2016-01-01

# On An Iteration Method For Solving A Class Of Nonlinear Matrix Equations

Mohamed Illafe
*University of Texas at El Paso*, millafe@miners.utep.edu

ON AN ITERATION METHOD FOR SOLVING A CLASS OF NONLINEAR MATRIX EQUATIONS

MOHAMED ILLAFE

Masters Program in Mathematical Sciences

APPROVED:

---

Mohamed Amine Khamsi, Ph.D., Chair

---

Emil Daniel Schwab, Ph.D.

---

Hector Gonzalez-Huizar, Ph.D.

---

Charles Ambler, Ph.D.
Dean of the Graduate School

*to my*

*MOTHER and FATHER*

*with love*

# ON AN ITERATION METHOD FOR SOLVING A CLASS OF NONLINEAR MATRIX EQUATIONS

by

MOHAMED ILLAFE

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Masters Program in Mathematical Sciences

THE UNIVERSITY OF TEXAS AT EL PASO

August 2016

# Acknowledgements

First and foremost, I would like to express my thanks to Almighty Allah, the Most Gracious and the Most Merciful, for giving me strength and wellbeing and for his blessing in finishing this thesis.

Then, it is a great pleasure to express my deep sense of thanks and gratitude to all of the people who helped me throughout my graduate study. Firstly, I would like to offer my special thanks to Dr. Mohamed Amine Khamsi, my thesis advisor, for his advices and guidance throughout my graduate experience. Also, I deeply appreciated his patience and encouragement.

Secondly, I would like to thank the faculty members at The University of Texas at El Paso for their support and encouragement, with special thanks to Dr. Joe A. Guthrie, and Dr. Rachid Skouta for being part of my thesis committee members.

Finally, with great gratitude, I want to thank my family for believing in me. I highly appreciate their tutelage and support.

# Table of Contents

# Chapter 1

# Introduction

Let X be a non-empty set and $T$ be a map from $X$ into itself. A point $x \in X$ is said to be a fixed point of $T$ if $T(x) = x$. The set of fixed points of $T$ will be denoted by $Fix(T)$. Fixed point theory studies the conditions on both $T$ and $X$ that guarantee the existence or both the existence and the uniqueness of a fixed point. Fixed point theorems are considered to be among the most important and powerful tools in mathematics and specially in nonlinear analysis, due to its use across large number of disciplines of mathematics, like analysis, topology, geometry, game theory and set theory, and its applications in other branches of science such as economics and population biology [14].

Historically, fixed point theory finds its root in 1886 in the work of Poincaré [16] . It can be divided into three main subfields namely topological, metric, and order-theoretic fixed point theories according to the discovery of Brouwer's, Banach's, Tarski's fixed point theorems respectively.

In 1912, Brouwer [12], proved that *any continuous function from the closed unit ball in the finite dimensional space* $\mathbb{R}^n$ *into itself has a fixed point.*
Although it is proved in an equivalent form in the work of Poincaré, the proof of the current form was given first in 1909, by Brouwer for n= 3. Then, it is proved for n arbitrary independently by Hadamard in 1910, and Brouwer in 1912. It is considered as one of the major fixed point theorems which is used considerably in solving differential and integral equations. One can see that convexity and compactness of the domain and continuity of the mapping are crucial in Brouwer's fixed point theorem. The drawback with Brouwer's fixed

point theorem is the lack of information to how to approximate or approach the fixed point.

In 1922, Banach [7] proved in his thesis the famous and interesting Banach fixed point theorem, also known as the Banach Contraction Principle. It gives conditions which guarantee that, if they are satisfied, the process of iterating a mapping produce a fixed point constructively. The conditions in Banach Contraction Principle are the contraction-Lipschitz condition of the map and the completeness of the metric space. By contrast with Brouwer's fixed point theorem, Banach Contraction Principle provides a constructive method to find a fixed point.

The development of fixed point theory in partially ordered sets was initiated by Knaster in 1927 [13]. He proved his fixed point theorem for monotone increasing mappings. In 1939, this result was extended to monotone increasing mappings on complete lattice by Tarski as follow:
*any monotone increasing mapping i.e., for all* $x, y \in X$ *we have* $x \preceq y$ *implies* $T(x) \preceq T(y)$, *on a complete lattice i.e., every subset* $Y \subseteq X$ *has a least upper bound (supremum) and a greatest lower bound (infimum) in X, has a fixed point. Moreover, the set of all fixed points of* $T$ *is a complete sublattice of* $X$.

Beside the extension of Knaster's theorem, Traski's fixed point theorem has some important applications in many areas of mathematics such as set theory and topology as well as in artificial intelligence. It is worth mentioning that Tarski's fixed point theorem was unpublished until 1955 [13].

Recently a new direction has been attracting some attention dealing with the extension of the Banach Contraction Principle to metric spaces endowed with a partial order or a directed graph. This excitement followed the publication of Ran and Reurings paper [17]. The ideas behind the main fixed point theorem of [17] are found in the original paper

[9]. In particular, the authors of [9, 17] showed how this extension is useful when solving some special matrix equations for which the technique involved in solving them is similar to the one used in the Banach Contraction Principle. Following the publication of [17], Nieto and Rodríguez-López [15] extended the conclusion of [17] by dropping the continuity assumption of the map. Then used such arguments in solving some differential equations. Jachymski [10] was the first one to give an extension of the Banach Contraction Principle in metric spaces endowed with a directed graph. Jachymski's work obviously extends the ideas developed by Ran and Reurings as well as Nieto and Rodríguez-López. Since then many mathematicians used Jachymski's framework to come up with more fixed point results [2, 3, 4, 5, 6, 11]. Before we close these historical facts, let us point out that the first attempt to generalize the Banach Contraction Principle to partially ordered metric spaces was carried by Turinici in [18, 19].

In this thesis, I looked at the most recent extension of the Banach Contraction Principle [1]. Then I show that a concept of Sorgenfrey space may be used to explain this extension as well as Ran and Reurings fixed point theorem.

# Chapter 2

# Preliminaries

The purpose of this chapter is to provide some basic concepts and results of metric spaces and metric spaces endowed with a partial order which will be used throughout this thesis.

## 2.1 Metric Spaces

### 2.1.1 Definitions and Basic Concepts

Since we are going to study the metric fixed point theory, particularly Banach Contraction Principle and its extensions, we will start by giving the definition of the underlying structure.

Let $X$ be a non-empty set. A mapping $d : X \times X \to [0, \infty)$ is called a **metric or a distance** on $X$ if the following conditions are satisfied:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(y, x) = d(x, y)$, for all $x, y \in X$; (symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ (the triangle inequality).

A set $X$ endowed with a metric d is called a **metric space** and it is denoted by $(X, d)$.

Let $(X, d)$ be a metric space

**Definition 1** *A sequence $\{x_n\}$ in $X$ is said to be* ***convergent*** *if there exists $x \in X$ such that $\lim_{n \to +\infty} d(x_n, x) = 0$.*

**Definition 2** *A sequence* $\{x_n\}$ *in* $X$ *is said to be* ***Cauchy*** *if for all* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $d(x_n, x_m) < \varepsilon$*, for all* $m, n \geq N$*.*

**Definition 3** $(X, d)$ *is said to be* ***complete*** *whenever any Cauchy sequence in* $X$ *is convergent in* $X$*.*

### 2.1.2 Banach Contraction Principle

The classical Banach Contraction Principle appeared first in Banach's thesis and was used to prove the existence of a solution to an integral equation. It is applicable to vast branches of mathematics. This is because the simplicity of its assumptions which are the completeness of the metric space and the Lipschitz condition of the mapping.

**Definition 4** *Let* $(X, d)$ *be a metric space. The map* $T : X \to X$ *is called* ***Lipschitzian*** *if there exists a constant* $k \geq 0$ *(called* ***Lipschitz constant****) such that*

$$d(T(x), T(y)) \leq k\ d(x, y),$$

*for all* $x, y \in X$*. When* $k < 1$*, we will say* $T$ *is a* ***contraction****.*

Now we are ready to state the well-known Banach Contraction Principle.

**Theorem 1** *[7] Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be a contraction mapping. Then* $T$ *has a unique fixed point* $x_0$*, and for each* $x \in X$*, we have*

$$\lim_{n\to\infty} T^n(x) = x_0.$$

*Moreover, for each* $x \in X$*, we have*

$$d(T^n(x), x_0) \leq \frac{k^n}{1-k} d(T(x), x).$$

**Proof:** Since $T$ is a contraction, there exists $k \in [0,1)$ such that

$$d(T(x), T(y)) \leq k \ d(x, y),$$

for all $x, y \in X$. Hence

$$d(T^n(x), T^n(y)) \leq k^n \ d(x, y),$$

for all $x, y \in X$ and $n \in \mathbb{N}$. In particular, we have

$$d(T^{n+1}(x), T^n(x)) \leq k^n \ d(T(x), x),$$

for all $x \in X$ and $n \in \mathbb{N}$. Using the triangle inequality of the distance $d$, we get

$$\begin{aligned} d(T^n(x), T^{n+h+1}(x)) &\leq \sum_{i=n}^{n+h} d(T^i(x), T^{i+1}(x)). \\ &\leq \sum_{i=n}^{n+h} k^i d(x, T(x)). \\ &= k^n \frac{1-k^h}{1-k} d(x, T(x)), \end{aligned}$$

for any $x \in X$ and $n, h \in \mathbb{N}$. Since $k < 1$, we get $1 - k^h < 1$, for any $h \in \mathbb{N}$. Hence

$$d(T^n(x), T^{n+h}(x)) \leq \frac{k^n}{1-k} d(x, T(x)), \tag{2.1}$$

for any $x \in X$ and $n, h \in \mathbb{N}$. Since $k < 1$, we have $\lim_{n \to +\infty} k^n = 0$. In combination with (2.1), we obtain that $\{T^n(x)\}$ is a Cauchy sequence. Since $X$ is complete, there exists $x_0 \in X$ such that $\lim_{n \to \infty} T^n(x) = x_0$. Next, we show that $x_0$ is fixed point of $T$. We have

$$d(T^{n+1}(x), T(x_0)) \leq k \ d(T^n(x), x_0),$$

for any $n \in \mathbb{N}$. This inequality will imply that $\{T^{n+1}(x)\}$ converges to $T(x_0)$. Since it also converges to $x_0$ being a subsequence of $\{T^n(x)\}$, the uniqueness of the limit will force $T(x_0) = x_0$, i.e., $x_0$ is a fixed point of $T$ as claimed. Next, we prove the uniqueness of the fixed point of $T$. Let $z \in X$ be another fixed point of $T$. Then we have

$$d(x_0, z) = d(T(x_0), T(z)) \leq kd(x_0, z)$$

Since $k < 1$, we obtain $d(x_0, z) = 0$, i.e., $x_0 = z$. Therefore, the fixed point $x_0$ is independent of the point $x \in X$. In other words, we have

$$\lim_{n \to \infty} T^n(x) = x_0,$$

for any $x \in X$. Moreover, if we let $h \to +\infty$ in the inequality (2.1), we obtain

$$d(T^n(x), x_0) \leq \frac{k^n}{1-k} d(T(x), x),$$

for all $x \in X$ and $n \in \mathbb{N}$. □

# Chapter 3

# Example

This chapter is based on the article by Ran and Reurings [17].

## 3.1 Basic notation

The following are some notations which are used throughout the chapter.

**The spaces**

| | |
|---|---|
| $\mathcal{M}(\mathbf{n})$ | The set of all $n \times n$ matrices. |
| $\mathcal{H}(\mathbf{n}) \subset \mathcal{M}(\mathbf{n})$ | The set of all $n \times n$ Hermitian matrices, partially ordered. In $\mathcal{H}(n)$ any two matrices has an upper bound and a lower bound. |
| $\mathcal{P}(\mathbf{n}) \subset \mathcal{H}(\mathbf{n})$ | The set of all $n \times n$ positive definite matrices. We say $X > 0$, for $X \in \mathcal{H}(n)$, if and only if $X \in \mathcal{P}(n)$. |

**The distance**

Ran and Reurings [17] used the norm defined by:

$$\|A\|_{1,Q} = \|Q^{\frac{1}{2}} A Q^{\frac{1}{2}}\|_1, \qquad \textit{for } Q \in \mathcal{P}(n),\ A \in \mathcal{M}(n)$$

and where the $\|.\|_1$ is the trace norm defined by:

$$\|A\|_1 = \sum_{j=1}^{n} S_j(A) = tr(A),$$

where $S_j(A)$, j=1,...,n, are the singular values. Recall that the singular values of a matrix are the square roots of the eigenvalues of the used matrix. Throughout, we will use the distance defined by $\|.\|_{1,Q}$, *i.e.*

$$d(X,Y) = \|X - Y\|_{1,Q}$$

Also, we denoted by $\|.\|$ the spectral norm, *i.e.*, $\|A\| = \sqrt{\lambda^+(A^*A)}$ where $\lambda^+(A^*A)$ is the largest eigenvalue of $(A^*A)$ and $A^*$ is the conjugate transpose of the matrix $A$. The $n \times n$ identity matrix will be written as $I_n$.

## 3.2 The discussed problem

Ran and Reurings introduced the following matricial equation:

**Problem.**

*Find $X \in H(n)$ such that*

$$X = Q \pm \sum_{j=1}^{m} {A^*}_j \mathcal{F}(X) A_j, \tag{3.1}$$

*where $\mathcal{F} : \mathcal{H}(n) \longrightarrow \mathcal{H}(n)$ is a function which maps $\mathcal{P}(n)$ into itself, $A_j \in \mathcal{M}(n)$, and $Q \in \mathcal{P}(n)$,* with an extra condition

$$0 \leq \sum_{j=1}^{m} A_j^* Q A_j < Q.$$

The reason behind studying this kind of matricial equations is that they often arise in the analysis of ladder networks, dynamic programming, control theory, stochastic filtering statistics and many other applications [9].

Now set the maps $\mathcal{G}_{\pm}(X) : \mathcal{H}(n) \longrightarrow \mathcal{H}(n)$ defined by:

$$\mathcal{G}_+(X) = Q + \sum_{j=1}^{m} A^*{}_j \mathcal{F}(X) A_j, \ and \qquad \mathcal{G}_-(X) = Q - \sum_{j=1}^{m} A^*{}_j \mathcal{F}(X) A_j.$$

Observe that, $X$ is a solution of (3.1) if and only if $\mathcal{G}_\pm(X) = X$, *i.e* $X$ is a fixed point of $\mathcal{G}_\pm$. Moreover, $\sum_{j=1}^{m} A^*{}_j \mathcal{F}(X) A_j$ is Hermitian, i.e. $\mathcal{G}_\pm(\mathcal{H}(n)) \subset \mathcal{H}(n)$. Indeed, it is known that the sum of any Hermitian matrices is Hermitian. So it remains to show that every term of the form $A^*\mathcal{F}(X)A$ is Hermitian. Indeed, we have:

$$(A^*\mathcal{F}(X)A)^* = A^*\mathcal{F}(X)^*(A^*)^* = A^*\mathcal{F}(X)A,$$

where we used the fact that $\mathcal{F}(X)$ is Hermitian since $\mathcal{F}(\mathcal{H}(n)) \subset \mathcal{H}(n)$. Therefore, we have $\sum_{j=1}^{m} A^*{}_j \mathcal{F}(X) A_j \in \mathcal{H}(n)$.

Now let us consider the following example where we study the existence of fixed points of the maps $\mathcal{G}_\pm$.

**Example 1** *Take $\mathcal{F}(X) = X$. Then the equation (3.1)becomes:*

$$X = Q \pm \sum_{j=1}^{m} A^*{}_j X A_j = \mathcal{G}_\pm(X). \tag{3.2}$$

For this particular example we will investigate some of the properties of the maps $\mathcal{G}_\pm$. First, notice that $\mathcal{G}_+$ is monotone increasing. Indeed, let $X,\ Y \in \mathcal{H}(n)$, such that $X \leq Y$ or $(Y - X \geq 0)$ which means $\langle \vec{u}, (Y - X)\vec{u}\rangle \geq 0$, for any $\vec{u}$. Thus, for any $\vec{u}$ we have

$$\langle \vec{u}, A^*(Y - X)A\vec{u}\rangle = \langle A\vec{u}, (Y - X)A\vec{u}\rangle \geq 0.$$

Hence, $A^*(Y - X)A \geq 0$. Since the sum of positive Hermitian matrices is positive, we conclude that $\sum_{j=1}^{m} A_j^*(Y - X)A_j \geq 0$. Therefore, $\sum_{j=1}^{m} A_j^* X A_j \leq \sum_{j=1}^{m} A_j^* Y A_j$. Hence, $\mathcal{G}_+(X)$ is monotone increasing. Similarly, $\mathcal{G}_-(X) = Q - \sum_{j=1}^{m} A^*{}_j X A_j$ is monotone decreasing. In addition, we have $\mathcal{G}_+(0) = \mathcal{G}_-(0) = Q > 0$ where the 0 here is the zero-matrix.

Next, we investigate the Lipschitzian behavior of both maps $\mathcal{G}_+$ and $\mathcal{G}_-$. The following lemma will be useful,

**Lemma 1** *[17] Let $A \geq 0$, and $B \geq 0$ be $n \times n$ matrices. Then we have $0 \leq tr(AB) \leq ||A|| tr(B)$.*

Let $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$. Therefore, we have:

$$\begin{aligned}
\|\mathcal{G}_\pm(Y) - \mathcal{G}_\pm(X)\|_{1,\tilde{Q}} &= \|Q \pm \sum_{j=1}^{m} A_j^* Y A_j - Q \mp \sum_{j=1}^{m} A^*{}_j X A_j\|_{1,\tilde{Q}} \\
&= \|\sum_{j=1}^{m} A_j^*(Y-X)A_j\|_{1,\tilde{Q}} = tr[\sum_{j=1}^{m} \tilde{Q}^{\frac{1}{2}} A_j^*(Y-X)A_j \tilde{Q}^{\frac{1}{2}}] \\
&= \sum_{j=1}^{m} tr[\tilde{Q}^{\frac{1}{2}} A_j^*(Y-X)A_j \tilde{Q}^{\frac{1}{2}}] = \sum_{j=1}^{m} tr[A_j \tilde{Q}^{\frac{1}{2}+\frac{1}{2}} A_j^*(Y-X)] \\
&= \sum_{j=1}^{m} tr[A_j \tilde{Q} A_j^*(Y-X)] = \sum_{j=1}^{m} tr[A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \tilde{Q}^{\frac{1}{2}} (Y-X) \tilde{Q}^{\frac{1}{2}} \tilde{Q}^{-\frac{1}{2}}] \\
&= \sum_{j=1}^{m} tr[\tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \tilde{Q}^{\frac{1}{2}} (Y-X) \tilde{Q}^{\frac{1}{2}}] \\
&= tr[\sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \tilde{Q}^{\frac{1}{2}} (Y-X) \tilde{Q}^{\frac{1}{2}}] \\
&= tr[(\sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}})(\tilde{Q}^{\frac{1}{2}} (Y-X) \tilde{Q}^{\frac{1}{2}})].
\end{aligned}$$

Using the technical Lemma 1, we get

$$\|\mathcal{G}_\pm(Y) - \mathcal{G}_\pm(X)\|_{1,\tilde{Q}} \leq \|\sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}}\|.\|(Y-X)\|_{1,\tilde{Q}} \tag{3.3}$$

Choose $c = \|\sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}}\|$. Assume that $\sum_{j=1}^{m} A_j^* \tilde{Q} A_j < \tilde{Q}$, for some $\tilde{Q} \in \mathcal{P}(n)$. In this case, we have $c \in [0, 1)$. Indeed, since $\tilde{Q} \in \mathcal{P}(n)$, then we have:

$$\| \sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \| = \| \tilde{Q}^{-\frac{1}{2}} \left( \sum_{j=1}^{m} A_j \tilde{Q} A_j^* \right) \tilde{Q}^{-\frac{1}{2}} \|.$$

Using the assumption on $\tilde{Q}$, we have:

$$\| \sum_{j=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_j \tilde{Q} A_j^* \tilde{Q}^{-\frac{1}{2}} \| < \| \tilde{Q}^{-\frac{1}{2}} \tilde{Q} \tilde{Q}^{-\frac{1}{2}} \| = \| I_n \| = 1.$$

Therefore, the maps $\mathcal{G}_\pm$ are contraction mappings on $\mathcal{H}(n)$ for comparable elements.

Next, fix $X_0 \in \mathcal{H}(n)$ such that $X_0$ and $\mathcal{G}_\pm(X_0)$ are comparable. Then by monotonicity of the maps $\mathcal{G}_\pm$, we have $\mathcal{G}_\pm^n(X_0)$ and $\mathcal{G}_\pm^{n+1}(X_0)$ are also comparable, for $n \in \mathbb{N}$, which implies

$$d(\mathcal{G}_\pm^{n+1}(X_0), \mathcal{G}_\pm^n(X_0)) \leq c^n d(\mathcal{G}_\pm(X_0), (X_0)), \tag{3.4}$$

for $n \geq 0$. This inequality implies that $\left\{ \mathcal{G}_\pm^n(X_0) \right\}_{n=1}^{\infty}$ is a Cauchy sequence. Indeed for $n < m$, we have:

$$\begin{aligned} d(\mathcal{G}_\pm^n(X_0), \mathcal{G}_\pm^m(X_0)) &\leq \sum_{i=n+1}^{m} d(\mathcal{G}_\pm^i(X_0), \mathcal{G}_\pm^{i-1}(X_0)) \\ &= d(\mathcal{G}_\pm^{n+1}(X_0), \mathcal{G}_\pm^n(X_0)) + ... + d(\mathcal{G}_\pm^m(X_0), \mathcal{G}_\pm^{m-1}(X_0)). \end{aligned}$$

Using the inequality (3.4), we have:

$$\begin{aligned} d(\mathcal{G}_\pm^n(X_0), \mathcal{G}_\pm^m(X_0)) &\leq (c^n + c^{n+1} + ... + c^{m-n-1}) d(\mathcal{G}_\pm(X_0), X_0) \\ &= c^n \frac{1 - c^{m-n-1}}{1 - c} d(\mathcal{G}_\pm(X_0), X_0) \leq \frac{c^n}{1 - c} d(\mathcal{G}_\pm(X_0), X_0). \end{aligned}$$

Since $c < 1$, then $\left\{ \mathcal{G}_\pm^n(X_0) \right\}_{n=1}^{\infty}$ is a Cauchy sequence as claimed. Since $\mathcal{H}(n)$ is complete, it follows that:

$$\lim_{n \to \infty} \mathcal{G}_+^n(X_0) = \hat{X}_+, \ and \qquad \lim_{n \to \infty} \mathcal{G}_-^n(X_0) = \hat{X}_-$$

exist. Since $\mathcal{G}_{\pm}$ are continuous, then $\hat{X}_{+}$ and $\hat{X}_{-}$ are fixed points of the maps $\mathcal{G}_{+}$, and $\mathcal{G}_{-}$ respectively.

Next, we investigate the uniqueness of the fixed point of $\mathcal{G}_{\pm}$, so we need to show that:

$$\lim_{n\to\infty} \mathcal{G}_{+}^{n}(X) = \hat{X}_{+}, \text{ and } \lim_{n\to\infty} \mathcal{G}_{-}^{n}(X) = \hat{X}_{-}$$

for every $X \in \mathcal{H}(n)$. Now we will do the proof for $\mathcal{G}_{+}$ and the same argument will apply for $\mathcal{G}_{-}$. Indeed, for comparable elements $X$ and $X_0$ in $\mathcal{H}(n)$, it is obvious because in both cases $\mathcal{G}_{+}^{n}(X)$ and $\mathcal{G}_{+}^{n}(X_0)$ are comparable. Thus, from (3.4), we have:

$$d(\mathcal{G}_{+}^{n}(X), \mathcal{G}_{+}^{n}(X_0)) \leq c^{n} d(X, X_0), \ \textit{for} \ n \in \mathbb{N},$$

now since $c < 1$, the right-hand side tends to 0 if $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \mathcal{G}_{+}^{n}(X) = \lim_{n\to\infty} \mathcal{G}_{+}^{n}(X_0) = \hat{X}_{+}.$$

Now, let $X \in \mathcal{H}(n)$ be arbitrary, i.e., $X$ is not comparable to $X_0$. Then, there exists $Z \in \mathcal{H}(n)$ such that:

$$(X \leq Z, \ \textit{and} \ X_0 \leq Z) \ \textit{or} \ (Z \leq X, \ \textit{and} \ Z \leq X_0).$$

by monotonicity, we have $\mathcal{G}_{+}(Z)$ is comparable to $\mathcal{G}_{+}^{n}(X)$ and $\mathcal{G}_{+}^{n}(X_0)$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} d(\mathcal{G}_{+}^{n}(X), \mathcal{G}_{+}^{n}(X_0)) &\leq d(\mathcal{G}_{+}^{n}(X), \mathcal{G}_{+}^{n}(Z)) + d(\mathcal{G}_{+}^{n}(Z), \mathcal{G}_{+}^{n}(X_0)) \\ &\leq c^{n}[d(X, Z) + d(Z, X_0)]. \end{aligned}$$

Since $c < 1$, we get $\lim_{n\to\infty} d(\mathcal{G}_{+}^{n}(X), \mathcal{G}_{+}^{n}(X_0)) = 0$. Hence,

$$\lim_{n\to\infty} \mathcal{G}_{+}^{n}(X) = \hat{X}_{+}, \qquad \textit{for every} \ X \in \mathcal{H}(n).$$

Therefore, each map $\mathcal{G}_{\pm}$ has a unique fixed point.

**Remark 1** *We have two cases:*

- **Increasing case** $\mathcal{G}_+$*:*

  *We consider for* $\mathcal{G}_+$ *the initial value* $X_0 = (0)$*. Then we have* $\mathcal{G}_+(X_0) = Q > 0 = X_0$*. By the iteration process the induced sequence* $\left\{\mathcal{G}_+^n(X_0)\right\}_{n=1}^{\infty}$ *is increasing. Hence the limit* $\hat{X}_+$ *is bigger than* $Q$ *and then* $\hat{X}_+ \geq Q > 0$*. Thus, limit* $\hat{X}_+$ *is positive and we have a positive definite solution for the discussed problem.*

- **Decreasing case** $\mathcal{G}_-$*:*

  *Again, we use the initial value* $X_0 = (0)$*. Then we have* $\mathcal{G}_-(0) = Q > 0$*. By the iteration process we have the following relations:*

$$\mathcal{G}_-(X_0) = Q > 0 = X_0$$
$$\mathcal{G}_-(X_0) > \mathcal{G}_-^2(X_0) = \mathcal{G}_-(\mathcal{G}_-(X_0))$$

*note that* $\mathcal{G}_-^2(X_0) > 0 = X_0$ *because*

$$\begin{aligned}\mathcal{G}_-^2(X_0) &= \mathcal{G}_-(\mathcal{G}_-(X_0)) = \mathcal{G}_-(Q) \\ &= Q - \sum_{j=1}^{m} A^*{}_j Q A_j > 0. \qquad (by\ (3.2))\end{aligned}$$

*Therefore, we have:*

$$\begin{aligned}&\mathcal{G}_-(X_0) > \mathcal{G}_-^2(X_0) = \mathcal{G}_-(\mathcal{G}_-(X_0)) > X_0 \\ &\mathcal{G}_-(X_0) > \mathcal{G}_-^3(X_0) > \mathcal{G}_-^2(X_0) > X_0 \\ &\mathcal{G}_-(X_0) > \mathcal{G}_-^3(X_0) > \mathcal{G}_-^4(X_0) > \mathcal{G}_-^2(X_0) > X_0. \\ &\qquad\vdots\end{aligned}$$

*Continuing in the iteration process, we have:*

$$X_0 < \mathcal{G}_-^2(X_0) < \mathcal{G}_-^4(X_0) < ...., \; and$$

$$\mathcal{G}_-(X_0) > \mathcal{G}_-^3(X_0) > \mathcal{G}_-^5(X_0) > ... \; .$$

*Therefore, by the assumption* $\mathcal{G}_-(Q) > 0$ *(by (3.2)), we get that the induced sequence of the even terms* $\left\{\mathcal{G}_-^{2n}(0)\right\}_{n=1}^{\infty}$ *is increasing and the sequence of the odd ones* $\left\{\mathcal{G}_-^{2n+1}(0)\right\}_{n=1}^{\infty}$ *is decreasing. Since* $\mathcal{G}_-^n(0) \to \hat{X}_-$*, we get*

$$\mathcal{G}_-^{2n}(X_0) \leq \hat{X}_- \leq \mathcal{G}_-^{2n+1}(X_0), \; for \; every \; n \in \mathbb{N}.$$

*Since for the even ones it holds* $\mathcal{G}_-^{2n}(0) \geq \mathcal{G}_-^2(X_0) = \mathcal{G}_-(Q)$ *and for the odd ones* $\mathcal{G}_-^{2n+1}(X_0) \leq \mathcal{G}_-(0) = Q$*, we get:*

$$0 = X_0 < \mathcal{G}_-(Q) \leq \hat{X}_- \leq Q.$$

*Therefore, the fixed point* $\hat{X}_-$ *is positive. Thus, we get positive definite solution for the discussed problem.*

## 3.3 Partially Ordered Metric Spaces

In some applications, the contraction condition satisfied by the mapping in the Banach Contraction Principle may not be satisfied on the entire space but only on some subset. The existence as well as the uniqueness of a fixed point is worthy of investigation. This is the case with Ran and Reurings' extension of the Banach Contraction Principle in metric spaces endowed with a partial order.

### 3.3.1 Basic Definitions and Notations

Let $X$ be a set. An **order** (or **partial order**) on $X$ is a binary relation $\preceq$ on $X$ such that, for all $x, y, z \in X$, we have

(i) $x \preceq x$ (reflexivity),

(ii) $x \preceq y$ and $y \preceq x$ implies $x = y$ (antisymmetry),

(iii) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ (transitivity).

A metric space $(X, d)$ endowed with a partial order $\preceq$ will be known as a **partially ordered metric space** and will be denoted by $(X, d, \preceq)$.Two points $x$ and $y$ are said to be comparable if $x \preceq y$ or $y \preceq x$.

**Definition 5** *$(X, d, \preceq)$ is a **lattice** if each two-element subset $\{x, y\} \subseteq$ has a least upper bound (supremum) and a greatest lower bound (infimum) in X.*

**Definition 6** *$(X, d, \preceq)$ is **complete lattice** if every subset $Y \subseteq X$ has a supremum and an infinimum in X.*

**Definition 7** *Let $(X, \preceq)$ be an ordered set. A mapping $T : X \to X$ is said to be **monotone increasing** (resp. **decreasing**) if for all $x, y \in X$, we have $x \preceq y$ implies $T(x) \preceq T(y)$ (resp. $T(y) \preceq T(x)$). $T$ is called monotone if it is monotone increasing or decreasing.*

### 3.3.2 Banach Contraction Principle in Partially Ordered Metric Spaces

In this subsection, we discuss the extension of the Banach Contraction Principle in partially ordered metric spaces based on Ran and Reurings [17] and Nieto and Rodríguez-Lopez [15] articles.

**Definition 8** *Let $(X, d, \preceq)$ be a partially ordered metric space. A mapping $T : X \to X$ is said to be* ***monotone Lipschitzian*** *if $T$ is monotone and there exists $k \geq 0$ such that*

$$d(T(x), T(y)) \leq k\ d(x, y),$$

*for any $x, y \in X$ such that $x \preceq y$. If $k < 1$, then $T$ is called a* ***monotone contraction****.*

As we said earlier, the contraction condition is not satisfied on the entire space but only on comparable elements. In particular, monotone Lipschitzian mappings may not be continuous while Lipschitzian mappings are not only continuous, but they are uniformly continuous.

Let $(X, d \preceq)$ be a partially ordered metric space and $T : X \to X$ be a monotone contraction. Clearly if $x_0$ is a fixed point of $T$, then we have $x_0 \preceq T(x_0)$ and $T(x_0) \preceq x_0$. Next, we build a simple example of a monotone contraction $T$ such that $T(x)$ and $x$ are not comparable for any $x \in X$. In this case $T$ has no fixed point.

**Example 2** *Consider the Euclidean plane $(\mathbb{R}^2, d)$. Define $L_i = \{(i, y);\ y \in \mathbb{R}\}$, for $i = 1, 2$. It is clear that $(L_1 \cup L_2, d)$ is a complete metric space. Define the partial order on $L_1 \cup L_2$ by*

$$(x_1, y_1) \preceq (x_2, y_2)\ if\ \text{and only}\ if x_1 = x_2\ \text{and}\ y_1 \leq y_2.$$

*Consider the mapping $T : L_1 \cup L_2 \to L_1 \cup L_2$ defined by*

$$\begin{cases} T(1, y) = (2, \frac{1}{2}y); \\ T(2, y) = (1, \frac{1}{2}y). \end{cases}$$

*Since any point in $L_1$ and any point in $L_2$ are not comparable, there is no element $(x, y) \in L_1 \cup L_2$ such that $(x, y)$ and $T(x, y)$ are comparable. Therefore, $T$ has no fixed point. Moreover, $T$ is a monotone contraction. Indeed, let $(x_1, y_1) \preceq (x_2, y_2)$. Then we have $x_1 = x_2$ and $y_1 \leq y_2$. Set $x = x_1 = x_2$. Hence*

$$d(T(x,y_1),T(x,y_2)) = \sqrt{((3-x)-(3-x))^2 + \left(\frac{1}{2}y_1 - \frac{1}{2}y_2\right)^2} = \frac{1}{2}\sqrt{(y_1-y_2)^2} = \frac{1}{2}d((x,y_1),(x,y_2)),$$

*i.e., $T$ is contraction on comparable elements with $k = \frac{1}{2}$. It is clear that $T$ is not a contraction on the whole $L_1 \cup L_2$. Let $x = (1, 1)$ and $y = (2, 1)$, then we have:*

$$T[(1,1)] = (2, \frac{1}{2}), \ \textit{and} \ T[(2,1)] = (1, \frac{1}{2})$$

*Therefore, we get:*

$$d(T(1,1),(2,1)) = 1, \ \textit{and} \ d(T(1,1),(2,1)) = 1.$$

Therefore, to hope for the existence of a fixed point of $T$, we must assume that there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Hence it is natural to ask what assumptions are needed to make sure that $T$ has a fixed point. In order to understand what was done for monotone contraction mappings, we will need the following technical lemma.

**Lemma 2** *Let $(X, d \preceq)$ be a partially ordered metric space and $T : X \to X$ be a monotone contraction. If $x$ and $y$ are comparable, then we have*

$$d(T^n(x), T^n(y)) \leq k^n \ d(x,y),$$

*for any $n \in \mathbb{N}$. In particular, if there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable, then the orbit $\{T^n(x_0)\}$ is a Cauchy sequence.*

**Proof:** Since $x$ and $y$ are comparable and $T$ is monotone, we deduce that $T^n(x)$ and $T^n(y)$ are comparable. Since $T$ is a monotone contraction, there exists $k \in [0, 1)$ such that

$$d(T(u), T(v)) \leq k \; d(u, v),$$

for any comparable elements $u, v \in X$. Hence

$$d(T^n(x), T^n(y)) \leq k \; d(T^{n-1}(x), T^{n-1}(y)) \leq k^n \; d(x, y),$$

for any $n \in \mathbb{N}$. Next, let $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Hence

$$d(T^n(x_0), T^{n+1}(x_0)) \leq k^n \; d(x_0, T(x_0)),$$

for any $n \in \mathbb{N}$. As we have done in the proof of the Banach Contraction Principle, we obtain

$$d(T^n(x_0), T^{n+h}(x_0)) \leq \frac{k^n}{1-k} \; d(x_0, T(x_0)),$$

for any $n, h \in \mathbb{N}$, which implies that $\{T^n(x_0)\}$ is a Cauchy sequence. □

Using Lemma 2, we obtain the following result.

**Theorem 2** *Let $(X, d \preceq)$ be a complete partially ordered metric space. Let $T : X \rightarrow X$ be a monotone contraction. Assume there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Then $\{T^n(x_0)\}$ converges to $\omega \in X$. Moreover, for any $x \in X$ comparable with $x_0$, we have $\lim_{n \rightarrow +\infty} T^n(x) = \omega$.*

**Proof:** Lemma 2 implies that $\{T^n(x_0)\}$ is a Cauchy sequence. Since $X$ is complete, then $\{T^n(x_0)\}$ is convergent. Set $\omega = \lim_{n \rightarrow +\infty} T^n(x_0)$. Let $x \in X$ be comparable to $x_0$. Again Lemma 2 implies

$$d(T^n(x_0), T^n(x)) \leq k^n \; d(x_0, x),$$

for an $n \in \mathbb{N}$. In particular, we have $\lim_{n\to+\infty} d(T^n(x_0), T^n(x)) = 0$ since $k < 1$, which implies $\lim_{n\to+\infty} T^n(x) = \omega$. $\square$

Following the conclusion of Theorem 2, it is natural to wonder when the limit of an orbit is a fixed point. The first such result is given by Ran and Reurings [17].

**Theorem 3** *Let $(X, d \preceq)$ be a complete partially ordered metric space. Let $T : X \to X$ be a continuous monotone contraction. Assume there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Then $\{T^n(x_0)\}$ converges to a fixed point $\omega$ of $T$.*

**Proof:** Theorem 2 implies that $\{T^n(x_0)\}$ converges to $\omega \in X$. Since $T$ is continuous, we get

$$\omega = \lim_{n\to+\infty} T^{n+1}(x_0) = \lim_{n\to+\infty} T(T^n(x_0)) = T(\omega),$$

i.e., $\omega$ is a fixed point of $T$. $\square$

In order to fully have an extension of the Banach Contraction Principle, we need to discuss the uniqueness of the fixed point of $T$. Note that under the assumptions of Theorem 3, we have

$$\lim_{n\to+\infty} T^n(x) = \omega,$$

for any $x \in X$ comparable with $x_0$. Therefore, if $X$ is lattice, then for any $x \in X$ not necessarily comparable to $x_0$, there exists $z \in X$ such that $x \preceq z$ and $x_0 \preceq z$ (here we only used the existence of an upper bound of two elements of $X$). Then we know that $\lim_{n\to+\infty} T^n(z) = \omega$. Using Lemma 2, we also have $\lim_{n\to+\infty} d(T^n(z), T^n(x)) = 0$ which implies $\lim_{n\to+\infty} T^n(x) = \omega$. In other words, we have $\lim_{n\to+\infty} T^n(x) = \omega$, for any $x \in X$. This is exactly the main conclusion of the Banach Contraction Principle discovered by Ran and Reurings [17].

**Theorem 4** *[17] Let $(X, d \preceq)$ be a complete partially ordered metric space. Assume that $(X, \preceq)$ is a lattice. Let $T : X \to X$ be a continuous monotone contraction. Assume there* *exists* $x_0 \in X$ *such that* $x_0$ *and* $T(x_0)$ *are comparable. Then* $\{T^n(x_0)\}$ *converges to the unique fixed point* $\omega$ *of* $T$*. Moreover, we have*

$$\lim_{n\to+\infty} T^n(x) = \omega,$$

*for any* $x \in X$*.*

# Chapter 4

# Banach Contraction Principle in Extended Quasi-metric Spaces

In this chapter we will show that the version of Banach contraction theorem that was proved in the paper [17] by Ran and Reurings on a metric space is a special case of the one that was proved in the paper [1] by Abdou, Aljohani and Khamsi in the extended quasi-metric spaces.
First, we give the definition of the concept of an extended quasi-metric space.

**Definition 9** *Let $X$ be an abstract set. The function $\bar{d} : X \times X \to [0, \infty]$ is called an* ***extended quasi-distance*** *if the following conditions are satisfied:*

*(i)* $\bar{d}(x, y) = 0 \Leftrightarrow x = y$,

*(ii)* $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$, *for all* $x, y, z \in X$ *(oriented triangle inequality).*

*In this case, the pair $(X, \bar{d})$ is called an extended quasi-metric space.*

For a list of examples on quasi-metric spaces one could look into paper [6]. The issue of a topology comes as one of the difficulties when we deal with quasi-metric spaces more precisely the concepts of convergence, Cauchy and completeness [7, 17, 18] [1].

**Definition 10** *Let $(X, \bar{d})$ be an extended quasi-metric space. A sequence $\{x_n\}$ in $X$ is said to* ***convergent*** *if there exists $x \in X$ such that $\lim_{n\to+\infty} \bar{d}(x_n, x) = 0$. $X$ is said to be a* $T_2$***-space*** *if and only if the limit of the sequence is unique, i.e., whenever $\lim_{n\to+\infty} \bar{d}(x_n, x) = \lim_{n\to+\infty} \bar{d}(x_n, y) = 0$ implies $x = y$. We will say that a subset $Y$ of $X$ is* ***closed*** *in $(X, \bar{d})$ if $Y$ contains the limit of any convergent sequence from $Y$.*

From now on, we will assume that any extended quasi-metric space $(X, \bar{d})$ is a $T_2-space$. For more details about the concept of Cauchy sequences in extended quasi-metric spaces, we refer the reader to paper [7].

**Definition 11** *Let $(X, \bar{d})$ be an extended quasi-metric space. A sequence $\{x_n\}$ in $X$ is said to be **Cauchy** if and only if there exists a sequence $\{y_n\}$ such that for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $n,\ m > N$ we have*

$$\bar{d}(x_n, y_m) < \varepsilon,$$

*i.e., $\lim_{n,m \to +\infty} \bar{d}(x_n, y_m) = 0$. The sequence $\{y_n\}$ is called a cosequence to $\{x_n\}$. The extended quasi-metric space $(X, \bar{d})$ is said to be **complete** if and only if any Cauchy sequence in $(X, \bar{d})$ is convergent.*

Even though this definition is kind of complicated, it permits the following to be true:

**Proposition 1** *Let $(X, \bar{d})$ be an extended quasi-metric space, then the following hold:*

*(i) Every convergent sequence in $(X, \bar{d})$ is a Cauchy sequence.*

*(ii) Every subsequence of a Cauchy sequence is a Cauchy sequence.*

*(iii) If $(X, \bar{d})$ is an extended metric space, then Definition 3 is equivalent to the usual definition of Cauchy sequences.*

Now, we will give an example of quasi- metric space in the positive real line $\mathbb{R}$.

**Example 3 (Sorgenfrey line)** *The Sorgenfrey line is a topology on $\mathbb{R}$ generated by a base of half open interval $[a, b)$ where $a < b$. Let $\bar{d} : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ defined by:*

$$\bar{d}(x, y) = \begin{cases} y - x, & x \leq y \\ +\infty, & otherwise. \end{cases}.$$

*Then $(\mathbb{R}, \bar{d})$ is the Sorgenfrey extended quasi-metric space on $\mathbb{R}$ because of the following:*

*(i)* $\bar{d}(x, y) = 0$ *if and only if* $y - x = 0 \Leftrightarrow x = y$.

*(ii)* *To show that*

$$\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y).$$

*Assume* $\bar{d}(x, z) < +\infty$ *and* $\bar{d}(z, y) < +\infty$. *Hence,* $x \geq z$, *and* $z \geq y$, *then we have* $x \geq y$. *Thus,* $\bar{d}(x, y) < +\infty$, *then we have* $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$.

*Clearly,* $(\mathbb{R}, \bar{d})$ *is a Cauchy. Now, we discuss if it is a* $T_2$*-space. By definition, there exist* $n_0$ *such that for all* $n \geq n_0$ *we have:*

$$\bar{d}(x_n, x) = x - x_n \Rightarrow x_n \to x, \text{ and}$$

$$\bar{d}(x_n, y) = y - x_n \Rightarrow x_n \to y.$$

*Hence,* $x = y$, *and* $(\mathbb{R}, \bar{d})$ *is a* $T_2$*-space.*

**Definition 12** *Let* $(X, \bar{d})$ *be an extended quasi-metric space. A mapping* $T : X \longrightarrow X$ *is said to be* ***Lipschitzian*** *if there exists* $k > 0$ *such that*

$$\bar{d}(T(x), T(y)) \leq \ k \ \bar{d}(x, y),$$

*for all* $x, y \in X$. *If* $k < 1$, *then* $T$ *is said to be a* ***contraction*** *mapping. A point* $x \in X$ *is called a fixed point of* $T$ *whenever* $T(x) = x$.

**Remark 2** *Let* $(X, \bar{d})$ *be an extended quasi-metric space and* $T : X \to X$ *a* $\bar{d}$*-contraction mapping. Then for any fixed points* $x$ *and* $y$ *of* $T$, *we have* $x = y$ *whenever* $\bar{d}(x, y) < \infty$.

The following technical lemma will be useful when studying the Banach contraction principle in extended metric spaces:

**Lemma 3** *[1] Let* $(X, \bar{d})$ *be a complete extended quasi-metric space. Let* $T : X \to X$ *be a contraction. Set* $X_T = \{x \in X; \ \bar{d}(x, T(x)) < \infty\}$. *For any* $x_0 \in X_T$ *, the orbit* $\{T_n(x_0)\}$ *is*

*Cauchy. Moreover if $\{T^n(x_0)\}$ converges to $x \in X$, then $T(x) = x$, i.e., $x$ is a fixed point of $T$.*

In the next example, we discuss how a partially ordered metric space may be endowed with an extended quasi-metric structure. In fact, this example will give a better understanding of the fixed point theorems of Ran and Reurings [17], Nieto and Rodriguez-Lopez [15], Jachymski [10] and Ben-El-Mechaiekh [8] in connection with Theorem 3.1 [1].

In the next example, we show that a partially ordered metric space is in fact an extended quasi-metric space. To do this, we introduce a new concept which we call Sorgenfrey space similar to the Sorgenfrey line.

**Example 4** **(Sorgenfrey space) Let $(X, d, \preceq)$ be a partially ordered metric space. Define $\bar{d} : X \times X \to [0, +\infty]$ by:**

$$\bar{d}(x,y) = \begin{cases} d(x,y), & x \preceq y \\ +\infty, & otherwise. \end{cases}$$

Now, observe that $(X, \bar{d})$ is an extended quasi-metric space because of the following:

(i) $\bar{d}(x,y) = 0$ if and only if $d(x,y) = 0 \Leftrightarrow x = y$.

(ii) To show that

$$\bar{d}(x,y) \leq \bar{d}(x,z) + \bar{d}(z,y).$$

Assume $\bar{d}(x,z) < +\infty$ and $\bar{d}(z,y) < +\infty$. Hence, $x \preceq z$, and $z \preceq y$, then we have $x \preceq y$. Thus, $\bar{d}(x,y) < +\infty$, and by the triangle inequality on $(X, d)$ we have $\bar{d}(x,y) \leq \bar{d}(x,z) + \bar{d}(z,y)$.

Note that if $(X, d)$ is complete and the order intervals are closed, then $(X, \bar{d})$ is a complete quasi-metric space [1]. Next, we investigate Lipschitzian mappings in $(X, \bar{d})$. Assume

$T : X \to X$ be a Lipschitzian mapping. Then there exists $k > 0$ such that

$$\bar{d}(T(x), T(y)) \leq k \; \bar{d}(x, y),$$

for any $x, y \in X$. Fix $x, y \in X$ such that $\bar{d}(x, y) < +\infty$, then $\bar{d}(T(x), T(y)) < +\infty$ holds. In other words, if $x \preceq y$, then $T(x) \preceq T(y)$. This is the definition of a monotone increasing mapping. Moreover, we have:

$$\bar{d}(T(x), T(y)) = d(T(x), T(y)) \leq k \; d(x, y) = k \; \bar{d}(x, y).$$

Therefore, if $T$ is a Lipschitizan mapping in $(X, \bar{d})$, then $T$ is a Lipchitzian monotone mapping in $(X, \preceq, d)$ with the same Lipchitz constant. The converse is also true. In particular, $T$ is a monotone contraction mapping in $(X, \preceq, d)$ if and only if $T$ is a contraction mapping in $(X, \bar{d})$. Putting all these results together, we get an analogue result to lemma 3 in partially ordered metric spaces.

**Theorem 5** *[1] Let $(X, d, \preceq)$be a complete partially ordered metric space. Assume that order intervals are closed. Let $T : X \to X$ be a contraction mapping. Set $X_T = \{x \in X; x \preceq T(x)\}$. Then $T$ has a fixed point provided $X_T \neq \phi$.*

# References


[1] A.A.N. Abdou, M. Aljohani, M.A. Khamsi, *A Novel Approach to Banach Contraction Principle in extended quasi-metric spaces* ,preprint.

[2] M.R. Alfuraidan, M.A. Khamsi, *Fixed Points of Monotone Nonexpansive Mappings on a Hyperbolic Metric Space with a Graph*, Fixed Point Theory and Applications (2015) 2014:44 doi: 10.1186/s13663-015-0294-5.

[3] M.R. Alfuraidan, *Fixed Points of Multivalued Mappings in Modular Function Spaces with a Graph*, Fixed Point Theory and Application 2015, 2015:42 doi: 10.1186/s13663-015-0292-7.

[4] M.R. Alfuraidan, *The Contraction Principle for Mappings on a Modular Metric Space with a Graph*, Fixed Point Theory and Applications (2015) 2015:46 doi: 10.1186/s13663-015-0296-3.

[5] M.R. Alfuraidan, *The Contraction Principle for Multivalued Mappings on a Modular Metric Space with a Graph*, Canad. Math. Bull. 59(2016), 3-12.

[6] M.R. Alfuraidan, *Fixed Points of Monotone Nonexpansive Mappings with a Graph*, Fixed Point Theory and Applications (2015) 2015:49 doi: 10.1186/s13663-015-0299-0.

[7] S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applications*, Fund. Math. 3(1922), 133-181.

[8] H. Ben-El-Mechaiekh, The Ran-Reurings Fixed Point Theorem without Partial Order: A Simple Proof, Journal of Fixed Point Theory and Application 16(2014), 373-383.

[9] S.M. El-Sayed, A.C.M. Ran, *On an Iteration Method for Solving a Class of Nonlinear Matrix Equations*, SIAM Journal on Matrix Analysis and Applications 23: 3 (2002), 632-645.

[10] J. Jachymski, *The Contraction Principle for Mappings on a Metric Space with a Graph,* Proc. Amer. Math. Soc. 136(2008), 1359–1373.

[11] J. Jachymski, G.G. Lukawska, *IFS on a Metric Space with a Graph Structure and Extension of the Kelisky-Rivlin Theorem*, J. Math. Anal. Appl. 356 (2009) 453-463.

[12] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory,* John Wiley, New York, 2001.

[13] W.A. Kirk, S. Brailey, *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publisher, Norwell,USA, 2001.

[14] T. S. Morrison, *The Banach and Brouwer Fixed Point Theorems and Applications*, Morgan State University, Baltimore, USA, 2007.

[15] J.J. Nieto, R. Rodríguez-López, *Contractive Mapping Theorems in Partially Ordered Sets and Applications to Ordinary Differential Equations,* Order 22 (2005), 223–239.

[16] H. Poincaré, Mmoire sur les courbes dfinies par une quation diffrentielle (IV), Journal de Math. Pures et Appl. 2 (1886), 151-218

[17] A.C.M. Ran, M.C.B. Reurings, *A Fixed Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations*, Proc. Amer. Math. Soc. 132 (2003) 1435–1443.

[18] M. Turinici, *Fixed Points for Monotone Iteratively Local Contractions*, Dem. Math., 19 (1986), 171-180.

[19] M. Turinici, *Ran and Reurings theorems in Ordered Metric Spaces*, J.Indian Math Soc. 78 (2011),207-2014.

# Curriculum Vitae

Mohamed Illafe received a B.S. (2005) in Mathematics from the University of Benghazi. His research interests include Fixed Point Theory.

Permanent address: 300 Eubank Ct Apt 4A El Paso, Texas 79902