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The Influence of Multiple Representations on Secondary Students' Understanding of Trigonometric Functions

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THE INFLUENCE OF MULTIPLE REPRESENTATIONS ON SECONDARY STUDENTS’ UNDERSTANDING OF TRIGONOMETRIC FUNCTIONS

MAYRA LIZETH ORTIZ GALARZA
Doctoral Program in Teaching, Learning, and Culture

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Charles Ambler, Ph.D.
Dean of the Graduate School
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by

Mayra Lizeth Ortiz Galarza

2017
Dedicated to my children Yair, Ángel and Valeria, for the love and life lessons they provide each day.

To my husband Edmundo, for his unconditional love and support

And to my parents Carolina and Jesús, for giving me life and teaching me to fight with love and courage for my dreams.

Their support and love made it all possible.
THE INFLUENCE OF MULTIPLE REPRESENTATIONS ON SECONDARY STUDENTS’ UNDERSTANDING OF TRIGONOMETRIC FUNCTIONS

by

MAYRA LIZETH ORTIZ GALARZA, B.S., M.S

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Abstract

Trigonometry is a critical subject in mathematics that both high school and undergraduate students need to learn in order to be prepared for advanced mathematics. Despite the importance of trigonometry in the mathematics curriculum, little is known about best practices for teaching trigonometric functions and what difficulties students face when learning the topic. Using a Grounded Theory approach, this dissertation presents the results of a design study (or teaching experiment) whose purpose was to examine the process by which students constructed the concept of trigonometric functions through multiple representations and how students developed meta-representational competence. The design study involved two stages. In the first stage, initial conditions and elements of the teaching experiment were constructed. In the second stage, proposed conjectures about teaching and learning trigonometric functions were both redefined and redesigned. Qualitative data, including classroom observations and field notes, video recordings of classroom interactions and debriefing sessions, student work (including notebooks and artifacts), student interviews, surveys, and blogs are the focus of analysis. The dissertation presents and analyzes mathematical themes within a framework supporting critical aspects related to learning trigonometric functions through multiple representations and the development of meta-representational competence, that is, the competence to represent trigonometric functions in multiple ways (e.g. ratios, tables, and graphs). Emergent themes connected to the construction and conception of trigonometric functions included students’ conceptions of ratio and proportion, students’ conception of angle, and students’ sense of Cartesian Connectedness. Implications for research and practice include the need to examine how multiple representations stimulate students’ conceptual construction and development of trigonometric functions within the context of inquiry-based instruction.
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Chapter 1: Introduction

1.1 The Importance of Learning Trigonometric Functions

Trigonometry is a critical subject in mathematics that both high school and undergraduate students need to learn in order to be prepared for advanced mathematics (Akkoc, 2008; Karjanto, 2011). Aside from its importance in school mathematics, a wide variety of non-mathematical fields rely on trigonometric functions, even if members of those fields are not aware that those functions are involved. The study of trigonometric functions contributed to advances in the fields of acoustics, architecture, cartography, civil engineering, geophysics, crystallography, electronics, medical imaging and pharmacology. Surveyors and engineers have been using trigonometry in their work for hundreds of years. Common practical, modern applications of trigonometry include its use in satellite navigation, naval and aviation industries, the composition of music, and all types of digital imaging. It has also become critical in the construction of modern buildings and bridges. Because of its significance in the secondary mathematics curriculum, trigonometry serves “as an important precursor to calculus as well as college level courses relating to Newtonian physics, surveying, architecture, and engineering (Weber, 2008, p. 144).” However, trigonometry is considered a highly complex subject for students to learn at the secondary level (Güntekin & Akgün, 2011; Thompson, 2008). Despite the importance of trigonometry in mathematics curricula, little is known about how to best teach trigonometric functions and what difficulties learners face when approaching the topic (Weber, 2005, 2008). The purpose of this study was to investigate the types of understandings about trigonometric functions that emerge through multiple representations.
1.2 Examining the Education Standards for Teaching Trigonometric Functions

In the United States, as well as in México, education standards for teaching and learning trigonometric functions as outlined by both the National Council of Teachers of Mathematics (NCTM) and the Secretaría de Educación Pública (SEP), respectively, are quite diverse comparatively in their scopes and sequences. Despite NCTM curriculum standards for grades 9-12 indicating how mathematics curriculum should include the study of trigonometry, the standards are limited to: 1) applying trigonometry to problem solving situations involving triangles; and 2) exploring periodic real-world phenomena using only the sine and cosine functions. Furthermore, standards calling for function-based reasoning are only applicable to students taking advanced coursework in high school mathematics in preparation for college (NCTM, 2000).

Texas Essential Knowledge and Skills (TEKS), created and endorsed by the Texas Education Agency (TEA, 2012), confines the study of trigonometry to the geometry curriculum for grades 9-12 without making critical connections between geometry and algebra, the mathematics domain in which the concept of function is traditionally developed. Students study trigonometry only within the context of constructing geometric proofs based on a minimal understanding of angle measure and proportionality. Similar to the NCTM standards, TEKS standards for students enrolled in Pre-Calculus (typically an elective course) are quite different. For example, pre-calculus students are expected to study the applications of trigonometric functions and use technology to nurture understanding of how trigonometric functions can model real-world situations. Furthermore, they are expected to “systematically work with multiple representations of functions (TEKS, provision 111.41, 2012).” However, little to no connection
is made between basic triangle trigonometry studied in geometry and the study of trigonometric functions in Pre-Calculus because of the missing connections between geometry and algebra.

Contrary to the U.S. approach, the SEP in Mexico, through the *Estructura Curricular del Bachillerato Tecnológico* integrates geometry and the study of trigonometric functions in the curriculum, which designates formative purposes for each subject separately, including the interpretation and solution of contextualized problems, and utilization of spatial sense. Additionally, the curriculum allows for the analysis and representation of problems through figures, and the solution of problems through geometric and algebraic procedures (see Figure 1.1). The scope and sequence clearly indicates the transition from geometry to a “function-based” approach in algebra (see Appendix A). The curriculum is divided into two main sections: 1) geometric figures; and 2) relations and functions in the triangle. The curriculum is designed to teach geometry and trigonometry simultaneously, making a solid transition to graphing in the coordinate plane, and subsequently presenting applications of trigonometric functions to real-world situations.

However, an important aspect of all three sets of standards (NCTM, TEA, and SEP) is that no significant evidence of effective didactic strategies exists for the teaching and learning of trigonometric functions. There exists no clear learning trajectory upon which teachers can rely to build appropriate pedagogical strategies and allow them to foster student construction and understanding of trigonometric functions. According to Weber (2005), traditional instruction in trigonometry merely promotes the procedural aspect of mathematics, which entails the “routine manipulation of objects (referencing Gray & Tall, 1994, p. 117)” while non-traditional instruction proposes a learning trajectory for the construction of conceptual knowledge by building and using multiple relationships. Within mathematics education, understanding the
obstacles students encounter through their learning processes plays a key aspect in developing a learning trajectory. In addition, it is necessary to identify the learning processes students construct in order to envision potential corridors for a learning trajectory (Clements & Samara, 2009; Simon & Tzur, 2004).

For many high school students, the type of reasoning required for learning trigonometric concepts has not been fully developed (Blackett & Tall, 1991). Weber (2008) explains that students are required to relate diagrams of triangles to numerical relationships as well as manipulate the symbols involved in such relationships (see Figure 1.1).

![Figure 1.1: Defining Trigonometric Ratios for a 3-4-5 Triangle.](image)

Specifically, students are expected to utilize the numerical values involved in a right triangle (e.g. measures of height, base, and hypotenuse) to represent and understand numerical relationships among them (e.g. height over hypotenuse, base over hypotenuse, and height over base). Research indicates that in traditional instruction, trigonometry is typically taught using this ratio method (Kendal & Stacey, 1996). Weber (2008) further states that many exercises students are asked to complete while studying trigonometry rarely require an understanding of trigonometric functions; most simply require “using a ratio conception of the trigonometric operations or applying algebraic techniques (p. 145)” (see Figure 1.2).
Such techniques rely on the perspective of algebra as generalized arithmetic. As a result, Weber (2005) argues that students who receive this standard instruction do not develop a strong understanding of trigonometric functions.

1.3 Theoretical Framework: An Epistemology of Multiple Representations

According to NCTM (2000), students should “create and use representations to organize, record, and communicate mathematical ideas; select, apply and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena (p. 64).” Ge (2012) states, “The use of a variety of representations in a flexible manner has the potential to make the learning of mathematics more meaningful and effective (p. 10).” Delice and Sevimli (2010) explain that these representations “can be used as a flexible tool for solving the same concepts or problems in the case of transitions in themselves or with each other (p. 138).” Dufour-Janvier, Berdnarz and Belanger further indicate the importance of having flexibility of transition between representations for students’ development of conceptual understanding (as cited in Delice & Sevimli, 2010).
Multiple representations can potentially help students not only understand mathematical concepts, but also develop solutions to problems in different ways (Keller & Hirsch, 1998).

The design study conducted was intended to create opportunities for students to develop an understanding and competence with at least three representational forms of a trigonometric function that are part of the foundational and conceptual understanding of all functions in mathematics – representations of ratio and proportion (e.g. similar triangles and tables) and graphical representations of trigonometric functions (see Figure 1.3).

![Figure 1.3: Multiple Representations Promoted.](image)

In order to analyze how students represent trigonometric functions and how they realize the implicit connections and consistency between those representations, a framework developed by Confrey and Smith (1991) was utilized. According to the authors, an epistemology of multiple representations is described as follows:

Knowledge evolves cyclically in relation to our conceptions (mental states) and notations. We may have numerous pairings of these conception-notations. The cognitive operations that allow us to move between and among these pairings constitute knowledge as well as the conceptions and understanding of the individual notations. Herein lies the importance of examining the belief systems of the person engaged in a mathematical pursuit and

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recognizing the impact of available notations in shaping the conceptions and vice versa. (p. 60)

Based on this epistemology, the design study focused on how students develop metarepresentational competence (Boester & Lehrer, 2008), that is, the ability to develop conceptual relationships between and among several different representational forms of a function (i.e. ratio and proportion and graphs). Boester and Lehrer state:

Although traditional accounts of learning mathematics tend to view representational forms as mere adjuncts to learning, we accord them a more central role. In our view, each representational form embodies a different conceptual sense, or resonance among these different senses and associated representations. (p. 212)

The conceptual corridor model for a learning trajectory proposed by Jere Confrey in 2006 (see Figure 1.4) was adopted for this design study. This model entails the design of a learning corridor by considering six elements: 1) students’ prior knowledge, 2) conceptual trajectories, 3) constraints, 4) landmarks, 5) obstacles, and 6) learned ideas. These elements provide meaningful information that serves to establishing the learning trajectory. Students’ prior knowledge was considered in each stage of the experiment. Constrains were defined in advance by the research team. Conceptual trajectories were documented by using students’ learning behavior during the activities. Moreover, landmarks were established by the multiple representations previously described, and obstacles were defined by students’ experiences. Finally, learned ideas were identified from students’ experiences.
Figure 1.4: Conceptual Corridor (Confrey, 2006, p. 146).
Chapter 2: Literature Review

Research on the teaching and learning of trigonometry is sparse (Moore, 2010; Weber, 2005). Despite the prominence and importance of trigonometric functions in secondary mathematics curricula, few researchers have investigated students’ understanding relating to different instructional approaches (Gur, 2009). What seems apparent from the review of existing relevant literature is that students who study trigonometry often lack a rich understanding of the nature of trigonometric functions. Gur (2009) posits that students have misconceptions and learning complexities attributed to trigonometric functions because they learn some pre-concepts “incorrectly or defectively (p. 69).” She further states that one of the main obstacles to effective learning of trigonometric functions is that the mathematical concepts involved in their construction (e.g. ratio and proportion, angle, graphical representations) are non-intuitive to students. Therefore, inaccurate development of prior and new knowledge in certain key mathematical topics result in many misconceptions including: misused data (quantitative measures); misinterpreted symbolic language, and technical and mechanical errors related to both algebra and geometry.

This chapter presents an analysis of the relevant literature focusing on students’ obstacles to learning trigonometric functions and the general concept of function. Following the analysis is a discussion addressing critical obstacles that were apparent throughout the reviewed literature. Following the discussion, the researcher presents an overview of the importance of developing function-based reasoning and the importance of adopting an epistemology of multiple representations when addressing the learning obstacles students encounter when studying trigonometry.
2.1 Teaching and Learning Trigonometric Functions

Trigonometric functions are different from other algebraic functions in that they cannot be easily evaluated or manipulated by carrying out simple arithmetic operations (Weber, 2008; Wongapiwatkul, Laosinchari, Ruenwongsa, & Panijpan, 2011). Students often face the dilemma of how to perform symbolic manipulations on algebraic equations that look very different from ones that they have encountered and manipulated in the past. Weber (2005) states, “Trigonometric functions are operations that cannot be expressed as algebraic formulae involving arithmetical procedures” (p. 91). For example, the referent and process of algebraic manipulation is very different for $y = \sin \Theta$ than $y = mx+b$ or $y= ax^2$. In fact, students need to construct and understand the definitions of sine, cosine, and tangent before they can move on to algebraic equations involving these definitions. Furthermore, in order to construct those definitions, students have to relate diagrams, numerical relationships and symbols as representations of the same mathematical object, specifically, the triangle. The body of research indicates that these required constructions are a primary cause for students’ inability to understand trigonometric functions as functions.

Weber’s research (2005) focuses on students’ understandings of trigonometric functions in the context of two college trigonometry courses. His study compares students from each course. In one group, students received standard instruction while the other group of students was exposed to an experimental instruction approach that proposed “a learning trajectory for how a student can successfully construct an understanding of a mathematical precept” (Weber, 2005, p. 94). According to Gray and Tall (1994), a procept is defined as “the amalgam of three components: a process that produces a mathematical object, and a symbol which is used to represent either process or object” (p. 121). Weber (2005) indicates that proceptual thinking
involves “the ability to think of mathematical operations and objects as procepts (p. 92).” He further states that trigonometric functions are mathematical procepts and explains that thinking about them in this way is essential for understanding them. Weber (2005) points out that when students solve a trigonometric problem, their reasoning should include estimation, mental and physical constructs, anticipating results of a geometric process, and measurement. He emphasizes, “When reasoning about the properties of the numeric values of trigonometric expressions, one often refers back to the geometric processes used to obtain those values” (Weber, 2005, p. 93). During this study, Weber (2005) utilized an Experimental Instruction approach that proposed “a learning trajectory for how a student can successfully construct an understanding of a mathematical precept” (p. 94). Weber (2005) proposes that student understanding of trigonometric functions involves a transition between three stages of reasoning: procedure, process, and procept.

During the Procedure Stage, students learn how to apply an operation or systematic algorithm. The Process Stage involves the student applying the procedure or algorithm multiple times, reflecting upon it as a meaningful method for accomplishing the mathematical objective at hand. Finally, in the Procept Stage, students are able to: a) understand an operation as a process; b) anticipate the result of this process without applying all the steps; and c) reason about the properties the output of the process (Weber, 2005, p. 95). Weber’s (2005) experimental instruction covered the following concepts and procedures (p. 95):

- Computing sine and cosine using the unit circle model;
- Computing tangents using a Cartesian graph;
- Computing sines, cosines, and tangents using right triangles;
- Computing sines, cosines, and tangents using reference angles (based on the unit circle); and
- Graphing the sine, cosine and tangent functions.
During the study, Weber (2005) utilized a series of activities within an “activities class period,” (p. 96), which involved group work and consisted of four parts:

1) Students were shown how to execute a procedure to accomplish a specific trigonometric task;
2) Students applied the previous procedure about five or six times;
3) Students were asked questions to reason about the result of the procedure without executing the procedure; and finally,
4) Students were asked to reason about the procedure itself. (p. 96):

Classroom discussions formed a key component of the experimental instruction. Weber (2005) indicates that during these sessions the instructor gave a lecture in which he stated declarative facts (e.g., \( \sec x \) is the reciprocal of \( \cos x \)) and demonstrated procedural skills (e.g., after learning about computing sines by using right triangles, the instructor demonstrated how the sine of an angle can be computed if an appropriate right triangle is provided) (p. 96).

In addition, Weber (2005) indicates that after the lessons students completed ‘standard homework exercises’ from the course textbook. Figure 2.1 shows some of the activities from the study.
Weber’s (2005) results indicate that students who received the standard instruction were unable to justify the properties of trigonometric functions, and estimate the output of a trigonometric function. During interviews, students in the control group did not demonstrate an understanding of trigonometric expressions as procepts. In addition, Weber (2005) indicates that students did not understand the role that geometric figures play in trigonometric functions and states, “What these students seemed to lack was the ability or inclination to mentally or physically construct geometric objects to help them deal with trigonometric functions” (p. 103).

Weber (2005) concludes that most students in his teaching experiment “were able to approximate the values of basic trigonometric expressions, determine properties of trigonometric functions, and justify why these functions have the properties that they do… they thought of trigonometric functions as meaningful processes” (p. 107). Although Weber (2005) suggests that “students’ successful performance… appeared to be due, in part, to their propensity to reason
about trigonometric functions using a unit circle model” (p. 107), he indicates that not all forms of trigonometry using the unit circle model will lead to successful student learning. In addition, he mentions that research studies (e.g., Kenday & Stacey, 1997) have found that “students who were taught trigonometry using a unit circle model learned less than students who were taught using a right triangle model” Weber, 2005, p. 107).

In subsequent research, Weber (2008) indicates that trigonometric functions are typically presented in one of two ways. On one hand, trigonometric functions are presented as ratios, which help students to determine $\sin A=3/5$, $\cos A=4/5$, and $\tan A=3/4$ (see Figure 2.2).

![Figure 2.2: A Labeled Right Triangle.](image)

He emphasizes that the ratio approach has critical limitations, indicating that a simple ratio understanding of sine with a fixed angle does not permit students to a) approximate the sine of any given angle, b) determine the quadrant in which a trigonometric function is increasing, or c) graph trigonometric functions such as $y = \sin 2x$ (Weber, 2008). He concludes that in order to address these tasks, it is highly critical to develop a function-based understanding of trigonometric operations (Weber, 2008, p. 145).

In traditional instruction, trigonometric operations are usually first taught using the ratio method. Students are asked to label right triangles, compute operations like $\sin \theta$, and then solve word problems. Weber (2008) proposes an alternative approach to teaching trigonometry based
on the idea that “trigonometric operations such as sine can be understood as geometric processes” (p. 146). His approach includes the construction of a unit circle on a Cartesian plane for computing sine (Figure 2.3 shows a typical lesson). Weber (2008) points out the importance of creating opportunities for students to experience and apply more geometric processes. He states that although “students have difficulty imagining the application of a process without the experience of actually applying it” (Weber, 2008, p. 146), if they are given the opportunity to reflect on sine and cosine as geometric processes, they will understand these operations at a much deeper level “regardless of the model used to teach these operations” (p. 147).

In summary, Weber’s research highlights the importance of understanding trigonometric functions through the theoretical approach of procept. This approach emphasizes the importance of students referring to geometric processes to obtain the numeric values of trigonometric expressions and gain richer understanding of a trigonometric function as a function. Although Weber’s research exposed students to both the triangle and unit circle approaches, his studies
showed a strong inclination to rely on and promote the unit circle model. Furthermore, he showed little evidence that students were exposed to the right triangle approach or the construction of trigonometric curves (graphs). There is also no evidence that students worked with tabular (table) representations. Despite Weber’s claim that not all forms of unit circle trigonometry lead to student learning, he strongly suggests students’ successful performance on trigonometric problems resulted from their predisposition to reason about trigonometric functions using the unit circle. In summary, Weber (2008) points out that “regardless of the model used to teach these operations” (p. 147), it is necessary to create opportunities where students can experience and apply trigonometric processes that lead them to develop a function-based understanding at a much deeper level.

In his work, Maor (2013) posits that trigonometric functions may be introduced in several ways: a) as ratios of sides in a right triangle; b) in terms of ‘x’ and ‘y’ coordinates on the unit circle; c) as wrapping functions; or d) as power series of the independent variable (‘x’). However, he indicates that “not all are equally suitable in the classroom” (Maor, 2013, p. 213). Maor (2013) describes that motivation of beginning students learning trigonometry has decreased because of the symbolic language and formalities of inherent in trigonometry being imposed upon them. He indicates that students’ lack of algebraic skills have also affected the level and depth of their trigonometric reasoning. Maor (2013) suggests returning to the historic perspective of interpreting trigonometric functions as projections of the unit circle. This approach reflects “a shift in emphasis from the abstract to the practical” (Maor, 2013, p. 213). Figure 2.4 shows the projections of three trigonometric functions from the unit circle.
Maor (2013) emphasizes that “The New Math has imposed on trigonometry the language and formalities of abstract set theory—certainly not the best way to motivate the beginning student” (p. 213). By proposing a return to interpreting trigonometric functions as projections of the unit circle, Maor provides a context for students to investigate a geometric process in-depth while attending to their learning of angle measure, arcs, and other content related to study of the circle.

Moore (2014) explains that the connection of angle measure to measuring arcs to conceiving the radius as a unit of measure enhance trigonometry learning, basing his argument that the historical development of trigonometric functions occurred within both the triangle and the circle context. While Moore (2014) indicates that it is important for students to construct definitions and meanings in both contexts (triangle and circle), he emphasizes that, “restrictive understandings of topics foundational to trigonometric functions have affected both students and teachers’ development of trigonometric meanings” (p. 103). Moore (2014) presents a brief literature review focusing focuses on three main themes: 1) angle measure, 2) students’ capacity
to move among various representational systems (multiple representations), and 3) modeling and the use of technology.

Moore (2014) asserts that predominant approaches or treatments of angle measure “create an unnecessary divide between circle and triangle contexts” (p. 103), resulting in a divide in the literature on the teaching and learning of trigonometry. He states, “Undeveloped angle measure understandings contribute to teachers’ difficulties with trigonometric functions” (Moore, 2014, p. 103). He further indicates that both preservice and in-service teachers’ reliance upon and predominant use of standard degree measures results in the predominant use of right triangle contexts when teaching trigonometry (Moore, 2014).

Regarding students’ capacity to move among various representations, Moore relies on Brown’s work (2005) that posits the teaching of trigonometric functions, is typically limited to the triangle context, and that numbers are not represented as ratios. Furthermore, Moore (citing Brown, 2005) also explains that conceptualizing angles in terms of rotations could enhance students’ connecting angles to the unit circle and, subsequently, to graphs of trigonometric functions, stressing the importance of teaching coordinates in the Cartesian plane. Concurring with Weber (2005), Moore (2014) stressed that, “regardless of context, it is necessary that students construct the geometric objects of trigonometry as tools for reasoning” (p. 104).

Moore (2014) explains that another topic related to understanding trigonometric functions is “modeling and the use of technology” (p. 104). He believes that it is important for students to be able to move among different representations and that a modeling approach may support students’ representational fluency. Despite this claim, Moore (2014) explains that students face many difficulties when using modeling approaches, and that “much is to be learned about how to support and draw on students thinking in these areas” (p. 104).
In his study, Moore (2014) investigates “how a student’s angle measure meaning influenced his construction of the sine function” (p. 105). He further conjectures that a student’s lack of understanding angle measure complicates both the student’s and the teacher’s understanding of trigonometric functions through covariational reasoning, which he claims is not fully addressed in U.S. mathematics curricula (Moore, 2014, p. 106). Citing Carlson, Jacobs, Coe, Larsen, and (2002), Moore indicates that covariational reasoning characterizes the “cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). Moore (2014) suggests that such reasoning supports students’ learning the concept of function and that, regrettably, students are more familiar with the acronym SOHCAHTOA (where SOH stands for Sine is equals Opposite over Hypotenuse; CAH stands for Cosine is equal to Adjacent over Hypotenuse; and TOA stands for Tangent is equal to Opposite over Adjacent) to develop definitions of trigonometric functions. As a result, even high-performing calculus students struggle when trying to connect calculated ratios to graphs of trigonometric functions (Moore, 2014).

For his study, Moore adopted a framework of five mental actions (Carlson et al., 2002) required of students when studying trigonometric functions. This framework also describes the verbal behaviors students should exhibit for each mental action (see Figure 2.5).
For the experiment, Moore approached angle measure by using circles and examining the multiplicative relationships between arcs and radius. In one of the activities (the fan), Moore asked students to change the radius of the fan and reconsider the resulting graph. When evaluating the sine function, only specific input values such as 0, π/2, π, 3π/2 were considered. Moore (2014) indicates that he “avoided an emphasis on specified numerical pairs of values, as [he] intended that the students develop the capacity to reason about indeterminate values when constructing the covariational relationships defined by the sine and cosine functions” (p. 120). This explains his intention to relate the sine function as a measure relative to the radius without using specific values.

To illustrate connections between covariational reasoning and the sine function, Moore (2014) utilizes one activity that models circular motion and represents the covariational
relationship between an angle measure and a directed length. The activity limits the increasing angle measure to the first quarter of a revolution (or Quadrant I in the Cartesian plane). Moore (2014) explains, “A similar line of reasoning can be used for the other three quarters of a revolution” (p. 108). To support his approach, Moore (2014) indicates that sometimes students “may conceive of angle measures as labels of geometric objects without an associated scheme for the structure of the measurement unit” (p. 109). He also explains that students could understand angle measure in terms of “a measurement process that defines a multiplicative relationship... between a subtended arc and a circle’s radius” (Moore, 2014, p. 109).

Another activity that formed part of the study was the Circle problem, which required participants to determine the ordered pair for a given position of a right triangle in the unit circle. Moore (2014) explains this task also required using the radius as a unit of measure. Figure 2.6 shows the activity and one participant’s response.

![Figure 2.6: Circle Activity and Participant’s Solution.](image)

Moore’s participants included three undergraduate students enrolled in a pre-calculus course who were not yet exposed to trigonometric functions or the unit circle, but had background knowledge from prior courses in high school algebra, pre-calculus, and calculus.
Participants were asked to share and discuss their thinking and to generate conversations. Moore facilitated discussions by posing questions and situations, and the study included ongoing and retrospective analysis. In addition, Moore (2014) “did not assume that [the participants] would engage in covariational reasoning” (p. 114).

Results of this study, which reported on only one of the three participants, indicate that the participant held “restrictive angle measure meanings” (Moore, 2014, p. 115). Moore (2014) indicates that although the participant conceived some angle measures as labels for geometric objects, he did not “quantify angle measure in terms of a process that involved systematically measuring and comparing attributes such as arcs and a circle’s circumference” (p. 115). Moore (2014) also indicates that after the first two sessions, the participant:

1) Understood angle measure regardless of unit.

2) Conceived radian measure as a multiplicative relationship.

3) Developed a preference for radian angle measures.

Furthermore, the participant was able to examine covariational relationships among the different activities.

Moore (2014) indicates that during the previous activity the participant demonstrated his ability to transition between units of measure, and to evaluate the sine and cosine function in the context of circles and explain the relationship between a vertical distance and a transversed arc length. During the activity on circular motion, the participant also discussed his reflections on directional change, amounts of change, and rate of change, relying on the Cartesian plane. Moore (2014) indicates that after being engaged in covariational reasoning activities the participant created a formula that corresponded to a previous diagram. However, inconsistencies
between the graph and the formula point out the implications about position and tables to evaluate rates of change.

Excerpts from the participant’s interview indicate that although he developed an incorrect equation for the sine function, his reasoning in transitioning from one context to the other (i.e. from triangle to circle) was appropriate (Moore, 2014, p. 127). During this interview, the participant demonstrated his multiplicative reasoning and ability to interpret ratios and proportions. The participant also indicates that he was originally taught with triangles, using circles helped him to realize the connection between the hypotenuse and the radius.

Although Moore (2014) concludes that the participant “constructed a system of trigonometric meanings involving angle measure, quantity, measurement, and covariation that incorporated both circle and triangle contexts” (p. 130), in the experiment, important implications resulting from the participant’s mathematics background should be considered. First, while Moore (2014) indicates the participant “developed a meaning for the sine function that entailed coordinating various mental actions associated with covariational reasoning” (p. 130), he does not clarify if this development can be attributed, at least in part, to the participant’s mathematical background in pre-calculus and calculus. Second, although the participant was able to describe the curvature of a graph “in terms of shape and continuous motion” (Moore, 2014, p. 130), Moore concedes that the participant did not necessarily “conceive graphs as emergent representations of covarying quantities” (p. 130). Third, Moore questions whether the participant’s prior knowledge in angle measure supported his construction of the sine function. Fourth, Moore acknowledges that the participant’s constructed meanings could be the result of certain teaching moments during the sessions. Finally, Moore (2014) indicates that the participant applied multiplicative reasoning (ratio and proportion) intuitively to understand how
circle and triangle contexts relate trigonometric functions to angle measure and comparison of lengths (p. 132).

In summary, Moore’s (2014) research suggests that further research about exploring a covariational approach to the sine function with a particular focus on angle measure should be considered (p. 134), along with further investigations on students understanding of measurement and estimation. Furthermore, he states that “working with students being introduced to trigonometry for the first time forms an especially important area for future research” (p. 135).

By relying on theoretical framework developed by Moschkovich, Schoenfeld, and Arcabi (1993), Marchi (2012) explores students’ understandings of the sine function through multiple representations, by using two different perspectives: 1) the process perspective, and 2) the object perspective. In the process perspective, a function is viewed as an ‘x’ value linked to a ‘y’ value, i.e. the input and the output of the function. This relationship “can be seen as a transformation of quantities” (Moschkovich et al., 1993, p. 72). In the object perspective, a function is viewed as an entity or object (e.g. a graph or a collection of ordered pairs) upon which to be operated (Marchi, 2012, p.3).

Marchi’s (2012) research also addresses the importance of leaning trigonometric functions through multiple representations. Marchi (2012) indicates that when most high school students are exposed to learning linear, quadratic, polynomial, and exponential functions, they are introduced to several representations for each including algebraic, graphical, and equations. However, “trigonometric functions share additional unique representations” (p. 1). For instance, students are introduced to the unit circle and right triangles representations to make sense of such functions.
While Marchi (2012) highlights the importance of understanding how students represent functions in mathematics and how they make connections among those representations, he also emphasizes the importance of students knowing and understanding the conceptual basis of a representation. He affirms that “utilizing both the process perspective and the object perspective and knowing when to use each is essential to learning” (Marchi, 2012, p. 6).

Marchi (2012) indicates that students’ inability to create accurate representations is a major obstacle in the learning process. That inability is further compounded by many mathematics teachers’ assumption that students can build connections among correct representations, once they are constructed. In the case of graphical representations, difficulties arise when students fail to visualize a graph as a relationship between points that constitute the graph. Marchi (2012) explains that students often visualize graphs as objects and are unable to see any connection with the Cartesian plane, which he terms the “Cartesian Connection” (p. 10).

Citing Brown (2005), Moschkovich and colleagues (1993) present the following definition or description of the Cartesian connection:

a. Connection A: The point \((x_0, y_0)\) is on the graph of the function \(y= f(x)\) if and only if the point satisfies the equation, that is \(y_0 = f(x_0)\).

b. Connection B: In the Cartesian plane, specific algebraic expressions have graphical identities. For example, \((y_2 - y_1)\) is a directed line segment with both direction and magnitude specified by mathematical convention.

Marchi (2012) highlights students’ tendency to be more comfortable working with algebraic representations rather than graphs and how many students “fail to recognize a graphical representation might be suitable for solving problems” (p. 11). He states, “One reason why
Cartesian Connections may give students so much difficulty is the absence of opportunities for them to fully connect graphs to equations” (Marchi, 2012, p. 12).

Marchi (2012) conducted his study in a nontraditional school context with six participants who were selected based on their mathematical ability\(^1\). Participants were enrolled in one of the three following courses: Algebra II and Trigonometry, Algebra III and Trigonometry or Calculus. Data collection took place during two one-on-one interviews, focused on investigating “how students understood and represented sine and what connections students had among their representations” (Marchi 2012), p. 43, namely, an algebraic equation, a right triangle, a graph (sine wave), and the unit circle. Results of the study indicate that students’ understanding of sine, one that relied on a right triangle representation, affected their ability to see it as a function. Furthermore, students could not connect \(y = \sin (x)\) with its graphical representation. Marchi (2012) explains that the problem “appeared to stem from the inability to see inputs and outputs in the triangle representation” (p. 210).

Another major problem that affected participants’ understandings was a reliance on memorizing much of the information for representations. Marchi (2012) states that students relying on memory developed a weak conceptual understanding of how to create representations and how to find connections among those representations. For instance, Marchi (2012) indicates that although some students memorized a coordinate and could say that \(\sin (45^\circ) = \sqrt{2}/2\) (by using the unit circle), they were unable to make the same conclusion on the graph for \(y = \sin(x)\). Moreover, other students “incorrectly recalled information and made false connections, particularly when trying to connect the graph for \(y = \sin(x)\) with the unit circle” (Marchi, 2012, p. 212).

\(^1\) The author did not specify what constituted high or low ability
Marchi’s (2012) research also concluded that students could not connect the equation to its graph even though they could make a connection between the graph and the unit circle. Furthermore, he indicates that student development of the Cartesian Connection seemed to be restricted (Marchi, 2012). His results reveal that for easier examples of sine graphs, students rely on the object perspective and “switch to a combination of object and process perspectives when the graphs are more complicated” (Marchi, 2012, p. 216). He states that students readily connected a point on the graph to being an angle and the y-coordinate on the unit circle, but there was little evidence that they also understood it to mean (x, sin(x)), other than in the context where the student is asked to find the sine of a specific angles measure. (Marchi, 2012, p. 217)

Demir’s (2012) work investigates students’ concept development and understanding of the sine and cosine functions. His study addresses an understanding model of trigonometry comprising three different trigonometric contexts. He also presents a new instructional approach of trigonometric functions that emphasizes the importance of connections among the contexts of: a) triangle trigonometry, based on ratio definitions derived from right triangles; b) unit circle trigonometry, based on angle rotation; and c) function graphs, based on the domain of real numbers (see Figure 2.7).
According to Demir (2012), separation of the three contexts based on the assumption that “conceptual development of each occurs in a linear order from the first context to the last one… promotes incomplete and disconnected understandings” (p. 1). His work is based on “a conceptual approach analysis of mathematical ideas within and among three context of trigonometry” (Demir, 2012, p. 35). Figure 2.8 shows Demir’s (2012) model of trigonometric understanding.
Demir’s (2012) proposed model describes the contexts of Triangle Trigonometry (TT), Unit Circle Trigonometry (UCT), and Trigonometric Functions Graphs (TFG). It suggests the connections between those representations and the desired trigonometric understanding of students (U). The model represents the understanding of different aspects within the three different contexts (line segments 1, 2, 3). Line segments 4 and 5 represent a deeper level of understanding of the connections among the contexts (p. 36). According to Demir (2012), the triangle context is considered “students’ prior knowledge” (p. 38). In addition, the unit circle context served to “facilitate the transition from trigonometric ratios in right triangles to the trigonometric functions of real numbers” Demir, 2012, (p. 38).

Demir (2012) proposes an alternative sequence or learning trajectory to traditional methods of teaching trigonometry. Demir’s (2012) proposed trajectory is described in terms of seven stages: 1) the unit square, 2) transition to the unit circle, 3) naming the graph as sine, 4) closing the gap with radians, 5) integrating triangle trigonometry to unit circle trigonometry, 6) introduction to the cosine function, and 7) elaborating on three contexts of sine and cosine. Although Demir (2012) claims his alternative learning trajectory can benefit students with the development and understanding of the sine and cosine functions, his results show that students are only making slight connections between the three contexts (triangles, unit circle and graphs).

Demir (2012) states, “[c]onnections between the graphs of the trigonometric functions and the unit circle were among the main issues in both lesson sequence and the interviews” (p. 104). Furthermore, Demir (2012) indicates that students’ difficulties with angle measure relate to a lack of understanding of both negative angles and angles larger than 360°, due to their lack of experience in constructing angles in the unit circle (p. 20). For example, he reports, “[a]lthough
many students (14 out of 24) were able to show a trigonometric relationship for specific angles, an overwhelming number of students (18 out of 24) were not able to prove a trigonometric relationship with a variable” (Demir, 2012, p. 113). Moreover, “[m]ost students could not calculate trigonometric values of well-known angles like 30°, 60°, 45° because they did not know or did not remember the side lengths of a right triangle with these angles” (Demir, 2012, p. 114).

Moreover, the results illustrate students’ undeveloped function-based reasoning, based on their tendency to navigate only between the triangle and unit circle contexts. Demir (2012) states that

[st]udents developed a good level of understanding regarding the aspects in the context of unit circle, but it was found that when they were asked to use the coordinate definitions in different kinds of tasks which were not familiar to them, they had problems… Although the students learnt well the coordinate definitions of sine and cosine, they could not use them in some different kinds of tasks. (p. 112)

Although Demir’s (2012) research claims that “the designed learning trajectory helped students to develop a good understanding of trigonometric functions, their properties, and graphs based on connections between the unit circle and their graphs” (p. 119), he admits that “[r]eal difficulty was about assessing students’ understanding of trigonometric functions” (p. 121) (see Figure 2.9).
2.2 Developing Function-Based Reasoning through Mathematical Modeling

Kaput (1989) argues, “[t]here are no absolute meanings for the mathematical word *function*, but rather a whole web of meanings woven out of the many physical and mental representations of functions and correspondences among representations” (p. 168). As stated by Weber (2005) the development of strong function-based reasoning requires teaching and learning certain skills that allow students not only to move from one representation to another, but also to understand the connections among those representations. Likewise, Even (1998) explains that conceptual understanding of functions entails the ability to “identify and represent the same thing in different representations” (p. 105). A sound understanding of function, therefore, should include the ability to work with the different representations confidently and realize that there is consistency among them when using more than one representation to answer a given questions. According to Kaput (1989), “the cognitive linking of representations creates a whole that is more than the sum of its parts” (p. 179).
Research points out the importance of organizing multiple representations into sequences in order to allow students to understand the different facets of a mathematical idea (Cuoco, 2001; Ge, 2012; Heinze, Star & Verschaffel, 2009; NCTM, 2000; Stylianou, 2010). Ge (2012) states “The sequences make it easy to show similarities, differences and other relationships among multiple representations” (p. 11). Herman (2007) found that students tend to identify a function with only one representation, namely the algebraic formula, and they often feel this is the only representation they can use to solve a problem. Not understanding the graphical representation of the formula causes students to think symbolically more often than visually and graphically (Knuth, 2000). Since graphs present visual representations of relationships among points, students may be unable to see the graph as a series of points and only as an object (Santos-Trigo, 2002; Knuth, 2000). In essence, they are unable to have a robust understanding of the mathematical space known as the Cartesian plane, which also involves an understanding of position and directionality. If these connections are not made possible, then students will not be able to extract vital information from the graphical representation of a function. Students may go as far as to treat algebraic and graphical representations as being independent entities (Moschkovich, Schoenfield, & Arcavi 1993; Van Dyke & White, 2004).

However, diverse studies suggest that the use of multiple representations can have a counterproductive effect if students do not understand the conventions that regulate the way one representation is used before moving on to another representation (Ainsworth, Bibby & Wood, 1998; Ge, 2012; Gruber, Graf, Mandl, Renkl & Stark, 1995; Nistal, Dooren, Clarebout, Elen & Verschaffel, 2009; Yerushalmy, 1991). As a result, one reason why linking graphs to equations may give students so much difficulty is the absence of opportunities for them to connect graphs to equations. Students spend a great deal of time learning how to graph an equation, but not how
to get an equation from a graph. Hirsch, Weindhold, and Nichols (1991) stated that writing the equation from a graph is a significant step in developing a student’s graphical sense. However, it seems apparent that students need to do more than make a connection between algebraic and graphical representations of a function by moving from the former to the latter. Students must be given opportunities to make connections between these representations in both directions in order to fully grasp the connections among them.

2.2.1 Mathematical Modeling and Multiple Representations

According to Kaput (1998), multiple representations are described as the concertation of abstract concepts and symbols in the real world through a modeling process. Since mathematics is "inherently representational in its intentions and methods’ (Kaput, 1989, p. 169), representations and symbols systems are fundamental to mathematics as a discipline. According to Vergnaud (1997), mathematical concepts are defined by three variables:

1) The situation that makes the concepts useful and meaningful

2) The operation that can be used to deal with the situation

3) The set of symbolic, linguistic, and graphic representation that can be used to represent situations and procedures.

Carrejo and Marshall (2007) explain that a robust understanding of a function “may involve a complementarity between representations” (p. 53). As they explain, the construction and use of these representations (equation, graph and data table) while immersed in a mathematical modeling process can provide students opportunities to make connections between representing, thereby developing a deeper understanding of the problem or phenomenon being modeled.
In their study about conceptual change Carrejo and Reinhartz (2014) suggest a series of modeling activities (i.e. graphing motion using multiple representations) to investigate students’ thinking about motion (which entails the understanding of rates of change, a function-based concept). Carrejo and Reinhartz (2014) defined the criteria to assess students reasoning. Through a rubric, they point out the role of multiple representations and the relationships among them for reasoning development. The authors state that in order to measure students reasoning development about motion, it is necessary to measure: 1) the extent in which students can plot measures of distance and time, graph them, and demonstrate an understanding of the coordinate axes; and 2) the degree in which a student could build the relationship between the slope of the distance-time graph to the ratio of distance over time (Carrejo & Reinhartz, 2014, p. 18).

Blanton and Kaput (1994) explain that development of function-based reasoning (functional thinking) occurs when students are able to: “(1) use representational forms such as t-charts, (2) articulate and symbolize patterns, from natural language descriptions of additive relationships to symbolic representations of multiplicative relationships, and (3) account for co-varying quantities” (p. 139). In their findings, Blanton and Kaput (1994) explain that by using t-charts students were able to notice patterns and how numerical values varied. Students were able to describe those patterns by using both additive and multiplicative relationships. They also indicate that by using data, graphs, and charts students in early grades (PK-2) began to think about how quantities co-varied.

Another important factor in the development of a function-based reasoning through modeling is the development of spatial sense. Lehrer, Jenkins, and Osana (1998) explain the implications of this particular reasoning in students’ ability to “recognize and measure planar angles to operations on the plane, so that angle measure is enabled by the mental equivalent of a
Cartesian coordinate system” (p. 146). They also explain that due to students’ conceptions of angles include multiple models, it is necessary to examine such models of angles in both dynamic-rotational and static-shape contexts (Lehrer et al., p. 146). In their research, Lehrer et al. (1998) investigated students’ development of similarity in terms of 1) similar orientation, 2) length of segments, and 3) distance between legs of an angle, 4) notion of similar sweeps or turns, 5) overall appearance, and 6) conventional angle measure (p. 148).

2.4 Discussion

The research presented in this literature review has provided theories of, and approaches to, the study of trigonometric functions. However, a viable approach to bridging the domains of right triangle trigonometry to trigonometric functions is not apparent. Considering what the literature review indicates about the possible development of a unit-circle model to foster function-based reasoning, the researcher claims that adopting an epistemology of multiple representations, developing meta-representational competence, and the role of modeling as inquiry are critically relevant issues in the teaching and learning of trigonometric functions. Discussion of these issues helps provide a rationale for the present study of students learning trigonometric functions through multiple representations to develop richer conceptual understanding of trigonometry. The following section presents a discussion on important themes that emerged from the literature.

2.4.1 Angle Measure

Multiple definitions of angles as well as methods of measuring angles have challenged students while learning trigonometry. Henderson and Taimina (2000) point out three definitions of angle that reflect different perspectives: a) angle as movement, b) angle as measure, and c) angle as geometric shape. These varied definitions of angle have an impact on the teaching and learning process. For example, understanding angle as movement could have implications
Learning trigonometric functions through the unit circle model while understanding angle as a measure or as a geometric shape could greatly benefit understanding the more-static right-triangle trigonometry. Maor (2013) describes angle as a geometric entity, claiming that the definition reveals a complementarity when trying to understand the concept because “[i]t describes both the qualitative idea of ‘separation’ between two intersecting lines, and the numerical value of this separation—the measure of the angle” (p. 15). Another important question is whether children perceive an angle as static or dynamic, for example, as a bend or a rotation.

The complementarity is important to examine because it has pedagogical implications and suggests that epistemological tensions between conceptions of angle may be involved in developing an understanding of angle. Although multiple definitions of angles are provided by research literature, little is known about their impact on the learning of trigonometric functions and more research is needed. An emphasis on developing physical geometric processes to compute sines, cosines, and tangents could potentially aid students in their construction of knowledge about angle and its role in understanding trigonometric functions.

In regards of angle measure, Moore (2014) states that it influences the construction of the sine function meaning. Besides, Moore (2014) indicates that in order to being able to make sense of relationships among representations, students need to develop strong understandings of angle measure and covariation. Moore (2014) also indicates that most students conceive angle as labels of geometric objects and struggle to understand them as a multiplicative relationship. By his part, Marchi (2012) explains that students struggle when asked to find the sine of a specific angle measure. Weber (2005) points out the use of reference angles to compute sines, cosines and tangents on the unit circle. Moreover, Demir (2012) explains that some difficulties encountered by students when working with angle measure refer to dealing with negative angles and angles
larger than 360°. Demir (2012) points out that students’ fragile concept of rotation of angles results on difficulties when drawing rotations of angles in the unit circle.

### 2.4.2 Geometric Processes

In addition to NCTM’s recommendations for integrating graphing during the study of trigonometry, multiple studies emphasize the importance of going back to geometric processes (Demir, 2012; Maor, 2013; Marchi, 2012; Moore, 2014; NCTM, 2000; Weber, 2005). As pointed out by Weber (2005) geometric process are critical to understand trigonometric functions. Weber (2005) suggests an alternative approach to teaching trigonometry based on the premise that “trigonometric operations such as sine can be understood as geometric processes” (p. 146). However, important implications regarding difficulties and obstacles need to be considered. For instance, Marchi (2012) indicates that when working with graphical representations, students often visualize them as objects, and not as relationships that can be connected to the Cartesian plane. In addition, Moore (2014) explains that students struggle with the geometric objects associated with trigonometric functions, especially with angle measure. By his part, Maor (2013) suggests to go back to the notion of geometric process by teaching and interpreting trigonometric functions as projections of the unit circle.

### 2.4.3 Cartesian Connections

Another important implication that emerged from the literature was the role of the Cartesian connection for the conceptualization of trigonometric functions as functions. Both Marchi (2012) and Demir (2012) indicate that most students cannot visualize graphical representations and connect them fully to the Cartesian plane. Despite the emphasis on using multiple representations for teaching and learning of trigonometric functions, the existing
literature does not provide evidence of the use of the table representation. For instance, Marchi (2012) explores students’ understandings of the sine function by using multiple representations, relying solely on the algebraic, graphical and formula (equation) representations. Weber (2005) focused on the unit circle approach without including the use of tables to evaluate the functions. Moore (2014) indicates that his participants compared changes in vertical distances, yet they did not utilize a table to aid them in their analysis of rates of change. Finally, Demir’s (2012) proposed contexts in which trigonometric functions need to be learned (namely, the triangle, circle and function graph contexts) conspicuously omits the table representation.

2.4.3 Conclusion

The literature shows that the teaching of trigonometry has focused on procedural techniques that have affected not only students’ opportunities to connect and apply the procedures they have learned (Marchi, 2012; Weber, 2008), but also their ability to construct geometric objects that help them learn trigonometric functions (Moore, 2014; Weber, 2005). Scholars in this review utilized multiple methodologies, including those identified as teaching experiments, to investigate students’ difficulties and obstacles when learning trigonometry. However, these methodologies do not necessarily fit the design study framework.

Designs studies provide “an extended investigation of educational interactions provoked by use of a carefully sequenced and typically novel set of designed curricular tasks studying how some conceptual field, or set of proficiencies and interests, are learned through interactions among learners with guidance” (Confrey, 2006, p. 135). Design studies are used to “investigate students’ mathematical development and to design more effective learning environments” (Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008, p. 219). In addition, design studies provide opportunities for teachers to be engaged in the development of artifacts “that reveal
aspects of their own thinking” (Zawojewski et al., 2008, p. 221). For these reasons, the current study utilizes a design study methodology to capture classroom interactions and determine core themes that emerge from the practice of constructing and learning trigonometric functions.

Chapter 3: Methodology

This chapter outlines the setting, participants, design, and procedures for this study. The study involved twenty-three secondary students enrolled in a Geometry class in a local high school, and one mathematics teacher. The researcher used a grounded theory framework (Charmaz, 2010; Cobb, Stephan, McClain, & Gravemeijer, 2001) to analyze student construction and learning of trigonometric functions. Methods of design, data collection, and analysis are discussed in detail. An explanation of the design study approach in classroom settings is highlighted in the data analysis section.

3.1 Setting

This study took place in a public school in a southwestern city on the US-Mexico border over the course of five weeks. According to the National Center for Education Statistics (2014), the school is considered a Title I school. The U.S. Department of Education (2016) defines as Title I those educational agencies and schools with “high numbers or high percentages of children from low-income families.” The school population is approximately 2,071, with Hispanics representing 96.18% of the population, while 80.98% of the population are considered economically disadvantaged (citation). Approximately 14% of the students did not meet the math standards on the Texas Assessment of Knowledge and Skills (TAKS, 2012) test.
3.2 Participants

Twenty-three students and their classroom teacher took part in this study. Of the twenty-three students, sixteen were female (69.6%) and seven were male (30.4%). Nine students (39.1%) reported an age of 14, thirteen (56.5%) reported an age of 15, and one student (4.3%) was 17 years old at the time of the study. Twenty-one participants (91.3%) identified themselves as Hispanic/Latino. Nineteen students (82.6%) and twenty-two students (95.7%) attended elementary and middle school respectively, in the same city as where this study took place. Regarding language, fifteen (65.2%) out of the twenty-three students considered English as their native language (L1). Three students (13%) reported Spanish as their L1, and five (21.7%) considered both English and Spanish as their native language. Demographics of the classroom population are shown in Table 3.1.

<table>
<thead>
<tr>
<th>Category</th>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gender</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>7</td>
<td>30.4</td>
</tr>
<tr>
<td>Female</td>
<td>16</td>
<td>69.6</td>
</tr>
<tr>
<td><strong>Age</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 years old</td>
<td>9</td>
<td>39.1</td>
</tr>
<tr>
<td>15 years old</td>
<td>13</td>
<td>56.5</td>
</tr>
<tr>
<td>17 years old</td>
<td>1</td>
<td>4.3</td>
</tr>
<tr>
<td><strong>Racial/Ethnic identity</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hispanic/Latino</td>
<td>21</td>
<td>91.3</td>
</tr>
<tr>
<td>White/Caucasian</td>
<td>1</td>
<td>4.3</td>
</tr>
<tr>
<td>Other</td>
<td>1</td>
<td>4.3</td>
</tr>
<tr>
<td><strong>Attended elementary school</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In same city of the study</td>
<td>19</td>
<td>82.6</td>
</tr>
<tr>
<td>On both sides of the border</td>
<td>1</td>
<td>4.3</td>
</tr>
<tr>
<td>Elsewhere</td>
<td>3</td>
<td>13.0</td>
</tr>
<tr>
<td><strong>Attended middle school in</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In same city of the study</td>
<td>22</td>
<td>95.7</td>
</tr>
<tr>
<td>Elsewhere</td>
<td>1</td>
<td>4.3</td>
</tr>
<tr>
<td><strong>Native language</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>English</td>
<td>15</td>
<td>65.2</td>
</tr>
<tr>
<td>Spanish</td>
<td>3</td>
<td>13.0</td>
</tr>
<tr>
<td>Both English and Spanish</td>
<td>5</td>
<td>21.7</td>
</tr>
<tr>
<td><strong>Second language</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>English</td>
<td>3</td>
<td>13.0</td>
</tr>
<tr>
<td>Spanish</td>
<td>12</td>
<td>52.2</td>
</tr>
<tr>
<td>Both English and Spanish</td>
<td>3</td>
<td>13.0</td>
</tr>
<tr>
<td>Not applicable</td>
<td>5</td>
<td>21.7</td>
</tr>
</tbody>
</table>
The classroom teacher identified himself as Hispanic/Latino and stated that he was born and raised in the same city in which this study took place. He reported English as his native language. The teacher holds a Bachelor’s degree in Accounting, a certification to teach mathematics in grades 8 through 12, and a Master’s degree in Mathematics Education. He has taught a wide array of mathematics courses including Algebra I, Algebra II, and Geometry for more than ten years.

3.3 Research Design

A design study (Confrey & Lachance, 2000) approach, also known as a teaching experiment, was utilized in order to answer the research questions. According to Cobb, Confrey, DiSessa, Lehrer and Schauble (2003) the purpose of design experimentation is “to develop a class of theories about both the process of learning and the means that are designed to support that learning” (p. 10). Design studies are often considered test-beds for innovation that contribute to the investigation of possibilities for educational development. The theoretical intent of design studies is the identification of successive patterns in student thinking. They specify conjectured starting points, elements of a trajectory, and prospective endpoints. By testing and revising conjectures and through the ongoing analysis of the learning environment and students’ reasoning, design studies improve the initial design (Cobb, et.al, 2003). As explicated by Cobb and colleagues (2003), design experiments are conducted in diverse settings that vary in both type and scope (i.e. One-on-one experiments, classroom experiments, pre-service teacher development experiment, in-service teacher experiment, school and school district restructuring experiment) (p. 9). The present study was conducted as a classroom experiment. As recommended by Cobb and colleagues (2003) a collaborative research team conformed by the
principal investigator, university professor/mentor, and classroom teacher, with the “expertise to accomplish the functions associated with developing an initial design, conducting the experiment, and carrying out a retrospective analysis” (p. 12) worked in this investigation.

Through the classroom experiment approach, the researcher explored the possibilities for educational improvement in the teaching and learning of trigonometric functions. The researcher established diverse conceptual corridors in order to explicate students’ learning trajectories.

This study encompassed two stages based on establishing an Iterative Design Cycle (IDC) (Cobb, et.al, 2003; Zawojewski, et al., 2008; Bowman, & Lesh, 2008) (see Figure 3.1).

**Figure 3.1: Iterative Design Cycle for a Design Study**

In the first stage, a planning stage, the researcher, under the mentorship of a university professor, worked collaboratively over a nine-month period to design a series of guided activities to implement during the experiment, subject to revision/refinement as the experiment continued, based on anticipated pathways for learning and possible contingencies that might arise. The researcher along with the classroom teacher designed the initial conditions and elements of the experiment based on norms of classroom participation, available tools and materials, and the teacher’s schedule.
In the second stage, the principal investigator, with support from her mentor professor, and the participant teacher worked collaboratively over a five-week period on the ongoing process of redefining and redesigning the proposed and tested conjectures (i.e. conducting the experiment). After each class period, the researcher conducted forty-five minute debriefing sessions with the classroom teacher and the professor. During these sessions, the original activities designed in the first stage were subjected to refinement and redesign. In some cases, new activities were designed, based on student outcomes and classroom interactions with the participating teacher for the day. As part of this process, the research team both, developed and refined more specialized conjectures to be framed and tested.

3.4 Procedure

A series of activities were implemented during this study for the purpose of developing students’ meta-representational competence within the context of learning trigonometric functions through the unit-circle model. These activities included: building a domino staircase, building a cable-stayed bridge, investigating triangles, and representing ratio and proportion.

3.4.1 Building a Domino Staircase

The purpose of this activity was to engage students in investigating certain mathematical concepts involved in trigonometry, specifically slope and angles. Students worked in groups of four and five to explore tilt, point of collapse, angles of elevation, and symmetry. Over three days, students were presented an optimization problem: how to build a domino structure of maximum length while maintaining its stability. Students participated in discussions, reflections, and presentations about ideal stair models. (see Figure 3.2).
3.4.2 Building a Cable-Stayed Bridge

The purpose of this activity was to expose students to a hands-on experience where they were able to explore right triangles from a modeling perspective. Moreover, this activity was considered a generative activity (Cobb, Boufi, McClain, & Whitenack, 1997) because it could afford students access to learn important trigonometric concepts such as geometric representations, similarity, proportion, and trigonometric tables through subsequent activities. Over two days, the participants worked with a set of bridge manipulatives. The overarching goal of this activity was to attain stability and balance for the bridge. On the first day, students were asked to select from a set of pre-cut, pre-measured cables to support the deck of their bridge (see Figure 3.3).
After the first day, restrictions for the bridge construction were incorporated as a result of the iterative design process and were intended to both optimize the design and minimize variability (i.e. tilted deck) within the design. All bridges were inspected and evaluated.

After their respective bridge assembly, each group of students was asked to create two sketches to represent their first and second attempts to achieve balance. Students received a template where they could represent their bridge construction (see Figure 3.4).

**Figure 3.3: Prototype of the bridge and an example of a student bridge.**

**Figure 3.4: Bridge sketch template.**
The template included two sections, one for each tower (side view) of the bridge. Students needed to represent (sketch) the position of the cables and presented their sketches during the class. By using the bridge sketch template, students were able to create a triangle-based representation or model of the cable-stayed bridge. This activity helped the researcher analyze how students visualized proportion geometrically and understood the concept of ratio related to right triangles.

3.4.3 Investigating Triangles

This activity was designed to analyze student understanding of certain mathematical concepts such as patterns and the geometric definition of similarity, thereby generating the foundational understanding required to study co-variation, and ratio and proportion. Each group of students was given a set of triangles and was asked what was unique about their given family of triangles (i.e. what characteristics of these triangles made them “look alike?”) Each family of triangles was constructed as a result of a dilation of a given triangle, and, therefore, served as a physical, geometric representation of similarity. Over two class periods, students worked with two sets of triangles. The first day students worked with a set of five similar triangles of different colors (see Figure 3.5).
Participants were asked to define or describe the characteristics in order to support their reasoning. They did not utilize protractors; rather, they utilized the triangles themselves as the sole measuring tool. The second day students received a different set of white triangles and repeated the task. Students presented, compared, and contrasted their arguments.

### 3.4.4 Representing Ratio and Proportion

The purpose of this activity was to compare three different representations of the bridge: the manipulatives, graph, and table. Students translated their bridge prototype to a graphic representation and completed two tables with the corresponded measurements for the heights and lengths. Figure 3.6 shows the template used by the students. Data recorded included the horizontal measurement from both towers to the deck’s points of attachment; the vertical measurement from the deck to the towers’ points of attachment; the length of the cables attached from the towers to the deck (the diagonal); and the angle measurements between the cables and the towers’ points of attachment.
Through this activity, students began exploring the connection between angle and ratios by using the data generated during the previous activity. Students completed the table by including a selected angle and the computed ratio and presented their results to the class. Working in teams, students geometrically represented the following ratios: 3:4, 4:3, 2:5, 5:2, 3:5, and 5:3 and completed a table where the heights, lengths, angles of elevation and depression, and the actual ratio were defined. The participants drew triangles to represent such ratios. Students presented and discussed their data. Figure 3.7 shows the ratio table designed as a result of class discussion.

_Tables_

![Figure 3.6: Template to Compare Graphs and Tables.](image)
Foam Board

This activity was a result of the re-design process. During three days, students worked with a foam board equipped with one of the bridge cables attached to the center of the board. It was implemented as an interactive manipulative for geometric representation of ratios. Students worked in teams of four to five and traced the triangles represented by the ratios: 3:4; 4:3; 2:5; 5:2; 3:5; and 5:3 using the cable attached to the board. As different representations emerged, a set of interesting discussions resulted. Figure 3.8 shows one of the foam boards utilized by students. The researcher used GeoGebra software to recreate each group’s representation of ratios from the foam board activity. The purpose was to present an interactive representation of the geometric constructions to facilitate class discussion.
The Circle

This activity emerged during the re-design process and involved the students using an auxiliary tool. The tool was a template including the geometric representations of the triangles created during the foam board activity along with a linear scale of angle measures ranging from 0 degrees to 360 degrees. Students named the template the “lollypop” due its graphic similarity to a lollypop. Over two days, students worked on the projection of the heights and lengths of their triangles. Students were asked to project (map) triangle heights and bases to the linear angle scale (see Figure 3.9). The activity concluded with a discussion on the position of the projected heights and lengths.
Through this activity, students approached a number of trigonometric concepts through multiple representations. The intent of this activity was to provide students with an environment where they could work with their geometric representations in a more interactive and dynamic way. By guiding students through this activity, the researcher was able to examine the role technology played in their reasoning about trigonometric functions. Figure 3.10 shows one dynamic representation in GeoGebra to which the students were exposed.

**GeoGebra Technology**

Figure 3.9: “Lollypop” Template.

Figure 3.10: Dynamic Ratio Representations in GeoGebra.
First, students completed a series of exercises focused on how to use the GeoGebra software. Students constructed mathematical objects such as lines, angles and figures. Through these exercises, students learned and applied a variety of computational utilities for modeling and simulations including dilations and tangent constructions. After the training, students worked over three days on the GeoGebra version of their foam board representations and constructed a unit circle model for trigonometric functions (see Figure 3.11).

![GeoGebra screenshot](image)

**Figure 3.11: A Student's Unit Circle Representation in GeoGebra.**

After constructing a GeoGebra file (i.e., a technology-based version of the foam board), students were asked to utilize the file as a visual tool to map the heights of the “tower(s)” from the model as they did earlier using the “lollypop” template. Students were required to use the GeoGebra file as a visual aid, rather than measuring by hand. As a result, participants were exposed to important geometric aspects of functions such as maximums, minimums, and slope. In addition, the researcher was able to analyze their understanding of the Cartesian plane and quadrants to through their understanding of directionality and position.
3.5 Data Sources and Collection

3.5.1 Classroom Observations and Field Notes

Classroom observations of twenty-five classroom sessions took place during the fall semester of the 2013-2014 school year. They included classroom situations, student’s interactions, teacher’s actions, dialogs between the participants, and impressions about the activities. Each observation lasted forty-five minutes for a total of 18.75 hours. All classroom sessions were video recorded. During each session, as students work individually and in teams, the researcher asked questions and took notes along with pictures of both the students’ and the teacher’s work. Field notes, both descriptive and reflective, were fully completed and typed within 24 hours of each session to recall and record as much information as possible and to start organizing students’ voice and experiences.

After each classroom observation the researcher, university mentor, and classroom teacher met for an hour to discuss what occurred during class. Twenty-five hours of debriefing sessions were video recorded to capture the voices and perspectives of the researchers and the classroom teacher. Field notes documented impressions and reflections of the teaching experiment. During these sessions, the researcher and the classroom teacher discussed and outlined plans for the next activity for the following class day as well as anticipated changes to the original agenda for the experiment based on contingencies that arose.

3.5.3 Student Work and Artifacts

Students’ interactions and work were captured on video and as digital images for further analysis. Video and digital representations of processes, sequences, procedures, and explanations
were captured along with actual student artifacts: models, activity templates, foam boards, and GeoGebra files. Student notebooks were also collected and scanned.

3.5.4 Surveys

Two electronic surveys using Qualtrics© software were administered during this study. The intent of the surveys was to attain the demographic background of the participants and to collect information regarding students’ perceptions and attitudes about mathematics and the use of GeoGebra technology before and after the experiment.

3.5.5 Blogs

Another source of data was online blogging through an established course online platform. Data from this source include responses to questions and problems posed by the teacher that emerged during class and were related to the day’s activities. Students participated in and completed nearly fifteen blogs. All blog data was downloaded and saved in electronic files for analysis.

3.5.6 Interviews

Individual interviews with thirteen student participants and the classroom teacher were conducted at the end of the teaching experiment. The researcher intended to interview at least half of the participants (total 23). The selection criteria resulted from multiple factors. First, during classroom observations students were identified as potential interviewees due to either their low or high performance on the activities. In addition, online blogs and notebooks served as tools to identify possible candidates for interviewing. Student attendance was another variable that determined selection. Some students were absent at the time of interviewing recruitment. In
addition, the researcher intended to interview an equal number of male and female participants. The researcher was also sure to interview at least one member from each group.

Interviews were semi-structured and the questions were developed based on what the researcher observed during classroom sessions (see Appendix B). Students were asked about their perspectives on the mathematics content presented and the activities implemented during the experiment. Students recalled and described their work during several of the classroom activities and explained their procedures or practices for approaching a given problem or task. Students were given ample time to reflect on each question. Their detailed responses helped the researcher to understand their learning trajectory as well as the obstacles they encountered during the study. Each interview lasted approximately 60 minutes and was video recorded and transcribed.

In addition, the classroom teacher was interviewed three times after the conclusion of the experiment. Each interview lasted approximately 60 minutes and used a semi-structured format to allow the teacher to reflect upon his experiences about teaching the content and with classroom interactions. In the first interview, questions focused on the teacher’s background and experience as well as certain teaching practices and their respective impact on students. In the second interview, the teacher reflected on classroom interactions and his content knowledge about trigonometric functions. Additionally, reflections about the use of technology (e.g. dynamic geometry) in the classroom emerged. In interview three, questions focused on the pedagogical implications for teaching mathematics, his students’ learning processes, and the differences between teaching through traditional versus non-traditional approaches. Finally, the teacher answered questions about the effectiveness of using multiple representations to teach trigonometric functions and functions in general.
3.6 Data Analysis

3.6.1 Grounded Theory

For this study, a Grounded Theory (GT) approach was used to construct the theoretical explanation of students’ learning processes through the analysis of data. Originally developed by Glaser and Strauss (1965, 1967, & 2009), a GT approach assists social scientists to develop methodological strategies for qualitative research practice (Charmaz, 2010). This approach involves the simultaneous processes of data collection and analysis; construction of analytic codes and categories; use of the constant comparative method; and development of advancing theory. As explained by Atkinson and Delamont (2005), the use of GT derives provisional understandings that lead to further empirical explorations. By using GT methods, the researcher was able to build inductive theories about participants’ conceptual development through sequential levels of data analysis.

Data analysis began with the examination of the first debriefing session. The researcher used an IDC to establish preliminary conjectures. During twenty-five days, debriefing sessions were utilized to perform a constant comparison of data. Field notes were utilized to open the conversation and identify emerging themes along the study. As a result of these conversations, potential episodes were identified and served for further analysis. At the end of classroom and debriefing sessions, field notes were reviewed and expanded. During this process, the researcher highlighted important quotes, concepts and emerging themes. In addition, reflections on several questions resulted from the preliminary analysis of data. A GT approach was utilized to start an open coding for field notes analysis. Charmaz (2010) explains that GT coding “requires us to stop and ask analytic questions of the data we have gathered” (p. 42). By following the IDC, the
researcher performed a constant comparison of data throughout the experiment. Glaser (1978) emphasizes the constant comparative method constitutes the cores of GT. This method helped to both: compare the conceptualized data on diverse levels; and generate conceptual and theoretical themes.

After the classroom observations, the interviews were conducted and the transcribing process started. A total of twenty-five video recordings from classroom observations were fully transcribed. Since the researcher did not record the videos, she had the opportunity to review every session and see the classroom interactions from a different perspective. The transcribing process allowed the researcher to identify important episodes and potential case studies. The researcher could analyze classroom interactions that occurred while she was observing either other group’s discussion or students’ interactions. In addition, while transcribing the researcher had the opportunity to compare the field notes with the videos. This process helped to extend the researcher’s notes and to enhance details.

Subsequent to the transcription of the classroom observations, the researcher moved forward to the interviews. During this process, she had the opportunity to recall participants’ responses and compare them with their participation during the study. The interviews review allowed the researcher to triangulate data from different sources: classroom observation, field notes, and interviews.

Besides, the researcher performed a preliminary analysis of the survey’s data. Surveys were created by using the Qualtrics© software. This format allowed the researcher to manage the data in a more efficient way. The software includes a tool to create data reports in different formats.
A report in the Microsoft Excel© format was created. After that, data was transferred to the Statistical Package for the Social Sciences (SPSS©) software. This software includes a set of tools to analyze quantitative data. The Descriptive Statistics tool was utilized to analyze the variables pertinent to two categories: demographics, and mathematics and technology attitudes. Two different reports were created and their data compared. The reports helped to examine students’ perceptions before and after the experiment.

In addition, a preliminary analysis of students’ blogs was conducted. This analysis consisted in the review of the online blogs that students posted during the experiment. The content of each blog was copied and pasted in a Microsoft Word© format. A two-column file was utilized to organize both, the information in the blogs, and the researcher’s comments. On the left column, data from the blogs was placed. On the right column wrote notes and comments were written for further reference. The researcher identified and highlighted important discussions and responses regarding function-based reasoning.

Video recordings and transcripts were analyzed to identify representative episodes. The representative episodes constituted the mathematical themes that emerged from the students’ learning trajectories. Cobb and colleagues (2001) explain, “the critical episodes are those that prove pivotal in either refuting a conjecture or substantiating an assertion (p. 147).” Although, the episodes could appear to be of little significance, it is necessary to place them within the context of the entire study to make sense of them.

3.6.2 Coding

In order to code the data, the researcher utilized a grounded theory approach similar to that described by Cobb, Stephan, McClain, and Gravemeijer (2001). The first stage of analysis
involved examining the video recordings and transcriptions chronologically (i.e., classroom observations, debriefing sessions, and interviews) to identify potential themes. As explained by Carrejo (2004), “[a]n episode was characterized as a segment in which a mathematical theme (or perhaps themes) is the focus of activity and discourse” (p. 57). During this process, a variety of observations and conjectures emerged. Cobb and colleagues (2001) explain, “[t]he result of this first phase of the analysis is a chain of conjectures, refutations, and revisions that is grounded in the details of the specific episodes” (p. 128). By relying in grounded theory, three types of coding were involved in data analysis: open, axial and selective coding. By using an open coding, potential codes emerged. Axial coding determined the dimensions of categories and the selective coding resulted in the selection of representative episodes. Figure 3.12 shows the process utilized for coding. Reflective memos were created from the identified codes. Initial codes were “provisional, comparative, and grounded in the data” (Charmaz, 2010, p. 48). The codes were kept short, simple, active, and analytic (Charmaz, 2010). Table 3.2 shows the categories resulting from preliminary coding.
Analysis of video recordings (classroom observations and interviews)

Close reading of transcriptions (classroom observations and interviews)

Identification of potential themes

Reflection and memo-writing about potential themes

Preliminary open coding

Classification of selected data

Definition of categories and sub-categories

Classification of categories and sub-categories

Episodes definition

Figure 3.12: Coding Process.
### Table 3.1: Initial Codes.

<table>
<thead>
<tr>
<th>Code</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perceptions on math</td>
<td>Students’ perceptions about mathematics</td>
</tr>
<tr>
<td>Favorite activity</td>
<td>Students’ preferred activity during the experiment</td>
</tr>
<tr>
<td>Measurement accuracy</td>
<td>Students’ perceptions about accuracy in measurement</td>
</tr>
<tr>
<td>The importance of visualize it</td>
<td>Students’ perceptions about visual representations of math</td>
</tr>
<tr>
<td>Similarity</td>
<td>Students’ knowledge about similarity</td>
</tr>
<tr>
<td>Dilation</td>
<td>Students’ knowledge about dilations</td>
</tr>
<tr>
<td>Tangent</td>
<td>Students’ knowledge about geometric representation of tangent</td>
</tr>
<tr>
<td>Ratio and visualization</td>
<td>Students’ reactions to visual representation of ratios</td>
</tr>
<tr>
<td>Proportion</td>
<td>Students’ knowledge about proportion</td>
</tr>
<tr>
<td>Mapping the heights</td>
<td>Students’ strategies about geometric representations of sine</td>
</tr>
<tr>
<td>Mapping the bases</td>
<td>Students’ strategies about geometric representations of cosine</td>
</tr>
<tr>
<td>Similarity and visualization</td>
<td>Students’ connections between similarity and geometric reps.</td>
</tr>
<tr>
<td>Dilation, projection &amp; transformation</td>
<td>Students’ connections between dilation, projection, and transform.</td>
</tr>
<tr>
<td>Importance of math</td>
<td>Students’ reflections on everyday mathematics</td>
</tr>
<tr>
<td>Estimation</td>
<td>Students’ knowledge about estimation</td>
</tr>
<tr>
<td>Perceptions about the experiment</td>
<td>Students’ perceptions about the experiment</td>
</tr>
<tr>
<td>Measurement tools</td>
<td>Students’ strategies for measurement</td>
</tr>
<tr>
<td>Linking activities</td>
<td>Students’ connections between activities</td>
</tr>
<tr>
<td>Struggling with technology</td>
<td>Students’ difficulties using technology</td>
</tr>
<tr>
<td>Perceptions about GeoGebra</td>
<td>Students’ perceptions on dynamic geometry software</td>
</tr>
</tbody>
</table>

After establishing the preliminary codes, focus coding began. These codes were more selective and conceptual. They helped me to synthesize and explain adequately the data (Glaser, 1978). Through focus coding, the researcher moved across classroom observations, interviews,
and students’ reflections to compare the codes (Charmaz, 2010). From the initial coding, selected data was reorganized into a separate file. The researcher classified the emerged themes into categories and sub-categories. A color-code approach was utilized to better visualize the data. This approach helped to reorganize the data in a more effective manner. Table 3.3 shows the reorganization of data.

### Table 3.2: Categories and Sub-categories.

<table>
<thead>
<tr>
<th>Category</th>
<th>Sub-category</th>
<th>Color code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Similarity</td>
<td>-</td>
<td>Yellow</td>
</tr>
<tr>
<td>Dilation</td>
<td>-</td>
<td>Aqua</td>
</tr>
<tr>
<td>Projections</td>
<td>-</td>
<td>Purple</td>
</tr>
<tr>
<td>Angle Measure</td>
<td></td>
<td>Green</td>
</tr>
<tr>
<td>Directionality</td>
<td></td>
<td>Purple</td>
</tr>
<tr>
<td>Measurement</td>
<td>Measurement Strategies</td>
<td>Purple</td>
</tr>
<tr>
<td>Accuracy</td>
<td></td>
<td>Purple</td>
</tr>
<tr>
<td>Complimentary Angles</td>
<td></td>
<td>Green</td>
</tr>
<tr>
<td>Tools</td>
<td></td>
<td>Aqua</td>
</tr>
<tr>
<td>Ratio and Proportion</td>
<td>-</td>
<td>Yellow</td>
</tr>
<tr>
<td>Dynamic Geometry</td>
<td></td>
<td>Yellow</td>
</tr>
<tr>
<td>Foam board</td>
<td></td>
<td>Aqua</td>
</tr>
<tr>
<td>Graphing</td>
<td>Coordinated System and Cartesian Plane</td>
<td>Aqua</td>
</tr>
<tr>
<td></td>
<td>Lollypop Trig</td>
<td>Green</td>
</tr>
<tr>
<td></td>
<td>The Unit Circle</td>
<td>Green</td>
</tr>
<tr>
<td>GeoGebra Impact</td>
<td>-</td>
<td>Aqua</td>
</tr>
<tr>
<td>Multiple Representations</td>
<td>-</td>
<td>Violet</td>
</tr>
<tr>
<td>Teacher Strategies</td>
<td>-</td>
<td>Aqua</td>
</tr>
</tbody>
</table>

After establishing the categories and sub-categories, representative episodes were identified. Figure 3.13 shows an initial classification of episodes related to multiple representations.
Additionally, students’ reflections were aligned with selected episodes. These reflections included direct excerpts from students and groups’ interactions. Figure 3.14 shows an example of students’ reflections.

Eleven episodes resulted from the focused coding of data (see Table 3.4) and summarize the learning trajectories students followed during the experiment. The episodes entailed four characteristics: students’ prior knowledge, landmarks, obstacles, and learned ideas. Data from classroom observations, interviews, online blogs, notebooks, and students’ work and artifacts defined these characteristics.
TABLE 3.3: EPISODES’ FORMAT

<table>
<thead>
<tr>
<th>Episode</th>
<th>Description</th>
<th>Episode</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Bridge Building</td>
<td>VII</td>
<td>Directionality and Position</td>
</tr>
<tr>
<td>II</td>
<td>Colored Triangles</td>
<td>VIII</td>
<td>Representing the Towers</td>
</tr>
<tr>
<td>III</td>
<td>Ratios table</td>
<td>IX</td>
<td>Representing the Bases</td>
</tr>
<tr>
<td>IV</td>
<td>Foam board</td>
<td>X</td>
<td>Dynamic Representations</td>
</tr>
<tr>
<td>V</td>
<td>Complementary Angles</td>
<td>XI</td>
<td>The Tangent Function</td>
</tr>
<tr>
<td>VI</td>
<td>Angles of Elevation &gt; than 90°</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Finally, the eleven episodes were condensed into five episodes (see Table 3.5). The data compiled on these episodes served to both, address the research questions and respond to the inquiry on how multiple representations enhanced student learning, and the types of understandings that emerged through these representations.

TABLE 3.4: FINAL EPISODES.

<table>
<thead>
<tr>
<th>Episode</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Approaching Similarity</td>
</tr>
<tr>
<td>II</td>
<td>Ratio and Proportion</td>
</tr>
<tr>
<td>III</td>
<td>Angle Measurement</td>
</tr>
<tr>
<td>IV</td>
<td>Directionality and Position</td>
</tr>
<tr>
<td>V</td>
<td>Dynamic Representation of Tangent Function</td>
</tr>
</tbody>
</table>

The design study methodology and Grounded Theory approach allowed the researcher to construct a theoretical explanation and describe how multiple representations enhance the learning of trigonometric functions. The Iterative Design Cycle and Grounded Theory were aligned both naturally and theoretically during the processes of planning, collecting and analyzing data. The simultaneous involvement in the collection and analysis of data resulted in developing provisional understandings at early stages that lead to further empirical explorations.
Chapter 4: Results

This chapter presents the results of qualitative analysis conducted on transcript data and other artifacts collected. Utilizing a grounded theory approach, the analysis established preliminary coding results, identification of core categories, and concluded with the selection of five meaningful episodes.

4.1 Episode 1

The first episode comes from a series of group discussions students had after they worked in groups and presented their optimal bridge configurations. Students worked on different activities and were exposed to multiple representations of triangles. Discussions during the activities were guided to approach the concept of similarity. The episode began with a discussion about achievement of the bridge’s balance, ratios and similarity, followed by a reflection on triangles’ characteristics and geometric understandings. The episode closed with a reflection on angle measure.

4.1.1 Approaching Similarity

The groups presented a sketch that included a first and second attempt of their bridge. Three groups (1, 3, and 5) managed to get the balance of their bridge on their first attempt by following a 2:1 ratio; two groups (3 and 5) achieved balance on their second attempt. After the five groups presented, the teacher opened the discussion to triangles. The teacher asked the students if there was anything as far as shape concern. Students manifested their thoughts:

A3: Triangles!
Teacher: And what about them? And that’s the next thing.
C1: They’re right triangles.
Teacher: How can you tell they’re right? What would we have to assume?
Group: They are all 90-degree angles.
Teacher: Which ones?
Group: All of them…

Students described the triangles formed on their sketches and affirmed that all of them were right triangles. The teacher asked students if that characteristic (right triangles) was sufficient in order to achieve balance of the bridge. The teacher started asking about their first attempts. Four out of the five groups affirmed that they could not achieve balance of the bridge. In addition, they mentioned that their bridges lifted.

The teacher moved to students’ second attempts and asked them about the differences between the two attempts (first and second). The teacher asked students about the patterns and compared the sketches created by groups 3 and 5. The conversation turned to describe what was so special about the triangles; the triangles followed a 2:1 ratio. The discussion continued and students described more characteristics about the triangles. From group 1 (the one achieving balance of the bridge on the first attempt by following a 2:1 ratio), participant J1 explained that the characteristic those triangles shared was one related to dilation.

Teacher: So you [participant J1] said, you mentioned something right now. You said…
J2: I don’t know the other word for dilation, but it’s like any… like with the little triangles … the little triangle inserted… but I don’t know the other word…
Teacher: So she said dilation.

From group 2, student A4 made her contribution and added proportional as another characteristic those triangles had in common. The discussion became more interactive and engaged the five groups. From group 4, participant C1 added the concept of ratio to the conversation and complemented what participants J1 and A4 stated.
Teacher: Ok, so now we talk about proportional and dilations, what else? Word association…
C1: Ratios!
Teacher: Ratios? Dilations, proportions, so what kind of triangles are they then? Is… they are as [participant] J2 said? Are they dilations of each other? Are they proportional? Do they have to…? What kind of triangles are they? Any idea, any clue as to what kind of triangles they may be?

During the discussion, the group agreed that the triangles they were working with were dilations of each other and all of them maintained a 2:1 ratio; then the teacher asked the class about the characteristics proportional figures share. Participant A4 stated that proportional could be described “as different sizes but equal ratios.” When the teacher asked them to define dilation participant C1 mentioned that can be described as “a decreasing size… as a change” and participant A3 explained that the change involved a 2:1 ratio.

Through the discussion, students manifested their conceptions about congruency, proportion and characteristics about triangles. The conversation turned to establish the differences between proportional and congruent. When the teacher asked students if the triangles were congruent, the response split among the five groups. Two groups affirmed that the triangles were congruent. For instance, participant C1 affirmed that the triangles were not congruent. By his part, participant A3 stated that the triangles were congruent “not like in size as I was looking but like in the ratio.” When the teacher asked them to define congruency students continued with different explanations. Participant A3 defined it as “same shape, but not size” while participant D1 as “same size and shape.” Table 4.1 summarizes groups’ perceptions (consensus) about congruency in the terms of shape and size.
### Table 4.1: Students Perceptions About Congruency.

<table>
<thead>
<tr>
<th>Group</th>
<th>Congruency in Terms of Shape</th>
<th>Congruency in Terms of Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Same shape</td>
<td>Same size</td>
</tr>
<tr>
<td>2</td>
<td>Same shape</td>
<td>Different size</td>
</tr>
<tr>
<td>3</td>
<td>Same shape</td>
<td>Same size</td>
</tr>
<tr>
<td>4</td>
<td>Same shape</td>
<td>Different size</td>
</tr>
<tr>
<td>5</td>
<td>Same shape</td>
<td>Same size</td>
</tr>
</tbody>
</table>

After some agreements and disagreements, the teacher clarified that “congruency is actually the same size, same shape.” He also asked students about the ‘other’ (same shape, different size) to make clear the definitions.

D1: See, I told you… [To group 2]
Teacher: So I want to talk about that other one [same shape, different size] then because it exists in your mind somewhere in there. What is that other when you’re talking about same shape, different size?
A3: Scale
D1: Scale
Group: Scale
Teacher: There is a scale involved, there is dilation
A3: Ratio

Students continued the discussion by identifying some of the characteristics proportional triangles share as well as types of triangles they were working with. Responses ranged from a variety of concepts. The teacher elaborated questions that guided students to think about similarity. Reflections about congruency, and proportion helped students to link characteristics of those triangles and define them as similar. Table 4.2 summarizes students’ perceptions.

Teacher: There’s a ratio, we have proportion, what kind of triangles are those?
D1: Triangular triangles… It’s a weird word; it’s a weird name, isn’t it?
C1: Well they’re all right triangles… doesn’t matter what size they are … they are all same angles…

---

2 Participant D1 affirmed “Same shape, same size” but response of majority of the group’s members was considered
Teacher: Ok, so we can establish that congruency is to be equal. So if they’re not equal, they’re not congruent. They are what?

C1: We all, I think I guess we agreed them to be all right triangles. They all have the same angles despite the length, right?

Teacher: So it’s that what you call them, just right triangles whether they have different proportions?

C1: No,  
A3: Well technically they’re all right [triangles] because of their 90° angle and all of them have the 90° angle.

Teacher: But did you…? So did you come up with anything that would identify these triangles as particular type; the fact that they’re all the same shape that right triangles shape, but different sizes?

C1: No  
Teacher: Ok, what about this group [I]?

K3: They’re similar figures.

Teacher: What did you say again?

K3: They’re similar figures.

<table>
<thead>
<tr>
<th>Table 4.2: Students’ Perceptions about Triangles.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Description</strong></td>
</tr>
<tr>
<td>Proportional Triangles</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

The previous discussion shows the learning trajectory students followed to approach the concept of similarity. Although students worked with visual representations, they struggled when having to describe similar triangles.
4.1.2 Modeling with Triangles

The following series of discussions depart from the introduction to the concept of similarity and by establishing some characteristics about right triangles. Through these discussions, students reinforced the concepts addressed during the bridge building activity. Students opened the conversation to right triangles’ assumptions and the importance of angle measure.

During six class sessions, students worked in small groups in different activities with the triangle manipulatives. The purpose of these activities was to aboard mathematical concepts (e.g. similarity, dilation, ratio, proportion, and angles) through didactic situations where students could visualize, reaffirm or discard prior conceptions, and construct knowledge. Students received a set of five colored triangles (white, red, blue, yellow, and green). Additionally, they were provided with a little red square that represented one square unit. The triangle manipulatives represented the ideal attachment of the five cables (from the tower to the deck) of the bridge and maintained a 2:1 ratio (height to base). Table 8 shows the triangle’s characteristics (height and base) and reflects the ideal attachment of the cables that achieve balance of the bridge. Table 4.3 shows the relationship between ideal attachment of the cables, and the heights and lengths of the triangles manipulatives.

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Color</th>
<th>Ratio</th>
<th>Height (units)</th>
<th>Base (units)</th>
<th>Angle α</th>
<th>Angle β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>White</td>
<td>2:1</td>
<td>10</td>
<td>5</td>
<td>65</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>Red</td>
<td>2:1</td>
<td>8</td>
<td>4</td>
<td>65</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>Blue</td>
<td>2:1</td>
<td>6</td>
<td>3</td>
<td>65</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>Yellow</td>
<td>2:1</td>
<td>4</td>
<td>2</td>
<td>65</td>
<td>25</td>
</tr>
<tr>
<td>5</td>
<td>Green</td>
<td>2:1</td>
<td>2</td>
<td>1</td>
<td>65</td>
<td>25</td>
</tr>
</tbody>
</table>
Students started playing with the manipulatives and the teacher began the conversation by asking students about the characteristics of the triangles. Exposure to the concept of similarity enabled students to identify such characteristic when working with the multicolored triangles.

Teacher: You’re already noticing that these triangles are not just any random triangles, what are they?
D1: They’re similar.
A4: Because they look the same, just…
D1: They’re dilated…

Half of the class overlapped the colored triangles with their bridge models. They compared them with the cables used in their bridges and recreated their configurations on the tables. The teacher asked students if they could see the triangles on their bridge designs, most of the students agreed (see Figure 4.1).

![Figure 4.1: Students’ Configuration of Triangles.](image)

The teacher then asked the class to utilize the learned concepts to stack up the triangles in a way that demonstrates they were similar. Different configurations emerged from this activity. Students overlapped their triangles and followed diverse strategies. Figure 4.2 shows students’ work with the triangles manipulatives.
Students were asked to develop a definition of similarity by using the provided triangle manipulatives. Besides, they were asked to explain the fact that those triangles were similar. Students summarized their responses in their notebooks; then post them on the class’ online blog. In addition, they had to define similar shapes and write their observations when stacking up the triangles. Twenty-two out of twenty-three students participated in this activity; one participant was absent. Table 4.4 summarizes students’ configurations of the triangles.
### Table 4.4: Students’ Configuration of Triangles.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Representation</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>![Diagram A]</td>
<td>16</td>
</tr>
<tr>
<td>B</td>
<td>![Diagram B]</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>![Diagram C]</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>![Diagram D]</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>![Diagram E]</td>
<td>1</td>
</tr>
</tbody>
</table>

Students’ configuration of triangles reflected a strong tendency to rely on right angles. Sixteen out of twenty-two participants employed configuration ‘A’ as their strategy to demonstrate that triangles were similar. Furthermore, students reinforced their tendency to rely in right angles when posting their definitions about similarity, and described their strategy to demonstrate that the triangles were similar. Twenty students formulated their responses in their
compos. Nineteen posted them on the online blog. Appendix B and C summarize students’ responses to the blogs.

Students’ responses evidenced the tendency to back up on right triangles when making assumptions about similar triangles. Although majority of the students relied on the 90° angle to state that the triangles were similar figures, they also utilized other strategies to support their response. From students’ responses to the blog, five participants (E1, D3, S3, K1, and L1) relied solely on the 90° angle to sustain the conjecture about similar triangles; thirteen participants complemented their response by considering other characteristics they noticed; one participant omitted the response.

One participant referred to the slope of the triangle as a way to demonstrate that the triangles were similar, despite the fact that they shared the 90° angle. Participant J1 stated at the blog that what she observed was that “they are all right triangles...the hypotenuse is going down the same way.” Considering the hypotenuse (diagonal) as a visual aid was a major strategy to figure out that triangles were similar.

Aligning the three angles of the triangles was a different strategy participants considered. Three students mentioned at the blog that they compared the three angles (edges) to prove similarity among the five triangles. These students relied on the geometric representation of angle and provided their observations. Participant’s A2 response stated, “when I ‘stacked’ the figures, I did it in such a way that the 90 degrees all matched. I also tried all of the other angles and all of them aligned perfectly.” For A2 ‘aligning’ the angles served as a strategy that helped him to prove that the three internal angles of the triangles where the same; therefore, they were similar figures. Likewise, participant K2 utilized this strategy to prove similarity among triangles; she stated, “when I stacked my triangles it showed that no matter how I stacked the
corners were all the same.” Finally, participant P1 described how stacking the triangles together helped her to visualize the similarity among them; Participant P1 stated, “I stacked the triangles together at the tip of the smallest angle I saw that they all fit together… I realize that they all have the same angle going through the top of the triangle and the other edges all kind of lined up together.”

Another participant considered the concept of ratio as a tool to demonstrate similarity among the five triangles. Participant A3 explained, “similar shapes should have the same basic shape and side to side ratio. The angles will come out the same because of the ratio.”

Furthermore, participants made use of their visual strategies to compare figures. Three participants utilized this strategy in order to explain similarity on shapes. Participant A4 explained, “I placed the triangles in a way where the 90° angles met up to show that all the triangles where similar. The angle met up and the shape stayed the same.” In addition, participant I1 pointed out that by observation she noticed that the angles were the same; I1 explained, “I saw how the angles were the same and how they were the same shape. The difference was just the size of the triangles. The lengths of the triangles are proportional though.” Likewise, participant C1 mentioned he could visualize similarity in terms of shape. He wrote at the blog “when I stacked my triangles, I noticed the similarity in shape [visually].”

Finally, two participants K3 and S1 relied on the concept of dilation to demonstrate that the triangles were dilations of each other therefore, similar figures. Student J1 mentioned that the triangles possessed congruent angles consequently they were similar. Participant J1 explained, “I noticed that when you stack the shapes on top of each other and put them on the same angle then you can tell that all the angles are congruent but the sides are different and that they’re proportional figures.” Another participant compared the distances between the edges of the
triangles when aligning them; this strategy could be related to the ratio strategy due to what he compared was the proportion of the triangles’ lengths. Participant D1 explained, “What I observed from stacking the triangles was that when stacked the triangles have equal distances between one another.” Participant D2 utilized a pyramid configuration as a tool to demonstrate that the triangles were proportional figures. Participant D2 stated in her response to the blog “When I stacked them I could also see a pyramid because a pyramid has everything the same but there are tons of different sizes of the shape that build it.”

During this activity, students made use of important mathematical knowledge to develop a ‘proof’ strategy to demonstrate similarity. Diverse strategies and methods emerged during the activity making evident the importance of visual representations of mathematical objects.

4.1.3 Angle Measure

As part of the triangle manipulatives activity, students were asked to work in groups and measure the three angles of the five triangles; and come into an agreement in order to determine the angles alpha (α) and beta (β) as the angles of elevation and depression respectively. The classroom teacher indicated the position of both angles to establish uniformity among students’ measurements (See Figure 4.4).

![Figure 4.4: Position of Angles Alpha and Beta.](image)
Students worked in teams of 2-4 members each to determine the angles’ measurement for the five triangles; come to an agreement on the measurements; and prove that they were similar. The ‘accurate’ measurements of the angles were 90°, 65° (alpha) and 25° (beta) respectively. Students were not provided with this information. Some group’s members could not agree on their measurements and provided two or more values for each angle. Table 4.5 summarizes students’ measurements.

**Table 4.5: Students’ Measurements.**

<table>
<thead>
<tr>
<th></th>
<th>Green</th>
<th>Yellow</th>
<th>Blue</th>
<th>Red</th>
<th>White</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alpha</td>
<td>Alpha</td>
<td>Alpha</td>
<td>Alpha</td>
<td>Alpha</td>
</tr>
<tr>
<td>Group 1</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>Group 2</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>Group 3</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>26</td>
<td>26</td>
<td>26</td>
<td>26</td>
</tr>
<tr>
<td>Group 4</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>Group 5</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>Group 6</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>26</td>
<td>26</td>
<td>26</td>
<td>26</td>
</tr>
</tbody>
</table>

The triangles resembled the ideal configuration of the five pairs of cables for the bridge. As the optimal configuration, triangles maintained a 2:1 ratio (height to base). The sets of triangles manipulatives were pre-cut, pre-measured and a small variance on the measurements (due to human error) was a reality. Students measured their triangles; the task challenged them. From group 1, J2 experienced difficulties while measuring one of the triangles (white). When she
measured the alpha angle her measurement corresponded to the expected angle of 65°; however, when she measured the beta angle she encountered a discrepancy when obtaining 27° instead of the expected value of 25° angle. Previous knowledge on right triangles created a non-sense situation when accepting 90°, 65°, and 27° as the expected angle values.

J2: What if like… we measured the angles but… they don’t adapt to… like they are more…?
Teacher: it was more?
J2: Yeah like, I got 65 [degrees] and then I got 27 [degrees].
Teacher: You got something else for this one? Let’s see, show me… isn’t that I don’t believe you but…
J2: [J2 measured the beta angle]
Teacher: Oh, ok,
J2: Because of…
Teacher: The cut?
J2: Yeah
Teacher: So yes, if you agree that this [alpha angle] is 65 [degrees] then what are you going to agree about this one?
J2: 25 [degrees]?
Teacher: Yeah it was the … [cut]

Prior knowledge on triangles was a factor during this activity. For J2 a conflict emerged when she obtained angle measurements of 65° and 27° as components of a right-angled triangle. For this student following the assumption that the third angle measured 90° (because they were right triangles) served as a red flag to discard at least one of the measurements (65°, 27°) as accurate. Similarly, participant A3 expressed his disagreement when measuring one of the angles. He argued that the measurement evidently did not correspond to a right triangle. Participant A3 affirmed to his group the measurement he obtained for the beta angle was 26° instead of 25° (the complementary angle for the 65° alpha angle). Taking into consideration that this group agreed on the alpha angle of 65° and assuming that they were working with right
triangles (at least one 90° angle) participant A3 expressed the incongruity of the situation. He claimed, “You guys broke the universe, and I got 26.”

4.1.4 Summary of Episode I

The sole exposure to the bridge building activity did not enable students to approach the concept of similarity. During the bridge activity, only one out of five groups evidenced the use of geometric knowledge (linked to similarity) by relying on symmetry. Students’ reasoning when presenting their bridge models did not demonstrate a developed concept of ratio (tower to deck). In addition, students did not make use of units’ count when presenting their physical models. Besides, when constructing their bridges, four out of five groups did not rely on ratios as a strategy. Although students were exposed to triangles through the bridge activity, it was until they sketched their bridge configurations that the concepts of ratio, proportion and similarity emerged; this validates the conceptions about the weak or even the absence of prior knowledge on these concepts at the beginning of the experiment.

Moreover, the data evidences that students were more able to identify triangles’ characteristics when they sketched their bridge configurations on a template; likewise, through this representation they were more able to think about angle and the relationship among triangles. Although students looked engaged with the bridge activity, they appeared more comfortable using paper and pencil modeling their bridges rather than working with the physical model. In addition, when sketching the bridges on the templates, students repeatedly make use of the counting strategy to figurate out where to ‘place the cables’; counting units horizontally and looking for their corresponded measures vertically evidenced that students started thinking on inclinations and relationships between measures (co-variation) the heart of function-based reasoning.
The triangles activity emerged from the need to bring the concept of angle that did not emerged during the bridge building nor when sketching the bridge. With the triangles manipulatives, students were exposed to a different representation that reinforced what they learned on the previous activities. This activity provided students the environment where they were able to learn about angle and ratio, and to reinforce their knowledge on triangles. Even though, students remained struggling with measure after the three previous activities (bridge, sketch, and triangles). Classroom observations show that at this point of the Design Experiment students demonstrated a deeper thinking about angles and ratios.

By building the triangles’ measurements table, students were exposed to another representation that enabled them to compare the angles alpha and beta as well as the lengths and heights of the five colored triangles. Students filled out the tables with the respective measurements and started noticing patterns along them. Students’ work indicates they were more willing to think about proportions once they started tabulating the ratios, and visualized (numerically) the increments. However, during this activity the concept of measurement continued challenging students. Specifically, students showed difficulties when they have to come into an agreement on the angle’s measurements. The debate came out when discrepancies on the cut of the triangles appeared and students’ lack of experience on measurement, estimation and error became evident.

4.1.5 Learning Trajectory for Episode I

The previous episode presented the learning trajectory followed by students immersed in an inquiry-based activity similarity. The concepts of proportionality, ratios, and similarity comprise the established landmarks. During this episode, students were exposed to geometric configurations and began developing measurement skills aimed at understanding the concept of
proportion. In order to approach the concept of ratio, students worked with triangles and reflected on their characteristics and the relationships existing between them. However, students struggled when working with the concept of scale. Furthermore, students showed a weak understanding on the concept of congruency. During the learning process, students experienced multiple obstacles that resulted in the addition of new sub-activities that helped them to approach this concept. For instance, while representing triangles in a template, students struggled when explaining why those triangles were proportional. They identified some relationships by comparing heights and lengths, but were not able to yet fully comprehend that comparing two measures constitutes a ratio. Students also struggled when measuring angles, including complementary angles. However, when students were given activities requiring them to record angles and ratios in tables, they were able to compare numerical values of geometric representations, identifying patterns and visualizing numerical increments. After resolving the issue of measurement estimation and error, students came closer to developing the concept of similarity. Figure 4.5 shows an interpretation and summary of the learning trajectory for the episode.
4.2 Episode II

The issue of measurement continued through the experiment. In the past episode, it was presented a series of situations where students interacted with angle measure while working with a fixed 2:1 ratio. The following episode presents students’ interactions and discussions about angle measurement while working with a variety of ratios. Students worked on groups of two to four members. Each group was required to measure the alpha and beta angles for the triangles constructed by the ratios of 2:5, 5:2, 3:4, 4:3, 5:3, and 3:5 (see Figure 4.6).
FIGURE 4.6: ASSIGNED RATIOS FOR EACH GROUP.

Each group was asked to draw the triangle corresponding to the given ratio; and come into an agreement on the values for alpha and beta. Students constructed their triangles and measured the angles. The teacher asked them about the correspondent angles to the ratios. Three out of six groups reported their angle measurements. Students continued constructing their triangles. When the teacher asked groups 2, 5, and 6 for their measurements, discrepancies regarding reciprocity of ratios arose. Group 5 presented the obtained values of 53° and 36° as the alpha and beta angles respectively. Considering the fact that they were constructing right triangles, participants from group 5 did not show evidence of taking into consideration the sum of those angles as a form of verification. Table 4.6 summarizes the measurements obtained by groups 1, 3, and 4.

<table>
<thead>
<tr>
<th>Group</th>
<th>Ratio</th>
<th>Angle α</th>
<th>Angle β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3:4</td>
<td>36°</td>
<td>54°</td>
</tr>
<tr>
<td>2</td>
<td>3:5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>5:3</td>
<td>59°</td>
<td>31°</td>
</tr>
<tr>
<td>4</td>
<td>2:5</td>
<td>38°</td>
<td>52°</td>
</tr>
<tr>
<td>5</td>
<td>4:3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>5:2</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

It was not until the teacher made an intervention to call for a reflection that group 5 reconsidered the values for the alpha angle. Additionally, group 5 also had the opportunity to compare their data with those obtained by group 1 (reciprocal 3:4 ratio) which presented their measurements previously.
Teacher: 4:3 you guys were my 4:3 your alpha was…
L1: 53 [degrees] and 36 [degrees]

The teacher asked group 5 if there was anything they wanted to change about their measurements; they looked at the table and asked to change the 53° angle by a 54°. Measurement discrepancies continued during the activity. For group 2 determining the alpha angle was a dilemma; while participants D1, A4, and E1 agreed to 30.5° as the alpha angle, participant A3 claimed 32° as the correct measurement. Yet again, the teacher asked if they could observe something based on the measurements. The teacher continued with group 6 (5:2 ratio) and asked them about their measurements. Group 6 reported measurements of 65° and 25° for alpha and beta respectively. Considering that groups 4 and 6 were working with the reciprocals of each other, their measurements did not reflect that. While group 2 obtained 38° and 52°, group 6 obtained 65° and 25°. Although participant C1 stated that he was able to see the pattern, he did not show the tendency to rely on the concept of reciprocity. The teacher went back to group 2 (3:5 ratio) and asked them about their measurements. Three out of four members agreed on 30.5° as the angle value for alpha.

A3: Thirty two!
D1: We just did it and we got like…
A3: Thirty two!
A4: No, it’s not thirty two!
A3: Thirty two.

Participants from group 2 continued with the dilemma on the measurements corresponding to the 3:5 ratio. Discussion and justification on angle measurement emerged.

E1: We did it and we got 61° and 31° but it doesn’t…
D1: But it doesn’t work because it is 182 [degrees]

Participant D1 from group 2 explained that the measurements they initially obtained were 31° and 61° for alpha and beta respectively. He also explained that because of their prior
knowledge on the sum of triangles’ internal angles (180° their measurements could not be correct
(90° + 61° + 31° = 182°) and they decided (three out of four members of the group) to switch to
31° and 59° respectively. The previous situation served as a learning opportunity to open the
discussion to measurement error.

Teacher: Ok so then you got 61 [degrees] and 31 [degrees] and there for one of them, now, here is what we have discussed based on these errors, are there going to be errors?
Group: Yes!
Teacher: Because of what?
E1: Approximation?
Teacher: There’s going to be errors on those cuts because of the approximation because of the triangles and the way they were cut. So now, your alpha based on what you got, you said you got an alpha at this…
A3: No! It’s 32 [degrees]! You are lying!

Participant D1 tried to convince his peer A3 that the angles corresponding to a ratio of 3:5 are 31° and 59°. D1 made a point relaying on the correspondence between the ratios 5:3 and 3:5 and affirmed that measurements for the angles of both ratios should correspond.

D1: Alpha is 31 degrees
A3: No it’s not
D1: And beta got 59 [degrees]
Teacher: Alright
A3: It’s thirty two!
D1: And it corresponds to the 5:3 over there because they are the same measures right?

The teacher opened the conversation about error by comparing the angle measurements obtained from six ratios. He asked the students to observe the table and make a reflection on the measurements obtained. Figure 4.7 shows the angles corresponding to the six ratios.
Teacher: All right, and that’s… here is the thing… guys look at these ratios you have…

C1: It’s all the way around, D1 is wrong!

D1: No it’s not, look at the ratio!

Teacher: The ratio of 2:5 up here is there a ratio that we don’t have an agreement with?

Group: Yes!

D2: That one [5:2 vs. 2:5]

Teacher: The 5:2 and the 2:5; the 3:4 and the 4:3, what was it that you explained \[participant\] D1?

D1: Oh, because the ratio look at it is uh, the 3 over 5 is the 5 over 3 wise then we flip the degrees, alpha and beta.

Participant D1 explained that correspondence of the ratios was like flipping the degrees, which implies that alpha and beta for the 3:5 ratio corresponded to the beta and alpha for the 5:3 ratio. The teacher supported participant’s D1 point and asked the group if participant’s D1 rationale of switching the ratios made sense.

C1: I think it’s like… it’s like uh, see that triangle up there? Let’s just says that’s 5:3 right? [See Figure 4.8a]
C1: Now we need 3:5, can you rotate that triangle just to flip it on the side?
Teacher: I can try
C1: Yeah, so then you have your ratio of 3:5 but your angles remain the same wide because when you have that kind of ratio it doesn’t change [see Figure 25b].

The previous discussion shows participants’ D1 and C1 perception and interpretation of reciprocity. They described how correspondence of angles for a certain ratio remains the same for its reciprocal ratio. The teacher concluded the discussion by asking the students if they were able to see that the angles pretty much just change positions. This episode presented important considerations not only on ratio and proportion but also on measurement and error. Analysis of classroom observations and students’ work shows the importance of guided discussions.

4.2.1 Summary of Episode II

During this episode, students were more able to analyze and make sense of the mathematical content presented. Evidence shows that students made use of their prior knowledge when relying on triangles. Unlike Episode I, students constructed the triangles based on ratios. Due to triangles were constructed by the students, discrepancies on angles resulted from
students’ errors on measurement. Afterward, the dilemma about the correct angles’ measurement corresponded to the 5:3 and the 3:5 ratios could be attributed to inaccuracy establishment of the heights and lengths measures that resulted on differences among angles. By creating didactic situations where multiple representations were present, students showed a better understanding on ratios, dilations, and similarity. During this episode, students struggled with the concept of reciprocity. Additionally, the issue of measurement continued.

4.2.2 Learning Trajectory for Episode II

The previous episode presented the learning trajectory followed by students learning the tangent function. The concepts of similarity, reciprocal ratios, and covariation comprise the established landmarks. Students continued to struggle with angle measure in terms of resolving error. Furthermore, students faced an obstacle in learning about reciprocal ratios in the context of covariation between angles and ratios. Difficulties with angles emerged when students needed to construct triangles based on predetermined ratios. When constructing the tangent table, students struggled with the concept of complementary angles. As a result, students were exposed to sub-activities that helped them make sense of ratios, dilations, and similarity. Consequently, students began to develop a more robust understanding of the tangent table and approached the concept of reciprocity. Figure 4.9 shows an interpretation and summary of the learning trajectory for the episode.
4.3  Episode III

The previous episodes showed the need for an auxiliary activity to reinforce the concepts of angle measure and complementary angles. This episode is the result of a series of discussions derived from a designed activity that allowed students to explore the concept of angle from a dynamic perspective. The activity resulted from the analysis of debriefing sessions and classroom observations. It was incorporated to the experiment by following the iterative design process. The activity consisted on drawing the triangles corresponding to the six ratios (2:5, 5:2, 4:3, 3:4, 5:3, and 3:5) on a foam board template. The template consisted of a section to draw, and a pre-cut, pre-measured fish string attached to the center of the board. The string represented the hypotenuse (fixed measure) of the triangles to build. Students received a protractor and a ruler.

Previously, the teacher asked students if by rotating a triangle from a ratio of 5:3 to convert it to one with a ratio of 3:5 (height to length), the hypotenuse’s length would change.
Most of the students responded that it (rotation of the triangle) does not affect. This episode starts when the teacher simulated the construction of one of the triangles (5:3 ratio) and asked the students about what they need to consider in order to build the correspondent triangle.

The teacher simulated the foam board at the white board and started rotating the string. He asked group 3 (5:3 ratio) about when should he stop rotating the string in order to represent the triangle with the 5:3 ratio; participant D2 suggested to use the protractor and measure the angles alpha and beta. The teacher then asked about another way to call the angle alpha. From group 2 participant D2 responded ‘elevation’ and the teacher asked him about the target for the angle of elevation. Participant D2 stated 59° angle. The teacher asked them about what they need to consider in order finish the 5:3 triangle. Participant I1 suggested ‘to measure’ the angle of depression which should be 31°. After constructing the triangle, the teacher asked students for a way to demonstrate that the triangle constructed corresponded to the 5:3 ratio.

L1: Using the angles!
K3: With the protractor!

Students’ rationale showed evidence they constructed knowledge related to co-variation.

While at the ratios’ table activity, students were asked to measure angles derived from the given ratios; during this activity, they had to rely on predetermined ratios to construct triangles with the corresponded angles. This opened the conversation to whether angle defines ratio or vice versa.

Teacher: How do I know that’s a 5:3 [triangle]? What about the angles?
I1: For 5:3 our elevation always is 59 [degrees] and then depression always will be 31 [degrees] so it just got to make sure…
Teacher: This is interesting because this angle being 59 [degrees] does it matter how far then I would pull that string?
I1: No! As long as the depression would be [31°].
Teacher: So as long as the depression stays at…
I2: 31 [degrees]
Teacher: And the angle of elevation stays at…
The previous discussion provides some evidence about students’ construction of knowledge of ratios and proportions. As the students worked with their foam boards and constructed their triangles important geometrical aspects emerged. Figure 4.10 shows students’ work about geometric representations of ratios.

4.3.1 Negative Ratios

A theme that emerged from the foam board activity was the representation of negative ratios. In Group 2, participant A3 made his argument about naming a negative relation (e.g. -2/5) in the specific case where a triangle was built on what traditionally would be considered the second quadrant of the Cartesian plane. It is important to clarify that the foam board was not marked with neither the ‘x’ nor ‘y’ axes.

Teacher: What is it that you are thinking about that makes it negative two over five?
A3: Because it’s going down this way.
Teacher: What is that? What do you symbolizing?
A3: It’s going that way.
Participant A3 referred to the direction of a triangle’s slope with a height of 2 and a length of 5 (see Figure 4.11). He indicated that a triangle constructed by a correspondence of 2:5 ratio from height to length located on quadrant II which entails a height of 2 (‘y’ axis) positive units and a length of 5 (‘x’ axis) negative units.

By considering important to correlate as a negative object, the diagonal built on that section of the foam board participant A3 showed his reliance on the Cartesian plane.

Students finished the construction of their triangles and the teacher asked them to complete the last column on the ratios’ table. The column corresponded to the mathematical computation of the height length for each of the six triangles; in other words, they were asked to divide the value of the height over length.

Teacher: What you are going to do next is... you are going to take the ratio of the height to the length. Now what should this value be?
C1: It should be 2 over 5, because if our ratios are correct then…they should be equal the previous ratios that we have.

Participant C1’s rationale provided some evidence for his understanding about co-variation and reliance on multiple representations to demonstrate accuracy on mathematic computations. Participant C1 explained that the computation resulted from dividing the height over length of the triangle constructed from the 2:5 ratio should be equal to the division of 2 over 5 (the actual ratio).

The conversation then moved to ‘different’ types of angles of elevation. By using group’s 3 work as an example, the teacher asked students if they could see different angles of elevation, and more specifically if they deal with angles of elevation greater than 90°.

Teacher: As you guys get to see everybody else’s designs… did you have different kinds or different angles of elevation? [see Figure 4.12]

A2: I don’t think so…

Teacher: Is it possible that you guys had or some groups were dealing with angles of elevation greater than 90 [degrees]?

The teacher asked the students to respond to the question (are you dealing with angles of elevation greater than 90 degrees?) on the online blog. Four out of twenty-three participants
responded. Responses reflected their position on whether or not accepting the possibility to have angles of elevation greater than 90°. Table 4.7 summarizes participants’ responses.

**Table 4.7: Participant Responses to the Online Blog.**

<table>
<thead>
<tr>
<th>Participant</th>
<th>Yes</th>
<th>No</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>D2</td>
<td></td>
<td>X</td>
<td>“… it can make another type of triangle”</td>
</tr>
<tr>
<td>I1</td>
<td>X</td>
<td></td>
<td>“… it can be possible but in our project with the bridge it is not possible...The cable wont connect to the tower”</td>
</tr>
<tr>
<td>D1</td>
<td>X</td>
<td></td>
<td>“… because the type of bridge we are building… If our cantilever type bridge had arms extending outward then it would be possible”</td>
</tr>
<tr>
<td>D3</td>
<td>X</td>
<td></td>
<td>“… for this kind of project you can't because it won’t correspond with the tower. The cable won’t connect to the tower the way it is supposed to”</td>
</tr>
</tbody>
</table>

The discussion continued next day and the teacher asked the students the same question: would it be possible to have an angle of elevation greater than 90°? The teacher opened the conversation and groups gave their explanations.

Teacher: Angles of elevation, is it possible for us to have an angle of elevation greater than 90°?
A3: Uh… No!
Teacher: Not possible?
C1: Yes!
A2: Well I guess it wouldn’t be possible because then you know, you have your 90° mark right there [he points above his head] well either side is going to be decreasing because you are decreasing all this… either side you know, of the 90°.
Teacher: So then you are doing this as far as what’s moving [head movement simulating the counter clockwise trajectory]
A2: The person’s angle of elevation.
Teacher: So once it reaches 90…
A2: It’s going to go down
Teacher: It’s going to decrease
A2: Yes sir!
Participant A2 referred to the situation where a person’s standing point is at the vertex of the angle. According to A2, by standing up at this point the angle measurement begins at the right side, reaches the vertical (90° angle) and start going down on the left side. Figure 4.13 shows participant’s (A2) rationale on angle measure and standing point reference.

![Figure 4.13: Participant A2’s Angle Measure: Rationale and Reference.](image)

The discussion continued and the teacher utilized another example to model the situation. The example referred to having one of the students pitching a cable to someone else at the top of a tower simulating a 3:5 ratio (see Figure 4.14).
The teacher asked them about the angle of elevation that corresponded to that ratio.

Teacher: What did you have for an angle of elevation for a 3:5 ratio?
A3: Was that the 32?
Teacher: That’s the famous 31 - 32, and then as you’re increasing your tower your ratio, let’s say go and jump over to the 5:3, so now we are there, what is the angle of elevation?
A1: 5:3? 59 [degrees]
Teacher: Fifty nine! So then as this angle continues to increase… can this angle of elevation there… be greater than 90 the way it seems to be increasing there?
A1: Yes!
C1: Well, do they have to be right triangles? One of those angles definitely has to be 90 [degrees]. Because I think if elevation goes over 90 it wouldn’t be, it wouldn’t look like a right triangle anymore.

Participant’s C1 interpretation of angles and his reliance to right triangles could be closely linked to the consideration of the standing point at the right side of the angle vertex (quadrant I). Figure 4.15 shows participant’s C1 position of the standing point.
The discussion continued and opinions about angles of elevation greater than 90 degrees were divided among the class. Two different interpretations of angles emerged from students’ reference to the standing point. However, majority of students related the concept of elevation to those angles embedded between 1 and 90 degrees; and had reliance to the angle vertex as the standing point. Likewise, students demonstrated a conflict when categorizing angles greater than 90°. Students presented their foam boards at an improvised gallery and had the opportunity to see each other group’s configuration. Students revealed enthusiasm during the presentations and were discussing ‘where’ the other groups placed their triangles.

The activity continued the next day and the teacher pointed out what participant C1 mentioned the day before. The teacher explained that in order to throw out the cable at an angle greater than 90 degrees, it would be necessary to do it backwards, implying that a person throwing the cable would standing at the vertex of the angle, or what could be considered the origin of the Cartesian Plane. After group discussion, the teacher presented a GeoGebra file that
included the six groups’ configurations of triangles, which opened the conversation to angles and position.

Teacher: All right so we start with this [group 6]. So this is group’s 6 [see Figure 4.16].

Teacher: So now if we start talking about those angles of elevation doing this, these three [see Figure 4.17] were what? As far of the measures in relation to 90°, these three are …

C1: Under 90 [degrees]

Teacher: Under 90? And then these [see Figure 4.18]?
C1: Greater than 90 [degrees]
A3: Oh! Over 90 [degrees]
Teacher: So then here goes, here is group’s I [Figure 4.19], and what about them?

Teacher: Higher or lower? All of them lower than 90 [degrees]? [Some students responded higher and other lower] … So all of these angles of elevation did they go passed 90 degrees?
C1: No!
Teacher: So then we go to another group… here is group 2 [see Figure 4.20] and this is pretty awesome I mean. How many cables were pitched that were less than 90 degrees?
A1: Five!
Teacher: Five, and then this one?
A1: It’s over 90 degrees!
Teacher: It went over 90 degrees, ok, so now we are seeing that relation.
Teacher: Now here comes group 3 [see Figure 4.21].

Teacher: Again, you can see the overlapping and some of them appear here in orange, some cables pitch this way [under 90°] some pitch this way [over 90°] but you see again that angle of elevation both, less than 90 and over here [left side] would be greater than 90. Well, then comes group 4 [see Figure 4.22].
A3: Oh this is cool!
D1: All those are over 180 right?
Teacher: Keep those thoughts [D1]; [Participant] C1, over here please.
C1: We are trying to point out something uh… something really cool, besides from the fact that this was cool, you know how, I guess it was [participant] A2 right? You know how he did those triangles underneath on this side. He was pitching all cables just from the reference point, he did throw them into the ground, but anyway; you see how all ones make like a 180 line? Right there, yeah! It’s a little bit off but more or less makes a 180 line. I guess would you say that those are like under 90 or how would you call them?
D1: I would say over… they just flipped.
C1: Yeah, those will be over 180 [degrees].
D1: All of them are over 180 [degrees].
Teacher: Okay well, the ones that are up here for that group. For group 4 are actually these [see Figure 4.23].
Teacher: Are these over 180?
Group: Yes! [Incorrect answer]
D1: And if you flip it upside down…
Teacher: Oh! Wait, let’s establish, where is 180 [degrees]?
A4: No, they are over 90. But the ones on the bottom are the ones over 180 [degrees]. [Participants A1 and C1 agreed]
Teacher: Ok, ok, so let’s come back, let’s come back 180 [degrees] would be where?
D1: The line!
C1: The floor!
Teacher: So then, are these [see Figure 4.24] over 180?
C1: Yes!

A4: Those are going all the way around…
C1: Those are over 180!
Teacher: So if you continue pitching, you continue going and pitching cables … These would be then, where would they been? As far as a degree measure, could you give me a degree measure?

C1: Yes!

A1: One…No! two-thirty [230 degrees] because it’s half, yeah the half is 90 [degrees]

Teacher: Come and show me… because I’m guessing what you guys are telling me.

A1: Because the half is right here [see Figure 4.25].

Figure 4.25: Reference Position I.

A1: Is 90 [degrees], an estimation should be like 80, 70. I guess right here so it’s going to be 180 [degrees] plus 70 [degrees] is…

Teacher: Like 250?

A1: [head movement indicating she agreed]

Participant’s A1 rationale on angle measure reflects a nontraditional form of measure. She started measuring the angle at an imaginary point (at 90°) and maintained the counter clockwise trajectory. She explained that from the starting point to the other extreme point (at the 270° mark) it would be considered and angle of 180°; then she added the estimated value of the angle (approximately 80° or 70°). Figure 4.26 shows participant’s A1 method to estimate an angle measurement.
By the time participant A1 conducted her explanation, participant C1 joined the discussion.

C1: No, wait! You can look at it like quadrants right? So you are doing the circle; has four quadrants. The first quadrant where a lot of the purple is, those are under 90 [degrees]; the second quadrant, those are over 90 [degrees]; third quadrant which has zero triangles that’s over 180 [degrees]; and the fourth quadrant is over 270 [degrees] all of those angles are going to be a little bit over 270 right [degrees]?

D1: That make sense, I like that one!

Teacher: [Does] that make sense? So the fact that he did it help out, this whole quadrant idea?

Group: Yes!

Participant’s C1 reliance on the Cartesian plane was evident. Even when the diagram presented did not include edges of reference. Participant C1 made use of his prior knowledge on quadrants to classify the triangles by their angle value. The teacher asked the students if this ‘quadrant’ idea served as reference to identify where the angles fell and started a review.

Teacher: Let’s say if we were talking about quadrants how would you do it? So we pull in the idea of a quadrant and you said these are all what? [Table 4.8 summarizes students’ responses]
After providing a review for the class, the teacher presented group’s 5 configuration, which displayed triangles on quadrants II, and III. Reflections about sectors emerged.

**Table 4.8: Students’ Responses.**

<table>
<thead>
<tr>
<th>Angles</th>
<th>Response and Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1:</td>
<td>Those are all under 90 [degrees]</td>
</tr>
<tr>
<td>C1:</td>
<td>Over 90 [degrees]</td>
</tr>
<tr>
<td></td>
<td>[there are no triangles in this section]</td>
</tr>
<tr>
<td>C1:</td>
<td>Over 180 [degrees].</td>
</tr>
<tr>
<td>Teacher:</td>
<td>But…</td>
</tr>
<tr>
<td>C1:</td>
<td>Under 270 [degrees].</td>
</tr>
<tr>
<td>A4:</td>
<td>Less than 270 [degrees].</td>
</tr>
<tr>
<td>Group:</td>
<td>Over 270 [degrees], under 360 [degrees].</td>
</tr>
</tbody>
</table>

Teacher: Okay, so and then we have one more group [see Figure 4.27] which I think they didn’t leave our quadrants lonely so that was pretty cool.
D1: Is there another word for it, other than quadrants?
C1: Sectors, that’s how you call it, because a quadrant is like a plane.

The previous example shows the importance of geometric representations of angle and students’ reliance to the Cartesian plane as a tool that helped them to make sense of mathematical concepts such as direction and position. Once students showed an understanding of angles in terms of position and direction, the teacher used a GeoGebra tool to show students the ‘x’ and ‘y’ axes; likewise, he overlapped a circle, which represented the angles’ trajectory. The file enclosed the six groups’ configurations. Figure 4.28 shows the final diagram. Students’ response to the visual reflected their awareness to what they were doing.
4.3.2 Summary of Episode III

Before the foam board activity students showed a weak understanding about angle in terms of direction and position. Data from this episode shows that there is a conflict not only on how to measure angles, but also on how to categorize them. Misunderstanding of the concepts of ‘angle of elevation’ and ‘angle of depression’ was evident. Students’ conceptions about ‘elevation’ reflected a correspondence to ‘going up’ and created a conflict when having to figure out angles of elevation greater than 90°. There is no evidence of students’ reliance on complementary angles. Despite this, the foam board activity enabled students to think about position and direction (two important considerations for angle measure) and to rely on the Cartesian plane. The ‘cable’ example, challenged students when instead of thinking about the supplementary angle, they changed the position and the direction of it.

This episode provides evidence about the importance of multiple representations, and what students can learn by making use of them. Through the foam board students had the opportunity to transfer the content of a table to a dynamic form. Students’ representation of ratios in the form of triangles represented a step on the learning trajectory ‘from static to dynamic.’ Due to exposure to multiple representations (i.e. tables, geometric manipulatives and dynamic contexts), students evidenced a strong reliance to mathematical concepts. Students’ prior knowledge about quadrants solidified their need for a reference in terms of position and direction. During this episode a reference point for angle measure was established (at the vertex
of the angle). Before the foam board, students showed how their preconceptions about the Cartesian plane limited their reasoning on angle measure to quadrant I. After the foam board, students showed a better spatial sense by thinking about the Cartesian plane in terms of direction and position.

Through visual representation using the GeoGebra file, students were more able to see symmetry and conventions about angle’s measure reference point emerged. This episode showed how students struggled when they worked with angles. Moreover, it illustrated a different learning trajectory followed by students. Contrary to direct instruction, students worked during an entire week on activities that helped them to construct their knowledge on angle measure and used the Cartesian plane in as a tool to conceptualize direction and position in dynamic contexts.

Instead of direct instruction, students were exposed to mathematical contexts where multiple representations played an important role on the construction of knowledge. They were able to observe, touch, play, create, tabulate, graph, measure, analyze, and connect geometric and algebraic concepts.

4.3.3 Learning Trajectory for Episode III

The previous episode showed the learning trajectory followed by students in order to obtain an understanding of geometric representations of ratios. During this episode, students were exposed to a new activity, The Foamboard, whose aim was to help students understand angle in terms of position and direction. However, while working with angle measure, students encountered a series of obstacles that had important implications on the learning trajectory. Students encountered difficulties when defining the starting point to measure an angle. In addition, students struggled with negative ratios based on their prior knowledge of the Cartesian plane. Other obstacles emerged when students had to define angles in terms of elevation and/or
depression. Students had to reconcile the possibility of angles of elevation greater than 90 degrees to understand the dynamic representation of ratios, thereby making connections to the Cartesian plane and developing their spatial sense. Figure 4.29 shows an interpretation and summary of the learning trajectory for the episode.

![Learning Trajectory Diagram]

**Figure 4.29: A Summary of the Learning Trajectory for Episode III.**

### 4.4 Episode IV

On the past episode, students reconciled the possibility of having angles of elevation greater than 90°. By referring to the Cartesian plane, students were able to classify angles of elevation using quadrants as reference. This type of reasoning opened the discussion to directionality. Students’ work and discussions are presented to show the learning trajectory for the conceptualization (in terms of direction and position) of angle measure. This episode begun when the teacher presents the GeoGebra file that showed all six group’s configurations together. He asked students to observe the figure and tell him what it looks like (see Figure 4.30).
Participant D2 pointed out that “we have all different places to put our triangles… But it looks like if we all decided to put them in one place.” Participant D2 referred to quadrant I due to it was at this region most groups placed their triangles (17 out of 36). The teacher conducted a briefly review about the correspondance of angles to quadrants and went back to the question about the possibility of having angles of elevation greater than 90. The teacher asked those students who responded the blog to make their point.

A2: I have said that at first I was wrong because like… and saying that no, there’s only up to 90 [degrees] because then you know its like … it’s like you have your tower over here [looking up].

Teacher: So you has being stuborning the fact that if you’re looking straight up, that’s already how many degrees?

Majority: 90

L1: Is zero isn’t it?

Teacher: Well… zero would start out where?

L1: At the origin!

The previous example shows the importance of visual representations for the teaching of angles. Through these representations students have the opportunity to reflect on their
geometrical arguments. For instance, participant A2 pointed out that by looking straight up the angle of elevation would be $90^\circ$. His rationale evidenced consideration of the standing point at the vertex of the angle. Likewise, participant’s L1 response suggests a different understanding regarding angle and position. Her response could be interpreted as she is considering that by looking straight up entails to be stanced at the origin of the Cartesian plane. The ‘origin’ represents the geometric center of the plane and serves as a reference point for position and direction. Colloquially, the origin is known as the ‘zero’ and could be represented by the ordered pair $(0,0)$. Participant’s L1 rationale shows a weak understanding on the implications for angle measure, especially for the standing point. The discussion continued and the teacher asked students about the construction of towers using predetermined angles.

Teacher: Where would we actually start the construction of these towers? Let’s say in $1^\circ$ [degree], where would $1^\circ$ [degree] look like?
C1: You would be able to see it in the right [quadrants I or IV]
Teacher: Would you be able to see it?
D1: No, not on this [visual limitations]
Teacher: Ok, so where? In what quadrant would that tower be?
Group: First [quadrant I]
Teacher: With $1^\circ \ldots$ quadrant I, would it be anywhere up here [pointing at $45^\circ$]? D1: No, it would be very low, low, low, low, low
Teacher: Do I cross [the ‘x’ axis] [see Figure 4.31]?
During the previous classroom discussion implications about directionality emerged. When the teacher asked students (in terms of ‘negative degrees’) about how many degrees would be if he passes the ‘x’ axis in 1° degree (clockwise direction), participant L1 questioned if would not be negative 1°. Her argument for the negative value of the angle was based on reversing the direction for the measurement. In other words, instead of start measuring the angle by following the traditional trajectory of angle measure (counter clockwise) she followed the clockwise direction. Although her rationale shows a correspondence of the negative value to the direction of the measure, the concept of angle is not fully developed. Establishing a measurement direction (passing the ‘x’ axis, clockwise) would not affect the magnitude of the measurement. The discussion continued and implications about geometry emerged.

Participant’s D1’s thinking showed evidence of a singular understanding regarding the position of the angle in the Cartesian plane. He explained that the angle below the ‘x’ axis could be a reflection of the one at quadrant I. His response revealed exposure to geometry, more
especific to symmetry concepts. Resulting from participants’ L1 and D1 responses the teacher made a pause and asked students to recapitulate what they previously have established concerning the Cartesian plane and the position of the angles. Students agreed on the correspondance of angles from $0^\circ$ to $90^\circ$ for quadrant I; from $90^\circ$ to $180^\circ$ for quadrant II; from $180^\circ$ to $270^\circ$ for quadrant III; and from $270^\circ$ to $360^\circ$ for quadrant IV.

Review about angles and quadrants opened the conversation about segments of 90 degrees that resulted from important considerations.

Teacher: All right! So then remember that. You went from 180, 270, and then from 270 through 360 [degrees]. So if I go back in this direction [counter clockwise], then what?

L1: Couldn’t it just start as 90 [degrees] too?

Teacher: But from here to here, it’s 90 [from $0^\circ$ to $90^\circ$, counter clockwise]

L1: So would it be 360 to 270 [degrees, clockwise]?

Teacher: 270 [degrees] go in this way [counter clockwise starting at $0^\circ$]?

L1: No, 360 and then 270 [clockwise]

Teacher: This way [clockwise]?

L1: Yes!

Teacher: So what you were talking about those degrees changing direction then?

Participant’s L1 reflections about directionality showed an inconsistency when pointed out that an angle of $1^\circ$ could be named as a negative $359^\circ$ angle if constructed by crossing the ‘x’ axis (clockwise) to quadrant IV. Additionally she did not rely on negative angles when referred to the action of going from the $360^\circ$ to $270^\circ$ angle. The discussion continued and different opinions arised.

A2: I was going to say that I don’t think all of us have the same point of view on how this is layered out. So that’s how we all are having a different… That’s why we all have different ideas and… Well, are viewing it differently than others because some others view it as like towers like that. Others’ view it’s like, you know the angles and stuff…
Reflections on angle as a measure and as an object emerged. Participant A2 pointed out that differences among the interpretation of angle (in terms of position and direction) could result from the different perspectives students have. He mentioned that while some students are viewing it (the angle) as towers (object) other view it like angles (turn).

A2: So, yeah, I just point it out because it’s like I know that why we all are having like different answers. It’s because of how we’re seeing it.

Teacher: So as far as what [participant] A2 is bringing up, do you guys…as far as the towers go, which onces are the towers?

L1: The pointing

D1: Is the outside…the outside of the triangle

Teacher: What do you mean outside of the triangle. Let’s try to be more specific and that’s actually good that you pointed that out. Which would be considered the tower?

A3: The 90 degree…

L1: Not the one that’s connected to the origin or the nine… [ninety] [referring to the hypotenuse]. The one that’s over here in the 90 [degrees], but the one on the top.

Teacher: More specific?

L1: I don’t know how to call that angle. It’s like…

Students showed difficulty to describe the segments representing the towers on the formed triangles. Despite their exposure to the Cartesian plane, students continued struggling with quadrats as reference for position. The teacher moved to ask them to identify the segments that represented the cables (of the bridge).

Teacher: Do we know which one the cable is?

D1: The one touching the origin

A2: The hypotenuse

Unlike the towers, students were able to rapidly identify which ones were the cables. Majority of the students named hypotenuses as the ones attached to the origin. Thinking about hypotenuses would fall on the acceptance of working merely with right triangles. The teacher continued with a reflection of towers and bases.
Teacher: All right, so if the hypotenuse is \([the]\) cable, which ones of these would be the towers then? How would you explain to me what the towers are?

D1: The sides!

A3: So the towers are always vertical

D1: Whatever is parallel to the ‘y’ axis is ok

Teacher: So is this a tower [see Figure 4.32]?

\[\text{Figure 3.32: Tower Representation I.}\]

Group: Yes!

Teacher: Is this a tower [see Figure 4.33]?

Group: Yes!

\[\text{Figure 4.33: Tower Representation II.}\]

Teacher: Is this a tower?
Teacher: What are these [see Figure 4.34]?
Group: Bases!

![Figure 4.34: Reference Segments.](image)

Teacher: Those are the bases. So then talking about what [participant] D1 brought up, the bases are then what? How could you explain?
D1: They’re always going to be on the ‘x’ axis
A3: They’re horizontal
Teacher: They’re horizontal or on the…
Group: ‘x’ axis
Teacher: So your bases, we relate to the ‘x’ axis, the towers…
D1: ‘y’ axis

After the discussion, using the Cartesian plane as a reference for position become more evident; at least in the geometric aspect where students utilized it to make sense of position.

Teacher: Ok, they’re [the towers] parallel to the ‘y’ axis. So then may I start talking about… What if we construct a tower with an angle of the elevation of 1°… I’m interested in that whole 1°. One degree and one direction and if is 1°…
A3: You can go either way…
Teacher: Ok, what do you mean?
A3: You can go either way. You can technically start from anywhere because it’s a circle [see Figure 4.35].
Teacher: Ok, so then where would I start from?
A3: Anywhere!
Teacher: Tell me!
A3: I don’t know, you can start form anywhere!
Teacher: Angle of elevation of 1 degree; so give me a place to start or if you want to point out.
A3: Ok, the fourth quadrant!

Despite of the constant reminder on quadrants and their corresponded angles, participant’s A3 conceptualization of the Cartesian plane differed from the rest of the class. His spatial sense enabled him to see the Cartesian plane as a ‘reference’ for position and direction; and not as a static context where the objects remain on the quadrants. His argument about “You can go either way. You can technically start from anywhere because it’s a circle” showed evidence of his spatial reasoning.

Once students established the towers and the bases in the diagram, they were asked to create a different representation of them. Students were provided with a template (see Figure 4.36) including the six groups’ visual representation of the foam boards and a numeric scale that represented the angles from 0° to 360°; the scale went from intervals of five degrees.
Teacher: Now, what I want you to do now is... Let’s go ahead and start comparing these heights...the first height we’re going to go ahead and place it to its corresponding degree... so this height right here [see Figure 4.37].

Participant L1 referred to the 23° angle as an estimation of position on the scale. She noticed the angle scale increased by 5 degrees each time, going from 20 to 25 degrees and, then,
from 25 to 30 degrees. The teacher asked students to measure all the towers and to represent them at the scale. The teacher also recalled the statement established by the students during a classroom discussion. The statement indicated the position of the towers regarding their corresponded angle. The statement read, “For towers built between 0° and 180° ‘above the x axis;’ for towers built between 180° and 360° ‘below the x axis’.”

Students worked on their diagrams and the teacher showed them how to project the first tower (see Figure 4.38). The teacher asked students to do the same with the rest of the towers and bring their completed diagram the next day.

![Figure 4.38: Projection of the First Tower.](image)

The activity continued the next day and some students showed their configurations and methods. Participant N1 brought her diagram without any mapped tower. However, her work showed that she constructed a table with the towers’ measurements and their corresponded angles (see Figure 4.39).
Reliance on tables reveals participant’s N1 reasoning about covariation and the use of different representations to make sense of a mathematical object. Students continued working on their diagrams. Different methods emerged and the teacher opened the discussion about those methods.

Teacher: What I would like to discuss right now is methods. What kind of methods did you use in other than your type of native methods?... did anybody else do anything different?

A3: [I did]

Teacher: Come up! Show us!

A3: Ok, so I did this and made my life easier...can I borrow this ruler ... I put the thing in whatever ... and I was like “Oh, I’ll just put this on over here” [see Figure 4.40].
Group: Oh!
Teacher: I see what you’re saying, so… Let me borrow one [ruler]…did you guys see what [participant] A3 did?
Group: Yeah!
Teacher: See, he measured the tower and then say this one the 23 [degree] it just pretty much map it out like this [see Figure 4.41 ].

Participant’s A3 method shows his reliance to indirect measurement by projecting the towers (geometrically). Students finished their projections and the teacher gave the instructions for the next activity.
Teacher: You’re going to go ahead and create pretty much the same mapping that we had with the heights. But now instead of with those heights, let’s go ahead and take a look at those bases… In other words, for the 23° angle… Which is the base? … From where to where?

D1: Oh, the origin to… where the vertical line is.

Students received a new template of the angle scale for the bases. This time the template did not include the configuration of the triangles. Students had to figure out how to work with the new template. Majority of students overlapped them in a way that the two angle scales coincided. This method would not work since geometrically in order to ‘project’ the bases it would be necessarily to rotate the angle scale (vertically). Figure 4.42 shows participants’ N1 and A4 incorrect overlapping of the templates while Figure 4.43 shows participants’ D1 and C1 correct overlapping of templates.

**Figure 4.42: Incorrect Overlap.**

**Figure 4.43: Correct Overlap.**
The teacher noticed the way students were overlapping the templates to project the bases and called for a reflection about the ‘axis’ as reference. He mentioned that when comparing the position of the towers some of them would go ‘above’ and others ‘below’ the ‘x’ axis.

C1: Mr. S, do we supposed to do this down?
Teacher: When we were comparing the position of the heights, all of these heights were where?
A3: Are you going to say anything about the ‘y’ axis and one side goes up and the other side goes down?
Teacher: Now as far as position goes…
A3: The right side goes up and the left side goes down.
Teacher: If you’re talking in the sense of “up” everything from where to where?
L1: Everything from… zero to 90 [degrees].

Participant L1 referred to bases in the section delimited (counter-clockwise trajectory) by the angles 0° and 90°. This bases in this section would be placed above on the angle scale (see Figure 4.44).
Despite the bases located on the right to the ‘y’ axis corresponded to the triangles formed either on quadrant I or IV, participant L1 only made reference to those on quadrant I by delimiting the section by the angles 0° and 90°.

C1: 90 to 270 [pointing the right side with respect the ‘y’ axis].

Participant C1 referred to the bases in the section delimited by the angles 90° and 270° as the bases to be placed above the angle scale. His response brings important implications on direction. He referred to the section on the right to the ‘y’ axis which entails the sections delimited by quadrants I and IV. Additionally, he geometrically delimited this section by going from 90° to 270° using a clockwise trajectory. Figure 4.45 shows the section (traditionally) delimited by the angles 90° and 270°. Figure 4.46 shows the section considered by participant C1.

![Figure 4.45: Section delimited by the angles 90° and 270°.](image)
Once the group agreed on the position of the bases located on the ‘right’ to the ‘y’ axis the teacher moved to the bases on the ‘left’ section. Students began projecting their bases. The next day the teacher conducted a discussion. The teacher projected participant’s D1 diagram of the bases. During this discussion, important ideas about position emerged (see Figure 4.47).

**Figure 4.46: Section Referred by C1.**

**Figure 4.47: Participant’s D1 Diagram of the Bases.**

D1: Ignore that string, the one on the left side [circled on Figure 4.47].

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Teacher: And why would we ignore this? I guess something that one is here, but why would we ignore it?
D1: Because it’s wrong!
Teacher: How do you know?
D1: Because it’s something we… It’s on the wrong side…

Participant D1’s awareness of position and direction was evident. Possibly, previous exposure to angle measure and trajectories contributed to his reasoning. The classroom discussion continued and the teacher compared and made observations on both diagrams, towers and bases. Likewise, he asked participant D2 for an important point she brought up. She made a point on continuity and visualized ‘the wave’, ‘the curve’ relying on the geometric aspect of function behavior.

Teacher: [Participant] D2, you said something about connecting the dots, what were you saying?
D2: Oh! that if you connect the dots from the bases or the heights until the end up and down… It makes the circle, but they’re like…
Teacher: Like what did you say? Say it louder, so they could hear back here
D2: Like a wave curve

The teacher opened up the discussion and asked the students about what they would need to do in order to create a ‘wave’ or a ‘curve.’ Figure 4.48 shows representations of tower and base mappings.

A3: You have to put in every single one
D1: Yeah, you have to put every single possible angle
The teacher utilized participants’ P1 and D1 diagrams of the towers and bases (respectively) to summarize where the towers and bases would belong to on the angle’s scale. By using these geometric representations, students had the opportunity to visualize the correspondence between lengths and angles. Through the diagrams students were making connections between their triangles’ representations on the foam board; their position and direction on the Cartesian plane; and the corresponding ‘mapping’ on the angle scale. Through this activity students were able to compare the differences among projections of both, the towers and bases. Being able to project them by using ‘sections’ that were delimited by the angles on the scale entails the ability to navigate from different representations of angles. Tables 4.9 and 4.10 present a summary of the classroom discussions on the projections of towers and bases respectively.
**Table 4.9: Projections of the Towers.**

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Constructed Towers</th>
<th>Projection on the Angle Scale</th>
</tr>
</thead>
<tbody>
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<td>I</td>
<td><img src="image1" alt="Quadrant I" /></td>
<td><img src="image2" alt="Quadrant I Projection" /></td>
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<td>II</td>
<td><img src="image3" alt="Quadrant II" /></td>
<td><img src="image4" alt="Quadrant II Projection" /></td>
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<tr>
<td>III</td>
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<td>IV</td>
<td><img src="image7" alt="Quadrant IV" /></td>
<td><img src="image8" alt="Quadrant IV Projection" /></td>
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<tr>
<td>Quadrant</td>
<td>Triangle Bases</td>
<td>Projection on the Angle Scale</td>
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<td>Quadrant I</td>
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<td>Quadrant II</td>
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<td>Quadrant III</td>
<td><img src="Image5" alt="Diagram" /></td>
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<td>Quadrant IV</td>
<td><img src="Image7" alt="Diagram" /></td>
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By using both diagrams, students showed a strong sense of direction when they had to figure out the ‘correct’ position of the towers and bases. For instance, participant L1 (who
previously struggled determining the starting point for angle measure, demonstrated a strong understanding about the position and direction of the towers and bases when projecting them on the angle scales. She opened the discussion for positive and negative correspondence.

L1: On first quadrant you’re going [to have] positive-posititve, and when you go to the other second quadrant, there’s positive and a negative.

Teacher: Ok, so let’s take that into consideration now. So then in the first quadrant, it’s… What do you mean by positive-positive?

L1: That the right… Like the rise and the run, to positive number, but on the other side…since you’re going opposite direction, it’s a positive up and then a negative going…

4.4.1 Summary of Episode IV

Previously to this episode, students reconciled the fact of angles of elevation greater than 90° but continued struggling with direction and position. During this episode, students were exposed to meaningful activities that enabled them to think about angle by considering position and direction. During the foam board activity, data showed students’ tendency to work on the first quadrant. However, during this episode important implications about direction and position emerged. In terms of position, students’ reference point for angle measure varied. Some students considered the origin of the Cartesian plane as the reference point while others considered the opposite side to the angle vertex as the standing point to measure an angle. Additionally, in terms of direction the theme of negative angles emerged. Some students considered as ‘negatives’ the angles measured when they followed the clockwise trajectory. Besides, quadrants were mostly utilized as reference for position but not for direction. During this episode, students relied on visual representations to make sense of correspondences between angles and lengths. They made use of the Cartesian plane as a reference for position and direction. Exposure to multiple representations played an important role to understand the geometric aspect of angles. During the activities presented in this episode students worked with tables and diagrams that helped them to
understand angle. Geometric representations of angles enabled students to comprehend angle in terms of direction and position.

4.4.2 Learning Trajectory for Episode IV

The previous episode highlighted the learning trajectory students followed to understand connections between and among different representations. The landmarks show the importance of reconciling the possibility of angles of elevation greater than 90 degrees, projections of geometric representations within a different context, and the role of multiple representations for connecting those mathematical concepts. Students encountered diverse obstacles when approaching these landmarks. For instance, students continued struggling with direction and position when projecting heights and lengths corresponding to the geometric representations of ratios. In addition, students struggled with the concept of negative angles. In order to understand projections as a different representation of ratios, students had to reconcile position and direction as two central concepts with understanding angle measure in order to make connections among the different representations that exemplify them. Figure 4.49 shows an interpretation and summary of the learning trajectory for the episode.
4.5 **Episode V**

In the previous episode, students reconciled position and direction as important facts for angle measure. They relied on the Cartesian plane for reference and visual representation. The following episode presents closure of the experiment and students’ reflections about angle, position, and directionality. The teacher then asked students to build a GeoGebra file. The activity consisted on creating a geometric tool to represent the tangent. Students had to build a circle with a radius of 9 units and an inscribed triangle with a hypotenuse’s length equal to the radius. The teacher guided students during the process for the file construction. Upon the construction of the file, the teacher asked students to build a ‘tower’ at nine (9) units (horizontally) from the center of the circle, and to connect the inscribed triangle to the new tower. He asked students about what they need to do in order to connect the triangle to the ‘outside’ tower. Students’ reasoning on projections emerged.
Teacher: We want to go ahead and construct a tower with a base at point D… Now what’s going to happen?

C1: … you have to make it longer so that it goes up longer.

Teacher: In other words the tower is going to be where? In relation to the circle.

C1: Outside!

Teacher: So what C1 was referring to is that you are going to have to take the cable and extend it.

Students built their GeoGebra files and the teacher asked them to construct a tower attached to the cable with an angle of elevation of 40 degrees. One student, A3 built his file rapidly. The teacher surprised called him to the front and asked him to reconstruct the file and project what he did on the board. Figure 4.50 shows participant’s A3 work.

![Figure 4: Participant A3’s Work.](image)

Participant A3 showed his work and students payed attention to the explanation. The teacher asked him if the file would work when changing the angle.

L1: What did you do?
A3: Mathematics!

Teacher: Now [participant] A3 does your design work if I change the angle?
A3: Yes!

Teacher: Yes? Would it actually change the tower height?
A3: Yeah, it works until is undefined.
Participant A3 pointed out an important characteristic of the tower ‘undefined.’ He referred to the situation where the tower is attached to a cable with an angle of elevation of 90 degrees. In which case the tower would overlap the cable.

Teacher: And then what happens when you are doing that? Put it in quadrant 4, what’s happening there? … is the tower existing?
A3: Yes!
Teacher: Ok so you have the tower underneath. And then there… go to quadrant II.
A3: It’s impossible cause […] that line
Teacher: Unless…
A3: Unless I put other line over here, or expand it this way [see Figure 4.51].

![Diagram](image)

**Figure 4.51: Referred Extension for the Cable.**

Participant A3 referred to extend the cable to the other side of the circle. In other words, to extend what it could be called ‘the radius’ and convert it to the diameter of the circle. Participant A3 stood up and created the extended line using the teacher’s computer. Figure 4.52 shows participant’s A3 work.
Students continued working on their GeoGebra files; they were asked to complete them by the next day. The teacher opened the conversation to ratio and proportion by using a geometric approach. Students needed to calculate the height of the new tower by considering any geometric construction in the GeoGebra file.

Teacher: So the question was, what is the height of the tower? … If we pitch this cable beyond this circle and it’s going to attach, if I say I want an angle of elevation of 70°. 

[Figure 4.52 shows the extended cable.] 

[Figure 4.53 shows the section referred to as the ‘Tower’.]
Teacher: What would you do?
A1: Find the...the one with the little one and then...
Teacher: Which one?
A1: The one that’s in the circle...find the height...and then place it on the other line ... many times and then multiply it by [...]
Teacher: No rulers but does this look familiar? As far as calculating this height? What does it look like?
A3: Like triangles
Teacher: Triangles!
A3: The dilation.

Participant’s A1 explanation revealed her reliance on ratio and proportion. This activity evidenced the impact of geometric representations for the understanding of ratios and covariation. In addition, concepts of dilations and similarity emerged during this activity.

Teacher: As far as what you said about the dilations, do we have similar triangles?
A3: Yes!
Teacher: Ok and what you guys are seeing, ... what do you call them C1?
C1: Projections!
L1: Isn’t it pretty much already like the smaller triangle right here already has to build the bigger one?
Teacher: In other words? What do you mean?
L1: You know how we started stacking the colorful triangles at first? Like that smaller triangle is like [the]other triangle ...the big one... because even though they aren’t the same height or length they are using the same angles.

Linking their GeoGebra constructions to the colored triangles revealed the impact of multiple representations in mathematics learning. Navigating from static to dynamic contexts provide students with diverse learning environments. The discussion moved to determine the position and direction of the constructed tower in predetermined angles. The teacher asked students to identify in which quadrant the tower would fell on each situation.

Teacher: What happens to the tower when you go beyond 90 degrees?
A3: It goes the other way!
Teacher: Which way? Ah what is it now?
Female: At the bottom!
Teacher: We are thinking about the same concept of the towers but now this tower is in which… you guys were saying it was negative?
Female: Yes!

Students showed their understandings about directionality and position. The geometric representations in GeoGebra served to visualize position in the Cartesian plane. The teacher moved the angle and asked students about the direction and position of the tower.

Teacher: When my angle enters this specific quadrant which quadrant is this? [see Figure 4.54].

![Figure 4.54: Reference Quadrant.](image)

Students: Two!
Teacher: Where is my tower?
L1: In quadrant IV!
Teacher: … is above or below?
Students: Below!
Teacher: … what happens when my angle goes or enters the third quadrant? It goes back up there! [see Figure 4.55].
Teacher: When I enter quadrant IV, where is my tower? … so is it above or below?
Students: Below!
Teacher: Did you guys experience this with the first wave? The heights way, did we have a situation where we have towers above and below?
Students: Yes!
Teacher: Did you experience this with the second wave? The base waves
A1: Yes!
Teacher: Did we have a situation with towers above and below?
Students: Yes!

The previous discussion evidences the way students made connections between multiple representations such as the templates and the GeoGebra file. In addition, this activity reinforced students’ conceptions about the position and direction of the towers. Subsequently, students were asked to use the GeoGebra file to figure out the way to project the outside-tower’s behavior in a new template. Besides, they were asked to re-draw the projections of the towers and bases of the inscribed triangle.

Teacher: Relating to this… 90 degrees would represent what kind of a tower here?
A3: The maximum!
Teacher: The maximum height because then what happens after 90 degrees?

A3: It starts going down!

Teacher: That’s what I want you guys to keep into consideration when sketching it.

The teacher asked students to take into consideration the GeoGebra file as a visual aid that helps them to understand the behavior of the towers and bases. The results show important implications on direction and position and the positive effect of using GeoGebra as a tool to visualize geometric and dynamic representations. Figure 4.56 summarizes student’s work using GeoGebra.

Figure 4.56: Students’ Work using GeoGebra.

Students continued working with their projections. While walking through the classroom, the teacher asked the teams questions about angles. The questions related to different situations.
Teacher: [to participant A1] These are the bases right? So if you put it at 90 [degrees] what happens to the bases?
A1: Oh...there is no base!
Teacher: Then put it passed 90 [degrees]. What happens to the base? Is it on the same position than this one?
A1: Oh... its bigger,
Teacher: [to participants A1 and C1] As far as at 0 [degrees], does a height exists? Do we have a height?
A1 and C1: No!
Teacher: [to participant C1]Ok, so then at .000000000001 do we have a height?
C1: Yes!
Teacher: [to participant A1]... when do these heights reach their maximum?
A1: At 90 degrees,
Teacher: At 90 degrees and then [to participant P1] when do these heights touch 0 again,
P1: 180 [degrees].
Teacher: And again, feel free to use your construction guys. Now [to participant D1] what happens to the heights when I pass 180.
D1: They go below the X axis!
Teacher: They go, they will go below, they change their position.
D1: 270 is your max!
Teacher: And then 270,
A3: Minimum!
Teacher: ... your minimum according to the wave, yeah? What are you thinking […] minimum [to participant A3]?
A3: Parabolas!

Students described different situations when they interpreted the behavior of the towers and bases. During these discussions, important concepts such as maximums and minimums surged. Reliance on those concepts showed the participants’ A3 and C1 understanding on directionality and position. The discussion about the towers and bases concluded and the teacher asked them to move out to sketch the ‘outside’ tower. Students completed their sketches and submitted their work the next day (last day of the study). Before the closure of the study, I interviewed some participants and asked them questions concerning the behavior of the ‘outside’ tower. Participant A1 revealed an understanding on ratio and proportion. However, she did not
demonstrate a well developed conception of directionality and position. When I asked her about what happen once the cable crosses the 90 degree angle, she immediately affirmed that the cable never crosses the line [(90 degrees) and consequently it could not be attached to the ‘line’ due to it is not on the same side.

Researcher: Ok, so let’s talk about this tower right here… how does this tower behave when you go from 0 to 90 [degrees], what’s happening?
A1: …it’s increasing!
Researcher: … [if] I would like to try like an [angle of] 89 or 88 or almost 90 right, where does the cable has to be attached? What is the heigh of this tower where the cable has to be attached?
A1: I think it’s an infinite number because if 90 [degrees] is the tallest on this one, then this one could be any number.
Researcher: Ok, now what’s happening when you cross the 90 degrees?
A1: It doesn’t cross it doesn’t attach to this line.
Researcher: why?
A1: Because it’s not on the same side.

While interviewing participant C1, I asked him the same question: what is the high of the tower at an angle of 89-90 degrees? Participant’s C1 response showed his reliance to the Cartesian plane and the implications it has on students’ interpretations of direction and position. Participant C1 affirmed that initially he visualized the tower as an object limited to the first quadrant. However, the data shows that the use of GeoGebra enabled him to picture the tower in a different quadrant with respect to the cable’s point of attachment. Additionally, important implications on direction emerged when he referred to quadrant I as the sector from ‘ninety to zero.’

C1: … when you see this [see Figure 4.57],… you’re going up, you’re going up, you’re going up; and even when it’s not a ninety degrees, it still going to go all the way up.
Figure 4.57: Outside-Tower in Quadrant I.

Researcher: When you go beyond the ninety degrees, what did you say about that?
C1: … earlier … I see this configuration, I keep thinking to myself “This tower is only in, limited to the first quadrant which is ninety; from ninety to zero degrees…But when you’re going here [see Figure 4.58], it’s like saying that this whole; like this giant triangle is gonna be your tower and that. It just goes under the ground.

Figure 4.58: Outside-Tower in Quadrant IV.

C1: And then when you go here [see Figure 4.59].
C1: Like if you can see this triangle here; that’s the kind of tower you’re gonna have. And this is like the top of the tower because which goes over like the ground though.

During the interview, participant D1 also demonstrated a clear understanding on direction and position. His reasoning evidenced the way he understands these concepts based on a determined angle. His reliance to the Cartesian plane supported his argument.

Researcher: What can you tell about this tower? How this tower behaves as you start moving …
D1: As it moves this upward? The tower is gonna be higher ‘till you get exactly ninety degrees or it’s impossible to reach it.
Researcher: Ok and what happen when you pass the ninety degrees?
D1: At one it [flip…] here automatically go down to this one instead [quadrant IV].

Dynamic representations in GeoGebra helped students not only visualize geometric models but also make mathematical conjectures that allowed them to understand mathematical concepts such as ratio, proportion, and angle (Bu & Schoen, 2010). In addition, dynamic contexts served as a tool to construct knowledge through multiple representations. The closure of
this experiment evidences the impact of multiple representations for the teaching of ratios, proportions, angles that serve as precursors to function-based reasoning.

During the conclusion of the experiment, the teacher showed students a GeoGebra file that dynamically represented the behavior of the towers, bases, and the outside-tower. The file had the property to show one by one the ‘waves’ formed by the projections of towers and bases as well as the simultaneous representations of these projections.

Teacher: ... did anybody care to... like to investigate beyond the class as far as what we were doing? ... You guys have been working on, this is your construction [see Figure 4.60], ..., what was build up here [GeoGebra file], ... this will be considered the lollypop but from the zero to three sixty degrees. when we animate this, and we actually have measured the towers, this is what happens.

![Figure 4.60: A Towers Graph.](image)

Teacher: Does that look familiar?
A3: Mind blow
D1: Mr. I found a picture at [my] iphone… I saved it, because I thought… can I show you?
P1: Is making the wave
Teacher: Ok so let’s check that out, so here goes the next one…so there is your basis… wave [see Figure 4.61], and… is that what it is?

FIGURE 4.61: BASES GRAPH.

Teacher: And then… the last one… the one you just finished constructing [see Figure 4.62].

FIGURE 4.62: OUTSIDE TOWER GRAPH.

D1: Impossible!
A3: Mind blowing!
C1: I drew that, I drew that!
A3: I drew it first!
Teacher: Well did you notice what’s happening as far as uh…?
A3: I was pretty close!

Students reacted to the GeoGebra file. The dynamic representation of their projections helped them to understand what they were doing in paper. These representations enabled them to see the behavior of the towers and bases. Additionally, the GeoGebra file helped them to visualize angles in a different manner. Figure 4.63 shows a comparison of students’ work and the GeoGebra representation.

The teacher continued with the discussion and asked students about what they thought they were studying. Up to this point of the experiment students had not hear about the ‘trigonometry’ word.
Teacher: What we’ve been studying guys? … Did you know this was trigonometry?

Group: No!

L1: I thought it is algebra and geometry all together.
D1: I thought it was geometry because we use similar figures but I looked it into google.

Teacher: Algebra and geometry all together, I mean that’s pretty much what it is.

The previous discussion showed important implications when students related trigonometry with algebra and geometry. Working with geometric representations and triangles facilitated the transition to trigonometric concepts such slope and angle measure. The discussion continued and the teacher called students to identify the ‘waves’.

C1: Oh, does the S stands for sine.
Teacher: Ok, and the C?
C1: Cosines
D1: and Tangents!
Teacher: … so which one is the sine curve?
A1: The blue one!
Teacher: What was the second? The C?
I1: The second was the red.
Teacher: And then the green one…
A4: Tangent!

The teacher asked students if they used any type of equation to create the waves and interpret the behavior of the projections. Students explained that they were not using equations but other sort of mathematical objects.

Teacher: … how do we create these waves? Did we come up with some sort of equations? … Do you see any algebraic equations out there?
C1: Yes! You got that slope, you have that ‘x, square, parabola, curve.’
Teacher: But did we use equations to come up with these curves?
Group: No!
Teacher: What did we use to come up with these curves?
C1: Angles, … Triangles!
L1: Points!
A3: Circles!
C1: Complementary and supplementary angles.
Teacher: Did we use the complementary and the supplementary angles?
C1: Yeah!
D2: Heights and bases! …Similar [triangles]
Teacher: I mean, well… think about it, think about it for a moment, when did you learn about all these things you just mentioned? Angles, points, triangles, angles…
Student: Algebra!
Teacher: Was really Algebra the first time you heard about these?
D2: Regular math!
D1: No!
A3: It was like in fourth grade!
D1: Yes!
Teacher: You’re calling it regular math, what is regular math?
E1: Basic!
D1: Two plus two [2+2].
Teacher: Two plus two, so are you telling me… here is the big question, are you telling me that in the fourth grade you guys could do trig?
Group: Yes!

4.5.1 Summary of Episode V

Students identified the mathematical objects and concepts they were using to create the geometric representation of the towers and bases. They described their reliance on geometry to represent the behavior of their projections in different angles. During this episode, students utilized a GeoGebra to visualize dynamic representations. Using this geometric tool allowed students to compare their static (paper-based) projections and understand the behavior of them when changing the angles. In addition, the dynamic representation of the towers and bases enabled them to comprehend direction and position. Besides, students’ explorations using dynamic contexts allowed them to realize the concept of ‘undefined’ when representing the ‘outside-tower’ at 90-degree angle. Students also made use of the Cartesian plane to understand directionality. GeoGebra served as a tool to not only visualize ratio and proportion, but also understand multiple representations.
4.5.2 Learning Trajectory for Episode V

The previous episode showed the learning trajectory followed by students as they developed their function-based reasoning. The landmarks point out the importance of making connections to the Cartesian plane, understanding projections, covariation, and the implications of directionality and position for a more robust development of the function concept. Throughout this learning trajectory, students encountered diverse obstacles. Students had to deal with geometric representations of maximums and minimums to understand the behavior of *The Wave* (graph). Students struggled with the concept of *Undefined* when projecting the geometric representation of the tangent function. In order to develop function-based reasoning, students needed to make connections between geometric representations of ratios and the Cartesian plane. They also had to grasp directionality and position in conjunction with angle measure in order to construct the graphical representations of trigonometric functions. Figure 4.64 shows an interpretation and summary of the learning trajectory for the episode.

![Figure 4.64: A Summary of the Learning Trajectory for Episode V.](image-url)
4.5.3 Summary Data

The researcher utilizes Confrey’s (2006) conceptual corridor to represent students’ learning trajectories during the process of developing function-based reasoning through multiple representations. During this process, students were exposed to a series of purposeful guided activities designed to approach determined landmarks along the learning trajectory. The students also encountered diverse obstacles through the process (see Table 4.11). While some of these obstacles were expected, the unexpected obstacles motivated the re-design of activities and provided opportunities for the researcher to investigate students’ conceptual development more closely. The study encompassed a retrospective analysis of this corridor model and presented alternative interpretations of students’ learning trajectories (Cobb et al., 2003).

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<th>Table 4.11: Observed Obstacles.</th>
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<td>Obstacles</td>
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<td>Congruency</td>
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<td>Angle Measure</td>
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<td>Complementary Angles</td>
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<td>Negative Ratios</td>
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<td>Angles of Elevation</td>
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<td>Directionality and Position</td>
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<td>Undefined Height of a Triangle (Tangent line)</td>
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Students’ learning trajectories delineated the conceptual corridor for developing function-based reasoning. Through the analysis of the conceptual corridor, resulting from students’ learning trajectories, seven selective codes emerged: Similarity, Projections, Coordinate System
and Cartesian Plane, Measurement Strategies, Complimentary Angles, Directionality, and Dynamic Geometry. The selective codes reflect mathematical constructs and students’ thinking throughout their immersion in the activities. The representative episodes highlight the emergent patterns and constructs that reflect students thinking throughout the study. Each episode highlights students’ learning processes including the obstacles they encountered. The representative episodes were compiled through selective coding and also reflect students’ feasible use of prior knowledge (Mann, 1993). From these episodes, three themes of significant importance for the development of function-based reasoning emerged:

1) Students Conceptions of Covariation
2) Students Conception of Angle
3) Students’ sense of Connecting to the Cartesian Plane

The themes emerged throughout the students’ learning trajectories and at multiple stages. This phenomenon indicates that neither student learning processes, nor student learning trajectories are linear. While students’ learning trajectories delineated a general conceptual corridor for developing function-based reasoning, each student’s prior knowledge, abilities, and ways of leaning (among other variables) defined their individual learning trajectories. Figure 4.65 shows a revised conceptual corridor model.
Prior knowledge: K-8th math

- Proportion
- Congruency
- Scales
- Ratios
- Right Triangles
- Angle Measure
- Similarity
- Complementary Angles
- Angle Measure
- Reciprocal
- Complementary Angles
- Tangent Table
- Complementary Angles
- Geometric Reps. of Ratios
- Angle Measure
- Negative Ratios
- Angles of Elevation
- Angle Representation
- Directionality
- Dynamic Rep. of Ratios
- Position
- Negative angles
- Maximums & Minimums
- Projections
- Undefined Height
- Covariation
- Geometric-Dynamic Rep of Trigonometric Functions

Learned ideas: Function-based reasoning

**Figure 4.65: A revised conceptual corridor model.**
Chapter 5: Summary and Discussion

5.1 Overview of Findings

The results of this study reveal the students’ learning trajectories and critical obstacles encountered when learning trigonometric functions. The historical genesis of trigonometric functions sheds light not only on the effectiveness of current teaching practices, but also on the learning trajectories designed for students (Clements & Sarama; 2009; Simon & Tzur; 2004). While recent reform curriculum in mathematics emphasize visualization and modeling, they still fall short in developing meta-representational competence. For example, graphs are seen as secondary facilitators that help one visualize an equation or numerical data. Most curricula (both reform and traditional) fail to complete a cognitive feedback loop where multiple representations, including physical dynamic geometry, are given fully equal status (Dennis, 1997).

The inquiry-based activities presented in this study were carefully designed and organized into sequences that allowed students to understand a mathematical idea through different facets (Cuoco, 2001; Ge, (2012); Heinze, Star & Verschaffel, 2009; NCTM, 2000; Stylianou, 2010), providing students access to learning opportunities that traditional instruction does not provide. Students’ work and interactions showed that the sequenced activities stimulated students’ conceptual development of function. The activities not only engaged students with the content, but also supported group work and provided them with different perspectives of key mathematical concepts in trigonometry. Through the domino activity, students were engaged in learning mathematical concepts such as slope and angle. The cable-stayed bridge provided a hands-on context where students explored similarity and proportion through physical models. Subsequent activities involving ratio and proportion provided students
with opportunities to develop multiplicative reasoning and meta-representational competence by comparing three different representations: triangle manipulatives, graphs, and tables. Constructing the tangent table helped students make connections between given data and computation of ratios. Furthermore, through triangle comparison, students were able to transition from numeric to geometric representations of ratios in the form of triangles. Both the foam board and the “lollypop” activity provided the dynamic context for students to develop Cartesian Connectedness: representing ratios, utilizing tables, and developing ideas about position and directionality. Finally, students utilized GeoGebra as a cognitive tool that helped them make sense of geometric relationships between circles, triangles, and a graph.

Results also reveal that sequencing the activities in a way that students could see the relationship between them played a critical role in this study. For example, during the clinical interview, participant L1 stated that “[it was] a good practice to being exposed first to the foam board and then moving to the technology.” He elaborated that if the technology had been introduced first, then, “it would be more difficult because you’re just looking at things and graphing things. But with the foam board, you actually get an idea of being able to fill the measurements yourself.” In addition, multiple representations enriched classroom interactions by providing a context that enabled students to see ‘others’ perspectives about the same mathematical idea. For instance, during a clinical interview, participant C1 was asked about his thoughts on the triangles activity. He stated, “I think this triangles activity was also really good because, you know… you got to see people’s different perspective on things and that really expands and enhances learning process.”

Student and teacher interactions also indicate both held strong perspectives on the usefulness about learning through multiple representations. For example, perceptions about
GeoGebra differed from one student to another. While some students perceive technology as a learning tool that helps them understand mathematical concepts, others found it more complicated. For example, while participant D1 firmly believed that he was not “good with computers,” he recognized that they could be useful to many others who want to understand mathematics. He stated, “I like the visual this right here with the paper, because I can see it easier, … I am not really good with computers, … [but] it really just helped me [the computer] to see like let’s say this was the line when we had the access and we used this line to make it like a 3D thing.” Likewise, participant L1 mentioned that technology helped her to visualize mathematical concepts in a dynamic context, something to which she had never been exposed.

The technology did help me see it because when I was moving it around to see where I could see the most” Students stated that thorough these activities the learning process is easier. Participant I1 stated “I think by doing stuff like this is easier to learn. Because when I noticed during half of the time we were doing the bridge, I noticed that this was going more beyond geometry… I think by this way, it was easier to learn it. I think it was easier to learning in this way than how our regular class teaching.

Likewise, participant C1 explained:

This program actually I really enjoyed because one of the styles of learning that I really appreciate is when teachers or when curriculum gets hands on, when you’re able to work with tools, when you’re able to explore; and math in that sense is really interesting because you don’t see that all in math… the bridge, and the dominoes, and all that. We were able to work with our hands and be able to, you know? Build things. I think that that’s really fun and it really makes you want to come to school and learn more about what’s going on.
The use of multiple representations provides students with opportunities to use different ways to solve a mathematical problem (Keller & Hirsch, 1998), and stimulate conceptual development of mathematical concepts (Dufour-Janvier, Berdnarz & Belanger, 1987; Panasuk, 2010). Additionally, multiple representations’ flexibility has the potential to make the learning of mathematics more meaningful and effective and enable students to see similarities, differences, and relationships among them (Ge, 2012). The results of this study reveal the influence of multiple representations for the understanding of trigonometric functions.

By utilizing a Design Study methodology, in which the designed activities did not rely on direct instruction methods or procedures, students became engaged in a learning process where they could construct their own knowledge. Students were exposed to different mathematical contexts where they were able to understand different representations of a mathematical idea and the relationships between those representations. Examining students’ experiences and interactions through this epistemic lens resulted in the emergence of three critical themes: 1) students’ conceptions of covariation, 2) students’ conceptions of angle, and 3) students’ sense of connecting to the Cartesian plane.

5.1.2 Students Conceptions of Covariation

Instruction in similarity and multiplicative reasoning without the use of formal definitions represents a challenge for many mathematics teachers, perhaps due to a lack of understanding the different stages of conceptual development through which students must pass in order to gain a robust understanding of ratio and proportion (Harel & Confrey, 1994). According to Lehrer, Strom, and Confrey (2002), a ratio can be expressed as a relation among variables and can be re-expressed in Cartesian coordinates both algebraically and graphically (p. 364). In addition, they
emphasize that ratio and proportional reasoning are developed “in concert with notions of similarity” (Lehrer, 2002, p. 367).

Results of the study indicate that even secondary students in a geometry class struggle with fundamental knowledge of the subject and that traditional teaching practice seems to be ineffective. Student participants explored ratio and proportion through similar triangles. By comparing triangle characteristics such as congruency and dilation, students were able to approach similarity and appreciate the conceptual underpinnings of ratio and proportion. Using multiple representations helped students make connections among different concepts related to similarity. Students utilized manipulatives (e.g. bridges, triangles), tables, and geometric representations (drawings) to understand ratio, proportion, congruency, and dilations. For example, students needed to visualize triangle measurements in a table in order to make assumptions on similarity (which leads to understanding proportionality). Likewise, students utilized geometric representations (drawings) to understand their bridge’s cables configuration and determine the ideal configuration based on proportional lengths. A learning trajectory for function-based reasoning necessarily entails an understanding of ratio and proportion, because such understanding is necessary to develop knowledge about rates of change and slope – two critical ideas when developing the concept of function (Harel & Confrey, 1994).

Lamon (1993) points out that students’ lack of understanding of ratio and proportion affects their ability to solve trigonometry problems. Furthermore, he conjectures that students face multiple challenges to developing multiplicative reasoning due to their misconceptions about angle. Results from the current study reveal that participants were challenged by the definition of angle as measure. For example, students’ difficulties arose when dealing with measurement estimation and error coupled with their strong tendency to rely on standard angle
measures such as 90° (i.e. rely on common angle measures they knew and with which they were comfortable). Furthermore, working with reciprocal ratios and angles proved challenging. However, being exposed to geometric representations of ratios provided students with opportunities to understand and interpret the relationship among triangles derived from reciprocal ratios (including their angles).

5.1.3 Students’ Conceptions of Angle

Laying a theoretical foundation for how children come to know and understand an angle concept is important for further research in geometry and trigonometry. Defining the concept of angle implies the consideration of the context and situation where it will be delineated. The concept of angle varies from a wide range of definitions (Freudenthal, 1973; Henderson & Taimina, 2000; Krainer, 1989; Maor, 2013; Mitchelmore, 1989; Mitchelmore & White, 2000; Roels, 1985; Schweiger, 1986) including:

1) Angle as an amount of turning
2) Angle as a pair of rays
3) Angle as a region formed by an intersection
4) Angle as a slope
5) Angle as a movement
6) Angle as a measure
7) Angle as a geometric shape or entity

Ambiguities in understanding the concept of angle include the qualitative idea of “separation between two intersecting lines, and the numerical value of this separation -- the measure of the angle” (Maor, 1998, p. 15). In addition, as evident from the research study, investigating whether students perceive an angle as static or dynamic is worthy of consideration within the context of teaching and learning trigonometry.
Understanding angle and its dynamic aspect represents a landmark for the development of function-based reasoning in trigonometry. According to Clements and Burns (2000), “the dynamic definition of angle introduces the concept of directionality” (p. 31). In this study, students were afforded the opportunity to explore the concept of angle from a dynamic perspective. Results show the effectiveness of the foam board as an instrument or tool to visualize angles. Additionally, students’ exposure to the foam board contributed to developing their spatial sense in terms of position and direction. Given the opportunity to construct geometric representations of ratios, students explored important concepts such as angle of elevation and angle of depression in a dynamic context, thereby allowing them to develop a deeper understanding of slope. The latter concept was developed in tandem with the representation of negative angles. Reliance on the Cartesian plane influenced participants’ interpretations of direction and position. For example, the “standing point” reference at the angle vertex versus at the right/left side influenced participants’ definition of angle of elevation (see Figures 4.13 and 4.15 in Chapter 4). By using the foam board, students were able to navigate among multiple representations (i.e. tables, manipulatives, geometric representations, and dynamic representations) and make sense of angles as a dynamic entity.

5.1.4 Students’ sense of Connecting to the Cartesian Plane

Directionality and position are two important concepts for angle representation and play a central role in the development of trigonometry function-based reasoning. This learning trajectory for conceptualizing trigonometric functions necessitates that students understand angle as a dynamic object and measure it in terms of position and direction. In this study, students continuously developed connections to the Cartesian plane when trying to make sense of ratio and proportion, angle, position, and directionality.
Results from this study indicate the importance of visual representations and the role of the Cartesian connection to understand the geometric relationships derived from angles, and the implications of direction and position. Students’ exposure to the foam board and to GeoGebra (dynamic software) provided the context to understand projections and indirect measurement. Participants were able to map out geometric objects (e.g. lines, segments, and angles) that are essential for the learning of trigonometric functions. Furthermore, they were able to visualize the relationships between them. For instance, students were able to project the heights and lengths of triangles by using a numeric scale consisting of angle measures. This activity required students to develop a Cartesian connection to understand direction and position and visualize the relationship among triangles’ heights and bases, and their correspondent angles (i.e. the trigonometric function’s geometric behavior).

Constructing a geometric representation of the tangent function implied a strong understanding of concepts of directionality and geometric correspondence. Being able to visualize projections of segments that comprise contradictory geometric relations (i.e. position and direction of the projected segment) was extremely complex for the majority of the students. However, GeoGebra played a critical role on facilitating the dynamic context where students could visualize the projections and their dynamic representation.

### 5.1.5 Revisiting the Conceptual Corridor

The emerging themes from students’ learning trajectories embrace the obstacles encountered by students in pursuing the outlined landmarks and are evident throughout the conceptual corridor. In addition, obstacles related to angle (i.e., Angle Measure, and Complimentary Angles) arose multiple times throughout the study. Table 5.1 shows the themes
emerging from the conceptual corridor and their correspondence to both landmarks and obstacles.

**Table 5.1: Themes emerging from the conceptual corridor**

<table>
<thead>
<tr>
<th>Landmarks</th>
<th>Obstacles</th>
<th>Themes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Proportion</td>
<td>Scales</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td>2 Ratios</td>
<td>Congruency</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td>3 Similarity</td>
<td>Right Triangles</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td></td>
<td>Angle Measure</td>
<td>2) Students’ Conceptions of Angles</td>
</tr>
<tr>
<td></td>
<td>Complimentary of Angles</td>
<td></td>
</tr>
<tr>
<td>4 Reciprocal Ratios</td>
<td>Angle Measure</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2) Students’ Conceptions of Angles</td>
</tr>
<tr>
<td>5 Tangent Table</td>
<td>Complimentary Angles</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td>6 Geometric Representations of Ratios</td>
<td>Complimentary Angles</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2) Students’ Conceptions of Angles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3) Students’ sense of Connecting to the Cartesian Plane</td>
</tr>
<tr>
<td>7 Angle Representation</td>
<td>Angle Measure</td>
<td>2) Students’ Conceptions of Angles</td>
</tr>
<tr>
<td></td>
<td>Negative Ratios</td>
<td>3) Students’ sense of Connecting to the Cartesian Plane</td>
</tr>
<tr>
<td></td>
<td>Angles of Elevation</td>
<td></td>
</tr>
<tr>
<td>8 Dynamic Representation of Ratios</td>
<td>Directionality</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3) Students’ sense of Connecting to the Cartesian Plane</td>
</tr>
<tr>
<td>9 Projections</td>
<td>Position</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td></td>
<td>Negative Angles</td>
<td>2) Students’ Conceptions of Angles</td>
</tr>
<tr>
<td></td>
<td>Max. &amp; Min.</td>
<td>3) Students’ sense of Connecting to the Cartesian Plane</td>
</tr>
<tr>
<td>10 Covariation</td>
<td>Undefined Height</td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td>11 Geometric-Dynamic</td>
<td></td>
<td>1) Students’ Conceptions of Covariation</td>
</tr>
<tr>
<td>Representations of Trigonometric</td>
<td></td>
<td>2) Students’ Conceptions of Angles</td>
</tr>
<tr>
<td>Functions</td>
<td></td>
<td>3) Students’ sense of Connecting to the Cartesian Plane</td>
</tr>
</tbody>
</table>
5.2 Implications for Current Theory

Multiple researchers have presented their argument on the positive implications that multiple representations have in the teaching and learning of trigonometry (Delice & Sevimli, 2010; Dreher & Kuntze, 2015; Dufour-Janvier, Berdnarz & Belanger, 1987; Flores, Koontz, & Inan, 2015; Ge, 2012; Golding & Shteingold, 2001; Izak & Sherin, 2003; Kaput, 1998; Keller & Hirsch, 1998; NCTM, 2000; Panasuk, 2010). Multiple representations connect abstract concepts and symbols to real-world situations through a modeling process (Carrejo & Marshall, 2007; Kaput, 1998) and each representation is used as a tool to solve the same mathematical problem. The capacity of multiple representations to promote concept development through problem solving is unlimited. However, despite NCTM’s (2000) recommendation that students should “create and use representations to organize, record, and communicate mathematical ideas; select, apply and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena” (p. 64), traditional teaching practices continue to prevail. By adopting an epistemology of multiple representations to develop students’ meta-representational competency, teachers may realize their full potential in making the mathematics learning process more meaningful and effective (Cuoco, 2001; Ge, 2012; Heinz, Star & Verschaffel, 2009; NCTM, 2000; Stylianou, 2010).

From an historical perspective, the genesis of multiple representations begins with Descartes Geometry. According to Dennis (1997), “The profound impact of Descartes' mathematics was rooted in the bold and fluid ways in which he shifted between geometrical and algebraic forms of representation, demonstrating the compatibility of these seemingly separate forms of expression” (p. 197). While Descartes is touted to students today as the
originator of analytic geometry, he never graphed an equation. He always started with a curve, constructed from geometrical actions. It was not until the curve was constructed that Descartes introduced coordinates and then analyzed the curve-drawing actions in order to arrive at an equation that represented the curve. In other words, equations did not create curves; curves gave rise to equations.

This study focuses on very important parts of the high school mathematics curriculum – trigonometry and the concept of function -- and explores other possibilities of presenting these topics based on an epistemology of multiple representation, which has its genesis in the history of mathematics. Trigonometric functions are presented and discussed by first examining equations. An example of each curve (sine, cosine, and tangent) is shown and then equation parameters are changed to show how the graph of each equation changes. Presenting trigonometric functions in this fashion creates the illusion that the curve is not, never has been, and might never be an entity deserving analysis. Furthermore, the practicality of studying the geometric processes that give rise to these equations is not necessarily presented to every student studying algebra, trigonometry, and analytic geometry. By studying a history of the trigonometric functions, their development and practicality, an instructor can revive and reexamine an otherwise mundane subject.

One must bear in mind, however, that Descartes intention was not to do away with "messy" geometrical approaches to problems by simply applying algebra. In fact, when algebra was used to analyze a curve, only half of the task had been accomplished (Bos, 1993). There are other considerations of Descartes's work that may be taken into account, including his use of the table representation. Perhaps a deeper application of multiple representation could offer a more balanced curriculum. When presented with a mathematical topic, students would not be
learning, in a sense, only part of the whole idea. If one applies such an approach to teaching and learning mathematics, then other arguments concerning curriculum, teacher development, and methodology come to the forefront.

One such argument about the rejection of an absolutist perspective on mathematical ideas concerns teacher confidence (Threlfall, 1996). Teachers may have a notion of which mathematical ideas or concepts are "right" or "true." If challenged, then they may feel that they can no longer teach. In a sense, the teacher is "de-skilled." However, teacher development can address a serious issue:

The ideas of a teacher are more important than the ideas of the author of a text, even if the latter is a mathematician, because the teacher can build on his/her insights into the child's conception to connect the child to a new way of seeing the situation. (Threlfall, 1996, p. 4)

Teachers should not be concerned with teaching the "right" ideas. Rather, they should ask a very important question -- Is a student's mathematical ideas sufficient for the demands and practices of the problems and situations with which they encounter (Threlfall, 1996)? Teachers can bear the responsibility for their own ideas while being ever cautious of trying to impose their ideas on their students. How teachers should use multiple representations in their classroom and how schools can develop a curriculum with such a foundation may attract different opinions and face certain challenges. However, one may argue that including this historical dimension in the mathematics classroom does not mean replacing any part of the curriculum. In this sense, the use of multiple representations should be adopted to make learning the basic skills more interesting and motivate students to sustain their interest in mathematics. According to Dennis and Confrey (1998):
If our curriculum is allowed to confront the uncertainties and ambiguities of how language interacts with the physical world; if mathematical language, symbols and notations are allowed to grow directly from experiences and be shaped by them, then this fully circular feedback loop could evolve into a powerful epistemological model based upon the coordination of multiple representations. (p. 317)

5.3 Limitations of The Study

5.3.1 Curriculum and Time Constraints

Research investigating the teaching of trigonometric functions indicates that trigonometry is an important subject addressed in the high school mathematics curriculum that requires the integration of diverse algebraic, geometric, and graphical reasoning (Demir, 2012). However, trigonometry as a single subject does not officially forms part of the mathematics course sequence. As a result, the researcher found it difficult to find a class that focused on incorporating trigonometry content into the current curriculum. Furthermore, finding a teacher who was both willing to participate in the experiment and was prepared to teach trigonometry content proved a challenge. Although trigonometry content can be taught as part of a geometry curriculum, it is common for geometry teachers to stick to the limited content established by current curriculum standards.

Another limitation of the study was the time frame established for the researcher to be present in the classroom. Originally, the research project encompassed a sequence of five activities that would take place within the period of twenty-five days. Although redesign of these activities was certainly expected, the constant redesign process inherent with a design study resulted on the design of eleven activities (including new sub-activities) that differed from the
original research agenda. Doubling up the number of activities represented a challenge for the research team because the time spent on each new activity reduced the amount of designated time for the introduction of activities that would have explored students’ construction of the formal equation representation of the basic trigonometric functions.

5.3.2 Limitations of the Methodology

According to Cobb, Stephan, McClain, and Gravemeijer (2003), “Design experiments are pragmatic as well as theoretical in orientation” (p.9). Their contexts are subject to test and revision. Design experiments “are conducted to develop theories, not to merely to empirically tune ‘what works’ (Cobb, et al., 2003, p. 9). The first limitation encountered deals with establishing the preliminary assumptions concerning both, the intellectual and social starting points for the anticipated forms of learning. The second limitation resulted from the use of the Grounded Theory approach to data collection and analysis. For instance, a distinctive characteristic of the design experiments methodology is that the research team expands its understandings of the investigation while the experiment is in progress (Cobb et al., 2003), and “the temptation to collect more data is especially strong in terms of wanting to either elaborate or confirm current findings” (Carrejo, 2004, p. 130). In order to discuss other limitations, the researcher relies on Cobb et al. (2001) framework that analyzes the classroom mathematical practices using a modified Grounded Theory approach.

Trustworthiness

According to Carrejo (2004), “the difficulty of presenting critical episodes in isolation cannot be overlooked” (p. 131). The presented episodes constitute mathematical themes that emerged from the students’ learning trajectories. In addition, Cobb et al. (2001) posits, “the
critical episodes are those that prove pivotal in either refuting a conjecture or substantiating an assertion” (p. 147). Initially, the episodes appeared to be of little significance. However, they make sense only “within the context of the entire study, and the reader must rely on the researcher’s claim that presented interferences of them span the entire data set” (Carrejo, 2004, p. 131). Episodes are presented in a profoundly way and the researcher hopes the reader to realize the full scope of the analysis for the selection of episodes that exemplify patterns of reasoning throughout the data set.

**Replicability and Commensurability**

Cobb et al. (2001) claim that the issue of replicability “rests on the assumption that the mathematical practices and associated patterns of learning documented during a teaching experiment can emerge when the instructional sequence is enacted in other classrooms” (p. 152). The researcher must explicate whether or not implementation of the same instructional sequence would yield the same findings and conclusions. As explained by Cobb et al, (2001), in contrast to traditional experimental research, the challenge is not that of replicating instructional treatments by ensuring that instructional sequences are enacted in exactly the same way in different classrooms. The conception of teachers as professionals who continually adjust their plans on the basis of ongoing assessments of their students’ reasoning would in fact suggest that complete replicability is neither desirable nor, perhaps, possible (Ball, 1993; Carpenter & Franke, 1998; Gravemeijer, 1994). The challenge for us is instead to develop ways of analyzing treatments so that their realizations in different classrooms can be made commensurable. (Cobb et al., 2001, p. 153).

In this study, students’ learning outcomes resulted from contextualized learning situations. By relying on Grounded Theory, the analytical generalizations established in this
study, derived from the students’ learning trajectories. Students’ learning trajectories delineated the conceptual corridor in which the learning process took place. The variables defined by the context, classroom culture, students’ prior knowledge, and teachers content knowledge (among others) defined the instructional sequences. Cobb and colleagues explain that “In contrast to traditional experimental research, the challenge as we see is not that of replicating instructional treatments by ensuring that instructional sequences are enacted in exactly the same way in different classrooms” (p. 153). Based on that premise, complete replicability is not feasible.

**Usefulness**

The study presented provides the means to support discussions on professional development of teachers. By documenting the learning trajectories of the classroom community, the study provides both, the suggested landmarks and encountered obstacles that students experienced during the learning process. In addition, the participant teacher played a significant role during the classroom experiment. His insights of the classroom community and contributions as a member of the research team resulted in important repercussions of the analytical approach that investigates “the manner in which it situates students’ mathematical activity and learning” (Cobb et al., 2001, p. 154).

### 5.4 Recommendations for Further Research

This study provides examples of the mathematics standards for trigonometry curriculum recommended by the National Council of Teachers of Mathematics (2012), Texas Essential Knowledge and Skills (2012), and the Secretaría de Educación Pública (Mexico) (2012). This study presents the learning trajectories followed by students on the pursuing of conceptual development of trigonometric functions through multiple representations. The study
investigates the types of understandings about trigonometric functions that emerge through multiple representations. Further studies investigating the influence of multiple representations for the teaching of trigonometry are needed. For instance, research on how multiple representations stimulate students’ multiplicative reasoning can be worthy of study; additionally, research regarding the role of multiple representations on modeling trigonometric problems is needed. It is of significant importance that more studies about angle measure and Cartesian connection would be conducted. Moreover, very few studies examine students’ perceptions of directionality and position. Furthermore, research on the extent to complementary angles and their dynamic perspective will be of great relevance for the field of mathematics. Additionally, more exploration on multiple representations and negative ratios is fundamental. Further investigation of students’ experiences on the transition to the formal algebraic representation of trigonometric functions is required. Design studies “investigate the possibilities for educational improvement” (Cobb et al., 2003, p. 10). They provide retrospective analysis and alternative interpretations of learning trajectories by combining the expertise and backgrounds among members of a research team and opportunities for innovation. However, the literature concerning design studies and Design Based Research (DBR) as well as their theoretical implications is scarce. According to Bakker and van Eerde (2015), DBR has “the potential to bridge the gap between educational practice and theory, because it aims both at developing theories about domain specific learning and the means that are designed to support that learning” (p. 430). The engineering nature of DBR explains how “[i]n the process of designing and improving educational materials… it does not make sense to wait until the end of the teaching experiment before changes can be made” (p.432). DBR is characterized as a form of didactical engineering that can be presented as a methodology whose purpose is to develop theories about learning,
thereby having an impact on didactics, or theories of teaching. DBR entails prospective and reflective components. Its cyclic nature (invention and revision) requires an iterative process (Bakker & van Eerde, 2015). By conducting more DBR, opportunities for educational improvement can emerge on a bigger scale. The iterative cycle allows both researchers and educators to continuously design and improve educational materials. The nature of DBR allow researchers to engineer their studies based on previously engineered designs resulting in a spiral phenomenon of an iterative cycle of iterative cycles (see Figure 5.1). The implications of this type of research for mathematics education include more robust theories about transferability in lieu of discussions focused solely on the limitations of replicability in DBR.

*Figure 5.1: A spiral model of an iterative design of an iterative design.*
References


Texas Education Agency (2012). Texas Academic Performance Reports. Retrieved from
http://tea.texas.gov/student.assessment/taks/


**Appendices**

**Appendix A: SEP Scope and Sequence**
Appendix B: Interview Protocols

Teacher Interview I
1. Tell me about you
2. Tell me about your teacher preparation
3. Tell me about your certifications
4. Why did you decide to become a teacher?
5. Could you share a life experience that influenced your teaching practices?
6. Could you describe your philosophy about education?
7. What do you enjoy the most about being a teacher?
8. What do you enjoy the less about being a teacher?
9. What does professionalism mean to you?
10. From your perspective, what are the three most important characteristics a teacher has to possess?

Teacher Interview II

1. Why did you decide to participate in this project?
2. Tell me about your thoughts regarding the project. What did you think during the project?

3. Tell me about your expectations from this project.

4. At the end, what was different?

5. What do you liked the most?

6. Is there something you did not like from this project?

7. How did you feel about participating in this project?

8. How did you see your students during the project? (Through the different stages).

9. What do you think about the project activities?

10. What do you think students enjoyed the most?

11. What did you learn from your students?

12. What did you learn from your own teaching practices?

13. During the project, do you think conflict moments (between your teaching practices/habits and the activities) emerged?

14. Could you tell me about some learning moments you encounter? (Before, during and after the class).

15. Did you learn new mathematical content during this project?

Teacher Interview III

1. Why did you decide to participate in this project?
2. How do you think teachers should be prepared pedagogically to teach mathematics?
3. Tell me your perceptions about how do students learn mathematics.
4. Tell me about your first year as a teacher.
5. During your first teaching year, did you receive any advice from your fellow teachers?
6. How did you learn mathematics?
7. Tell be about an “ah ha moment” as a teacher?
8. Tell me about your experience on teaching trigonometry through multiple representations.
9. Regarding the different activities, tell me your perceptions.
10. Tell me about your students’ reactions/comments to the different activities, and what do you think they learned.

Student Interview

1. Tell me a little about you.
2. Do you like mathematics? How do you feel with mathematics?

3. Do you have family members in any STEM professions? (i.e., science, technology engineering, or mathematics).

4. What is your perspective about the importance of mathematics?

5. Tell me about your experience during these weeks, how did you feel? Is there something you liked the most?

6. Regarding the activities, which one did you enjoyed the most?

7. Let’s talk about the triangle activity, could you show me your configuration? Why did you chose that configuration? What can you say about this configuration?

8. Tell me about some of the mathematical concepts you used during this activity.

9. What can you say about the triangle’s height and base? Is there any relation among them?

10. What can you say about the little red square tool?

11. Regarding triangles, perspective, and the GeoGebra file, what do you remember from the activity?

12. Using the bridge activity as reference: use the Geoboard to simulate the bridge; ask the student to place a rubber band to represent a cable on the bridge; present different scenarios and ask the student to maintain the balance of the bridge; ask about ratio and proportion.

13. Using the GeoGebra file: ask what the student remember from the configuration; how the heights and bases behave; what happens when going beyond the 90 degrees; when does it increase and decrease; what can you say about the angles?

Appendix C: Students’ Definition of Similar Shapes

<table>
<thead>
<tr>
<th>What is your definition of similar shapes?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Participant</td>
</tr>
<tr>
<td>------------</td>
</tr>
<tr>
<td>J1</td>
</tr>
<tr>
<td>E1</td>
</tr>
<tr>
<td>D3</td>
</tr>
<tr>
<td>S3</td>
</tr>
<tr>
<td>K1</td>
</tr>
<tr>
<td>L1</td>
</tr>
<tr>
<td>A2</td>
</tr>
<tr>
<td>A4</td>
</tr>
<tr>
<td>I1</td>
</tr>
<tr>
<td>C1</td>
</tr>
<tr>
<td>A3</td>
</tr>
<tr>
<td>B1</td>
</tr>
<tr>
<td>K2</td>
</tr>
<tr>
<td>K3</td>
</tr>
<tr>
<td>S1</td>
</tr>
<tr>
<td>P1</td>
</tr>
<tr>
<td>J1</td>
</tr>
<tr>
<td>D2</td>
</tr>
<tr>
<td>D1</td>
</tr>
</tbody>
</table>
### Appendix D: Students’ Demonstration of Similarity

Write about what you observed when you “stacked” your triangles:

<table>
<thead>
<tr>
<th>Participant</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>J1</td>
<td>“…they are all right triangles...the hypotenuse is going down the same way.”</td>
</tr>
<tr>
<td>E1</td>
<td>“I stacked my triangles connecting them at the 90 degree angle to show that they all have the same shape and angles.”</td>
</tr>
<tr>
<td>D3</td>
<td>“The way I stacked the triangles was by the 90 degree angle because you can see that the angle was the same on all of them.”</td>
</tr>
<tr>
<td>S3</td>
<td>“I stacked my triangles at the 90 degree point and I realized they all have the same shape and the angles are the same degrees.”</td>
</tr>
<tr>
<td>K1</td>
<td>“…when I stacked the triangles they all were similar in the way of having a 90 degree angle.”</td>
</tr>
<tr>
<td>L1</td>
<td>“I noticed the angles were the same and you could really tell by looking at the 90 degree angle it had so I stacked them by the right angle of the triangle to be able to see all the angles alongside the 90 degree angle.”</td>
</tr>
<tr>
<td>A2</td>
<td>“When I &quot;stacked&quot; the figures, I did it in such a way that the 90 degrees all matched. I also tried all of the other angles and all of them aligned perfectly. Another way I figured out how they were similar is that me and my group used the little squares to check if they were proportional and so they were.”</td>
</tr>
<tr>
<td>A4</td>
<td>“I placed the triangles in a way where the 90° angles met up to show that all the triangles where similar. The angle met up and the shape stayed the same.”</td>
</tr>
<tr>
<td>I1</td>
<td>“I saw how the angles were the same and how they were the same shape. The difference was just the size of the triangles. The lengths of the triangles are proportional though.”</td>
</tr>
<tr>
<td>C1</td>
<td>“When I stacked my triangles, I noticed the similarity in shape [visually].”</td>
</tr>
<tr>
<td>A3</td>
<td>“Similar shapes should have the same basic shape and side to side ratio. The angles will come out the same because of the ratio.”</td>
</tr>
<tr>
<td>B1</td>
<td>No responded</td>
</tr>
<tr>
<td>K2</td>
<td>“When I stacked my triangles it showed that no matter how I stacked the corners were all the same.”</td>
</tr>
<tr>
<td>K3</td>
<td>“When we stacked them we noticed the smaller ones were dilations of the biggest triangle.”</td>
</tr>
<tr>
<td>S1</td>
<td>“I saw the shape fit kind of together but in a smaller size.”</td>
</tr>
<tr>
<td>P1</td>
<td>“I stacked the triangles together at the tip of the smallest angle I saw that they all fit together… I realize that they all have the same angle going through the top of the triangle and the other edges all kind of lined up together.”</td>
</tr>
<tr>
<td>J1</td>
<td>“I noticed that when you stack the shapes on top of each other and put them on the same angle then you can tell that all the angles are congruent but the sides are different and that they’re proportional figures.”</td>
</tr>
<tr>
<td>D2</td>
<td>“When I stacked them I could also see a pyramid because a pyramid has everything the same but there are tons of different sizes of the shape that build it.”</td>
</tr>
<tr>
<td>D1</td>
<td>“What I observed from stacking the triangles was that when stacked the triangles have equal distances between one another.”</td>
</tr>
</tbody>
</table>
Vita

Mayra L. Ortiz Galarza was born in Ciudad Juarez Chihuahua, on November 26, 1980, the oldest of six children of Jesus Ortiz and Carolina Galarza. She attended the Universidad Autónoma de Ciudad Juárez (UACJ) where she earned a Bachelor of Science in Manufacturing Engineering in 2007 and a Master of Science in Mathematics Education in 2009. While earning her degrees, she taught pre-calculus and algebra at secondary levels and probability and statistics at the UACJ. In 2009, she enrolled at the University of Texas at El Paso to begin work on a doctoral degree in Teaching, Learning, and Culture in the STEM strand. Her education at UT El Paso has included experience as a graduate research assistant in several research areas including mathematical classroom practices, cross-cultural teacher practices in constructivist settings, translingual practices of engineering students in the U.S.- Mexico border, and linguistic capital among transfronterizo engineering students.

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