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Strings Lead to Lattice-Type Causality

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Abstract

Traditional causality relation of special relativity is a lattice in the simplest case of 1-D space (2-D space-time), but it is no longer a lattice in the actual 3-D space (and 4-D space-time). We show that if we take into account effects of string theory, then we get a lattice-type causality relation even for the 4-D space-time.

Keywords: causality, string theory, lattice

1 Causality in Special Relativity

Causality in special relativity: a brief reminder. According to Special Relativity Theory, all processes propagate with a speed that does not exceed the speed of light c; see, e.g., [1]. Thus, an event $e = (t, x)$ occurring at a spatial location x at moment t can influence an event $e' = (t', x')$ occurring at moment t' at a spatial location x' only if $t \le t'$ and the speed $\dfrac{d(x, x')}{t' - t}$ with which a signal can get from x to x' during the time interval $[t, t']$ does not exceed c:

$$(t \le t')\,\&\,\left(\frac{d(x, x')}{t' - t} \le c\right); \tag{1}$$

(here, $d(x, x')$ denotes the distance between the two spatial points x and x').

The relation "an event e can influence the event e'" is called a *causality relation*; we will denote it by $e \preceq e'$. From the mathematical viewpoint, this is an ordering relation: e.g., if e can influence e' and e' can influence e'', then e can therefore influence e'' – so this relation is transitive.

For each event e, the set $e^+ \stackrel{\text{def}}{=} \{e' : e \preceq e'\}$ of all the events that can be influenced by e is called the *future cone*, and the set $e^- \stackrel{\text{def}}{=} \{e' : e' \preceq e\}$ of all the events that can influence e is called the *past cone*.

In the special relativity theory, causality relation is described by the formula (1). By multiplying both sides of the equivalent form (1) by a positive number $t' - t$, we get an equivalent form

$$(t, x) \preceq (t', x') \Leftrightarrow c \cdot (t' - t) \ge d(x, x'). \tag{2}$$

Since the distance $d(x, x')$ is always non-negative, this inequality automatically implies that $t' \ge t$.

In the Euclidean space, the distance $d(x, x')$ between the two points $x = (x_1, x_2, x_3)$ and $x' = (x'_1, x'_2, x'_3)$ is determined by the usual formula

$$d(x, x') = \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}. \tag{3}$$

Thus, the formula (2) takes the form

$$(t, x) \preceq (t', x') \Leftrightarrow c \cdot (t' - t) \ge \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}. \tag{4}$$

This formula can be simplified if we square both sides of the inequality:

$$(t, x) \preceq (t', x') \Leftrightarrow ((t' \ge t)\,\&\,(c^2 \cdot (t' - t)^2 - (x_1 - x'_1)^2 - (x_2 - x'_2)^2 - (x_3 - x'_3)^2 \ge 0)). \tag{5}$$

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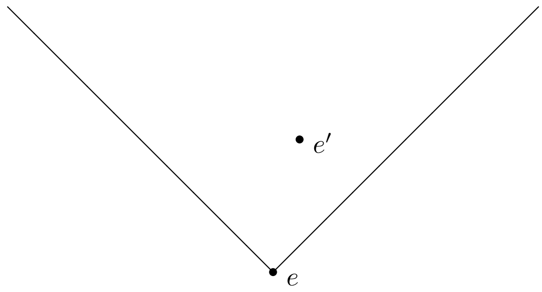
The set e^+ of all the events e' that can be influenced by the event e is thus described by the inequalities

$$(t' \geq t) \,\&\, (c^2 \cdot (t'-t)^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2 - (x_3 - x_3')^2 \geq 0). \tag{6}$$

From the geometric viewpoint, this set is a *cone*, that is why the set of all the future events is usually called the "future cone".



Simplest case of the 1-D space (2-D space-time). Let us start the analysis of the causality relation with the simplest case, when we only consider motions in one spatial direction. In other words, we consider a 1-D space in which every spatial location is characterized by a single coordinate x_1. In this description, a point (t, x) in space-time (i.e., an event) is characterized by two parameters t and x_1, so the space-time is 2-dimensional.

In this case of a 1-D space (or, equivalently, of a 2-D space-time), the distance $d(x, x')$ is simply equal to $|x_1 - x_1'|$ and therefore, the causality relation (2) takes the form

$$c \cdot (t' - t) \geq |x - x'|. \tag{7}$$

One can easily check that for every real number z, we have $|z| = \max(z, -z)$. Thus, the condition $t' - t \geq |x_1 - x_1'|$ is equivalent to $t' - t \geq \max(x_1 - x_1', x_1' - x_1)$. A number is larger than the largest of the two numbers if and only if it is larger than both of them. Thus, the above inequality (7) is equivalent to two inequalities

$$(t' - t \geq x_1 - x_1') \,\&\, (t' - t \geq x_1' - x_1). \tag{8}$$

In each of these two inequalities, by moving the terms corresponding to $e = (t, x_1)$ to one side and the terms corresponding to $e' = (t', x_1')$ to the other side, we get the following equivalent form of the causality relation:

$$(t + x_1 \leq t' + x_1') \,\&\, (t - x_1 \geq t' - x_1'). \tag{9}$$

Thus, if, instead of the natural coordinates (t, x_1), we use auxiliary coordinates $u = t + x_1$ and $v = t - x_1$ to describe each event, we get a simple formula for the causality relation:

$$(e = (u, v) \preceq e' = (u', v')) \Leftrightarrow (u \leq u' \,\&\, v \leq v'). \tag{10}$$

For 1-D space (2-D space-time), the causality relation of special relativity is a lattice. The 2-D space-time has an interesting *lattice property*. To describe this property, let us recall a few definitions.

We say that an element e'' from an ordered set is an *upper bound* of two elements e an e' if $e \preceq e''$ and $e' \preceq e''$. For some pairs e and e', there exists the *least upper bound*, i.e., an upper bound e'' which precedes all other upper bounds. An ordered set for which every two elements have the least upper bound is called an *upper semi-lattice.*

Similarly, we say that an element e'' from an ordered set is a *lower bound* of two elements e an e' if $e'' \preceq e$ and $e'' \preceq e'$. For some pairs e and e', there exists the *greatest lower bound*, i.e., a lower bound e'' which is preceded by all other lower bounds. An ordered set for which every two elements have the greatest lower bound is called an *lower semi-lattice.*

An ordered set which is both an upper semi-lattice and a lower semi-lattice is called a *lattice.* The formula (10) shows that the 2-D space-time of special relativity is a lattice. Indeed, let us show that it is an upper semi-lattice. Let $e = (u, v)$ and $e' = (u', v')$ be two arbitrary events. Then, due to (10), the condition that an event $E = (U, V)$ is an upper bound for e and e' is equivalent to

$$(e \preceq E \,\&\, e' \preceq E) \Leftrightarrow (u \le U \,\&\, u' \le U \,\&\, v \le V \,\&\, v' \le V). \tag{11}$$

The value U is larger than both numbers u and u' if and only if it is larger than the largest of them $u'' \stackrel{\text{def}}{=} \max(u, u')$. Similarly, the value V is larger than both numbers v and v' if and only if it is larger than the largest of them $v'' \stackrel{\text{def}}{=} \max(v, v')$. Thus, the condition (11) is equivalent to

$$(e \preceq E \,\&\, e' \preceq E) \Leftrightarrow (u'' \le U \,\&\, v'' \le V), \tag{12}$$

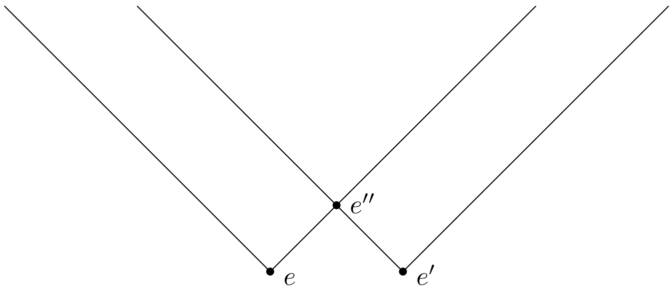
i.e., to

$$(e \preceq E \,\&\, e' \preceq E) \Leftrightarrow e'' \preceq E, \tag{13}$$

where $e'' \stackrel{\text{def}}{=} (u'', v'')$. Thus, the event e'' is the desired least upper bound for the given events e and e' – and thus, the space-time of 2-D special relativity is indeed an upper semi-lattice.

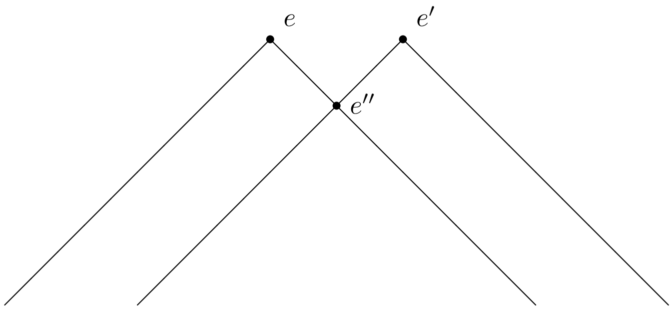
The condition (13) can be reformulated in terms of future cones. Indeed, the relation $e \preceq E$ means that E belongs to the future cone e^+ of the event e, and the relation $e' \preceq E$ means that E belongs to the future cone $(e')^+$ of the event e'. Thus, the event E satisfies the condition $e \preceq E \,\&\, e' \preceq E$ from the left-hand side of this condition (13) if and only if it belongs to the intersection $e^+ \cap (e')^+$ of these two future cones. The condition (13) then states that this intersection coincides with the future cone $(e'')^+$ of some event e''.

The condition (13) can thus be formulated in the following way: for every two events e and e', the intersection $e^+ \cap (e')^+$ of their future cones is also a future cone, of some event e'':



Similarly, for every two events $e = (u, v)$ and $e' = (u', v')$, the event $e'' = (\min(u, u'), \min(v, v'))$ is the greatest lower bound. Thus, the space-time of 2-D special relativity is also a lower semi-lattice – and thus, a lattice.

The property of being a lower semi-lattice can also be reformulated in terms of the cones (this time in terms of past cones): for every two events e and e', the intersection $e^+ \cap (e')^+$ of their past cones is also a past cone, of some event e'':



In general, the causality relation of special relativity is not a lattice. For 3-D space (= 4-D space-time), the causality relation is no longer a lattice – since for 3-D future (and past) cones, the intersection of two cones is no longer a cone.

2 Strings and M-Branes

In the traditional relativistic physics, elementary particles are points. The *elementary* particle is something that cannot be further divided into parts, something that acts as a whole.

If we have a non-point body, then, due to the fact that any interaction can only spread with a speed of light, affecting one spatial point x in the body does not immediately affect other points $x' \neq x$: it takes time at least $\dfrac{d(x,x')}{c}$ for this disturbance to get to the point x'. Thus, in terms of the reaction to this perturbation, the points x and x' act separately. In line with the above meaning of an elementary particle, this means that the body containing these two points is not an elementary particle. So, an elementary particle should be a single point.

Point-wise particles lead to physically meaningless infinities. It is known that the existence of point particles lead to serious problems of "divergence" (see, e.g., [1]) – i.e., to the occurrence of physically meaningless infinite values.

Let us give one simple example: let us consider an electrically charged elementary particle (e.g., an electron or a proton), and let us estimate the total energy T of its electrostatic field. It is known that the energy density ρ of an electrostatic field E is proportional to E^2: $\rho(x) = c_1 \cdot E^2$ (for some constant c_1). According to Coulomb's law, the field generated by a charged point particle depends on the distance r to this particle as $c_2 \cdot r^{-2}$ (for some constant c_2). Thus, $\rho(x) = c_3 \cdot r^{-4}$, where we denoted $c_3 \stackrel{\text{def}}{=} c_1 \cdot c_2^2$. The total energy of this field can be thus computed by combining the energies of different parts of this field, as

$$\int_{\mathbb{R}^3} c_3 \cdot r^{-4}\,dx = c_3 \cdot \int_{\mathbb{R}^3} r^{-4}\,dx. \tag{14}$$

Since the field only depends on the distance, it make sense to use polar coordinates. In these coordinates, we have $dx = 4 \cdot \pi \cdot r^2 \cdot dr\,d\Omega$, where $d\Omega$ corresponds to angular coordinates the integral over which is 1, so

$$\int_{\mathbb{R}^3} r^{-4}\,dx = 4\pi \cdot \int_0^\infty r^2 \cdot r^{-4}\,dr = 4\pi \cdot \int_0^\infty r^{-2}\,dr. \tag{15}$$

This integral can be explicitly computed, as $\int r^{-1}\,dr = -r^{-1} = -\dfrac{1}{r}$, so

$$\int_{\mathbb{R}^3} r^{-4}\,dx = 4\pi \cdot \int_0^\infty r^{-2}\,dr = -4 \cdot \pi \cdot \frac{1}{r}\bigg|_0^\infty = 4 \cdot \pi \cdot \left(\frac{1}{0} - \frac{1}{\infty}\right) = \infty. \tag{16}$$

This result is disturbing since, according to General Relativity, energy means mass, so an infinite energy would mean infinite mass – and thus, infinite gravitational attraction.

A similar problem occurs in quantum physics. One may conjecture that the problem appears since we consider classical (non-quantum) particles, thus assuming that we can measure everything – including the location of the elementary particle – with an arbitrary accuracy. In quantum mechanics, due to the uncertainty principle, there are limitations on measurement accuracy, so we may hope that the infinities will disappear. Alas, the same infinities appear in quantum theories as well [1].

Sometimes, there is a solution to this divergence problem. For some physical fields, there is a solution that enables us to avoid meaningless physical infinities. The main idea is that instead of starting with a point-wise particle of charge $q > 0$ and radius $r_0 = 0$, we start with a particle of finite radius r_0 and then tend r_0 to 0.

In electrodynamics, the coefficient c_2 is proportional to the charge q of the particle, so energy density is proportional to q^2. For a particle of radius $r_0 > 0$, the total energy is thus proportional to $\dfrac{q_0^2}{r_0}$. Thus, if we tend r_0 to 0 and then tend q to 0 accordingly, in the limit, we get a finite expression for the total energy. (This is just a raw idea, we also need to change other parameters, to make sure that the observed electric charge of the particle is not 0.)

However, this trick (called *renormalization*) only works for some physical fields. For other fields, the equations are more complex, and it is not possible to avoid all the infinities by simply selecting proper values of the parameters.

Main idea of the string theory. String theory (see, e.g. [2, 5]) resolves the divergence problem by assuming that elementary particles are no point-wise, that each elementary particle occupies several different spatial locations.

In the original version of a theory, it was assumed that an elementary particle occupies all the points along a 1-D curve ("string"). Later, it turned out that higher-dimensional locations (called *M-branes*) are also reasonable. So, in general, we can say that an event involving an elementary particle cannot be described by a single spatio-temporal location (t,x), it involves a while *set* of such locations.

How does this assumption affect causality? Since the main reason for considering point particles was to preserve the standard causality, clearly, non-point particles change the causality relation. In this paper, we analyze how causality is changed.

3 Strings Lead to Lattice-Type Causality: Main Result

Causality relation between the particle and an event. Let S be the set of all the points in space-time corresponding to an event in the life of an elementary particle, and let e be a point event. Since the particle is elementary, if we affect any point $s \in S$, we thus influence the particle as a whole. Thus, we can say that e influences S if e can causally influence at least one event from S. So, we arrive at the following definition.

Definition 1. *Let $(X, \preceq)$ be an ordered set. We will call this set X* space-time*, its element* events*, and $\preceq$* causality relation*. For every event $e \in X$ and for every set $S \subseteq X$, we say that e* can influence S *(and denote it by $e \preceq S$ if $e \preceq s$ for some $s \in S$:*

$$e \preceq S \stackrel{\text{def}}{=} \exists s \in S\,(e \preceq s). \tag{17}$$

Our main result is that, with respect to this definition, sets S have the following lattice-like property:

Proposition 1. *Let $(X, \preceq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $e \in X$,*

$$((e \preceq S)\,\&\,(e \preceq S')) \Leftrightarrow (e \preceq S''). \tag{18}$$

Comment. All the proofs are given in a special proofs section.

Similarly, we can say that S affects e if one of the points $s \in S$ affects e, then we get a similar result:

Definition 2. *Let $(X, \preceq)$ be a space-time. For every event $e \in X$ and for every set $S \subseteq X$, we say that S* can influence e *(and denote it by $S \preceq e$ if $s \preceq s$ for some $s \in S$:*

$$S \preceq e \stackrel{\text{def}}{=} \exists s \in S\,(s \preceq e). \tag{19}$$

Proposition 2. *Let $(X, \preceq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $e \in X$,*

$$((S \preceq e)\,\&\,(S' \preceq e)) \Leftrightarrow (S'' \preceq e). \tag{20}$$

Discussion: this is not exactly the lattice property. In the formulas (17) and (18), we only consider relations between events $S \subseteq X$ and point-wise events $e \in X$. In principle, we can also define a similar relation $S \preceq S'$ between two different events $S, S' \subseteq X$:

$$S \preceq S' \stackrel{\text{def}}{=} \exists s \in S\, \exists s' \in S'\, (s \preceq s'). \tag{21}$$

This relation can be described in terms of the above relations $e \preceq S$ and $S \preceq e$:

$$S \preceq S' \stackrel{\text{def}}{=} \exists s \in S\, (s \preceq S'); \tag{22}$$

and

$$S \preceq S' \stackrel{\text{def}}{=} \exists s' \in S'\, (S \preceq s'). \tag{23}$$

However, in general, this new relation $S \preceq S'$ is no longer transitive (and thus, no longer an order; see, e.g., [6]): for example, even when X is the real line with the usual order, we have:

- $S = \{0, 1\} \preceq S' = \{-2, 2\}$ since $1 \preceq 2$ for $1 \in S$ and $2 \in S'$;
- $S' = \{-2, 2\} \preceq S'' = \{-1\}$ since $-2 \preceq -1$ for $-2 \in S'$ and $-1 \in S''$;
- however, $S = \{0, 1\} \not\preceq S'' = \{-1\}$, since neither of the two elements of S (0 and 1) precedes the only element of S'' (-1).

However, from Propositions 1 and 2, once can easily conclude that similar lattice-like properties hold for the relation $S \preceq S'$ as well:

Corollary 1. *Let $(X, \preceq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $E \subseteq X$,*

$$((E \preceq S)\,\&\,(E \preceq S')) \Leftrightarrow (E \preceq S''). \tag{24}$$

Corollary 2. *Let $(X, \preceq)$ be a space-time. Then, for every two sets S and S', there exists a set S'' for which, for every $E \subseteq X$,*

$$((S \preceq E)\,\&\,(S' \preceq E)) \Leftrightarrow (S'' \preceq E). \tag{25}$$

Specifically, as S'', we can take the exact same sets as in Propositions 1 and 2.

Possible relation to Berwald-Moor Finsler causality. Several physicists are currently pursuing the idea that the actual causality relation may be different from the causality relation of special relativity, and that this actual relation is a lattice. Specifically, they conjecture that, similarly to the 2-D space-time case, in appropriate coordinates u_1, u_2, u_3, u_4, causality relation between two elements $u = (u_1, u_2, u_3, u_4)$ and $u' = (u'_1, u'_2, u'_3, u'_4)$ takes the form

$$u \preceq u' \Leftrightarrow ((u_1 \le u'_1)\,\&\,(u_2 \le u'_2)\,\&\,(u_3 \le u'_3)\,\&\,(u_4 \le u'_4)). \tag{26}$$

This research is related to a special *Berwald-Moor Finsler* related to this causality relation; see, e.g., [3, 4].

Our result provides a reasonable physical justification for the lattice character of the causality – and thus, for this research direction.

4 Proofs of Propositions 1 and 2

Without losing generality, it is sufficient to prove Proposition 1. For every set S, let us define its *past cone* as the set of all the events which precede S: $S^- \stackrel{\text{def}}{=} \{e : e \preceq S\}$. Let us prove that the desired formula (18) holds for $S'' = S^- \cap (S')^-$.

If $e \preceq S$ and $e \preceq S'$, then, by definition of the past cone, we conclude that $e \in S^-$ and $e \in (S')^-$. Thus, the event e belongs to the intersection $S'' = S^- \cap (S')^-$ of these past cones: $e \in S''$. Since $\preceq$ is an order relation, we have $e \preceq e$, so $e \preceq s''$ for some $s'' \in S''$ – namely, for $s'' = e$. Thus, $e \preceq S''$.

Vice versa, let us assume that $e \preceq S''$. By definition, this means that $e \preceq s''$ for some element $s'' \in S'' = S^- \cap (S')^-$. Since the element s'' belongs to the intersection of the two past cones, it thus belongs to both of them: $s'' \in S^-$ and $s'' \in (S')^-$. By definition of the past cone, the condition $s'' \in S^-$ means that $s'' \preceq S$, i.e., that $s'' \preceq s$ for some $s \in S$. Now, from $e \preceq s''$ and $s'' \preceq s$, we conclude that $e \preceq s$ for $s \in S$. This means that $e \preceq S$. Similarly, from $e \preceq s''$ and $s'' \in (S')^-$, we conclude that $e \preceq S'$.

The proposition is proven.

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