possible wavelengths, i.e., for radio observations.

There is a physical limit on the size of a single antenna; to overcome this limit, instead of a *single* antenna, we can use a telescope consisting of *several* antennas. If we use n antennas located at points $x_1, \ldots, x_n$, then we can detect the details of the image that corresponds to the difference vectors $x_i - x_j$, $1 \le i, j \le n$, $i \neq j$.

If two of these differences coincide, then we get fewer information that we potentially could. So, a natural question is: where should we place the antennas so that these difference vectors are different?

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Continuous approximation to the problem. Usually, an multi-antenna telescope consists of many antennas located along a certain curve $x(t)$, $0 \le t \le 1$ (usually, a smooth one). As soon as the curve is selected, it is reasonable to place the antennas on this curve at (approximately) equal distance from each other. Thus, the main problem is to choose the curve.

Since we are assuming that the antennas fill the curve rather densely, it is natural to reformulate the above different-ness requirement as follows: all the difference vectors $x(t) - x(t')$ are different.

Definition. *We say that a smooth planar curve $x(t)$ has unique differences if all the vectors $x(t) - x(t')$ that correspond to different pairs (t, t'), $t \neq t'$, are different.*

This property can be easily translated into the geometric language:

Proposition. *If a smooth planar curve has unique differences, then it is a convex curve.*

Vice versa, one can easily see that a portion of the border curve of a convex body has unique differences, if this portion does not go into parallel tangents (e.g., up to half a circle is still OK, but, say, the whole circle does not have unique differences anymore).

Comment. A circle is indeed one of the known-to-be-efficient antenna placements.

Proof of the Proposition. Since the curve is smooth, at any point, there is a unit tangent vector $\tau(t)$. By definition of a tangent vector, we can approximate it by secant: Namely, for each t, unless we are at the very end of the curve, for each sufficiently small $h > 0$, there exists the first point $f_h(t) > t$ on the curve for which the distance $|x(f_h(t)) - x(t)|$ is equal to h. For this t_h, the difference $x(f_h(t)) - x(t)$ is a secant of length h, and hence, the normalized difference $\tau_h(t) = (x(f_h(t)) - x(t))/h$ is a unit vector in the direction of the secant. As $h \to 0$, we have $f_h(t) \to t$ and hence, the normalized difference $\tau_h(t)$ tends to the unit tangent vector $\tau(t)$.

For sufficiently small h, the curve is close to its tangent, and therefore, the value $\tau_h(t)$ is continuously depending on t.

On a plane, a unit vector can be uniquely characterized by its angle; let us denote the angle that corresponds to the tangent vector $\tau(t)$ by $\theta(t)$, and the unit angle that corresponds to $\tau_h(t)$ by $\theta_h(t)$. From $\tau_h(t) \to \tau(t)$, it follows that $\theta_h(t) \to \theta(t)$.

In terms of the unit tangent vectors $\tau(t)$, convexity means that the corresponding angle $\theta(t)$ is monotonically depending on t (i.e., either non-increasing, or non-decreasing). Let us prove, by reduction to a contradiction, that this dependence is indeed monotonic. Indeed, let us assume it is not. It means that there exist values $t < t'$ and $s < s'$ for which $\theta(t) < \theta(t')$ and $\theta(s) > \theta(s')$. By considering all possible orders of t, t', s, s' and all possible orders of the angles $\theta(t), \theta(t'), \theta(s), \theta(s')$, one can see that there exist the values $t_1 < t_2 < t_3$ for which either $\theta(t_1) < \theta(t_2) > \theta(t_3)$ or $\theta(t_1) > \theta(t_2) < \theta(t_3)$. Without losing generality, let us consider the first case. Since $\theta_h(t) \to_{h\to 0} \theta(t)$, for sufficiently small h, we have $\theta_h(t_1) < \theta_h(t_2) > \theta_h(t_3)$. Therefore, $\max(\theta_h(t_1), \theta_h(t_3)) < \theta(t_2)$. Let us pick a value $\theta \in (\max(\theta_h(t_1), \theta_h(t_3)), \theta_h(t_2))$.

Since the dependence $\theta_h(t)$ on t is continuous, there exist values $t'_1 \in (t_1, t_2)$ and $t'_3 \in (t_2, t_3)$ for which $\theta_h(t'_1) = \theta_h(t'_3) = \theta$. Thus, $t'_1 \neq t'_3$, but $\tau_h(t'_1) = \tau_h(t'_3)$ and therefore, $x(f_h(t'_1)) - x(t'_1) = x(f_h(t'_3)) - x(t'_3)$. So, contrary to our initial assumption, we do not have unique differences. Thus, the possibility that a curve is not smooth leads to a contradiction. Hence, it is smooth. The proposition is proven.

Continuous case: open problem. As an approximation to the number of antennas, we can take the length of the curve X. As a criterion of choosing the best placement, we can take the area of the set formed by the differences $x(t) - x(t')$ (i.e., the area of the Minkowski difference $X - X$). The question is: among all curves of fixed length, for which is this area the largest possible? For a half-circle?

Discrete case: open problem. What if we take into consideration that there are only finitely many points $x_1, \ldots, x_n$? To formulate the exact problem in this case, we can fix an accuracy δ, and consider the set S of all the points that are δ-close to one of the differences $x_i - x_j$. If the set S is connected, then it is natural to consider its area as the desired characteristic of the antenna placements. If the set S is not connected, we consider the area of its largest connected component. The natural question is: where should we place the antennas so as to make this area the largest possible?

Comment. A similar 1D question – how to find integers x_i with the largest connected component covered by difference – leads to interesting placements and open problems (see references below).

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