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Note on Fair Price under Interval Uncertainty

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Abstract

Often, in decision making situations, we do not know the exact value of a gain resulting from making each decision, we only know the bounds on this gain. To make a reasonable decision under such interval uncertainty, it makes sense to estimate the fair price of each alternative, and then to select the alternative with the highest price. In this paper, we show that the value of the fair price can be uniquely determined from some reasonable requirements: e.g., the additivity requirement, that the fair price of two objects together should be equal to the sum of the fair prices of both objects.

Keywords: decision making, interval uncertainty, fair price, Hurwicz optimism-pessimism criterion

1 Decision Making under Interval Uncertainty: an Important Practical Problem

In many practical situations, we need to make a decision in situations when we do not know the exact consequences of each action. In some such situations, we know the probabilities of different consequences. However, often, we do not know these probabilities, all we know is a lower bound $\underline{x}$ and an upper bound $\overline{x}$ on the possible gain; see, e.g., [2]. In such only-bounds situations, the only information that we have about the resulting gain x is that this value x belongs to the known interval $\mathbf{x} = [\underline{x}, \overline{x}]$; because of this, such uncertainty is known as *interval uncertainty.*

If we knew the exact gain x_i resulting from selecting the i-th alternative, then the solution would be easy: select an alternative for which the expected gain x_i is the largest. Under interval uncertainty, we do not know the exact gains x_i, we only know the intervals $[\underline{x}_i, \overline{x}_i]$ of possible values of these gains. Based on this information, we must select a reasonable alternative i.

2 The General Problem of Decision Making under Uncertainty Can Be Reduced to the Problem of Fair Price under Interval Uncertainty

In the market economy, there is no need to compare different pairs of possible objects, it is sufficient to set an appropriate price of each object. Once the prices are established, it is easy to predict when a possible barter will be accepted by a customer: when the total price of all objects that he or she gains as a result of this exchange is equal to (or exceeds) the total price of all the objects that this customer gives away in this exchange.

From this viewpoint, in order to decide when an alternative with a gain range $[\underline{x}_i, \overline{x}_i]$ is better than an alternative with a gain range $[\underline{x}_j, \overline{x}_j]$, it is reasonable to find a fair price for each of these interval-valued alternatives. Then, in the decision making process, we select an alternative with the largest equivalent price.

Thus, the problem of decision making under uncertainty is reduced to the problem of finding the fair price under interval uncertainty.

3 Selecting a Fair Price under Interval Uncertainty: Towards a Formalization of the Problem

In principle, we can have all possible interval ranges $[\underline{x},\overline{x}]$. Thus, we need to be able, given any such interval, to estimate its fair price. Let us denote the fair price corresponding to the interval $\mathbf{x}=[\underline{x},\overline{x}]$ by $f(\mathbf{x})$. In these terms, the problem of selecting a fair price under interval uncertainty can be described as the problem of finding an appropriate function f from the set of all intervals to the set of real numbers.

What are the natural properties of this function? First, since the resulting gain x is somewhere between $\underline{x}$ and $\overline{x}$, this means that we are sure that the gain will be not larger than $\overline{x}$. Hence, the price that a reasonable person is willing to pay for this alternative should not exceed $\overline{x}$: $f([\underline{x},\overline{x}])\le\overline{x}$.

Similarly, we are sure that the gain will be not smaller than $\underline{x}$. Hence, the price that a reasonable person is willing to accept for selling his or her participation in this alternative should not smaller than $\underline{x}$: $f([\underline{x},\overline{x}])\ge\underline{x}$.

Another reasonable property is that if the fair price for the object with a price range $[\underline{x},\overline{x}]$ is $p=f([\underline{x},\overline{x}])$, and the fair price for the object with a gain range $[\underline{y},\overline{y}]$ is $q=f([\underline{y},\overline{y}])$, then the fair price of these two objects together should be $p+q$. For this combination of two objects, with price ranges $[\underline{x},\overline{x}]$ and $[\underline{y},\overline{y}]$, the total actual (unknown) price range from $\underline{x}+\underline{y}$ (in the most pessimistic case) to $\overline{x}+\overline{y}$ (in the most optimistic case). Thus, for this combination, the range of actual prices is $[\underline{x}+\underline{y},\overline{x}+\overline{y}]$. So, we conclude that for this combined range, the fair price should be equal to the sum $p+q=f([\underline{x},\overline{x}])+f([\underline{y},\overline{y}])$ of fair prices of individual objects:

$$f([\underline{x}+\underline{y},\overline{x}+\overline{y}])=f([\underline{x},\overline{x}])+f([\underline{y},\overline{y}]).$$

It turns out that these reasonable conditions lead to a reasonable expression for a fair price under interval uncertainty.

4 Main Result

The above reasonable properties of the desired function $f([\underline{x},\overline{x}])$ result in the following definition.

Definition 1. *We say that a function $f([\underline{x},\overline{x}])$ from intervals to real numbers* describes fair price under interval uncertainty *if this function satisfies the following two properties:*

- *for all $\underline{x}$ and $\overline{x}$, we have $\underline{x}\le f([\underline{x},\overline{x}])\le\overline{x}$; and*
- *for all $\underline{x}$, $\overline{x}$, $\underline{y}$, and $\overline{y}$, we have: $f([\underline{x}+\underline{y},\overline{x}+\overline{y}])=f([\underline{x},\overline{x}])+f([\underline{y},\overline{y}])$.*

Proposition 1. *A function $f([\underline{x},\overline{x}])$ describes fair price under interval uncertainty if and only if this function has the form*

$$f([\underline{x},\overline{x}])=\alpha\cdot\overline{x}+(1-\alpha)\cdot\underline{x}$$

for some real number $\alpha\in[0,1]$.

Comment. The above formula was originally proposed by the future Nobelist L. Hurwicz; it is known as Hurwicz's *optimism-pessimism criterion* [1, 2]. This name comes from the following analysis.

When $\alpha=1$, this means that the fair price $f([\underline{x},\overline{x}])$ of an object with the price range $[\underline{x},\overline{x}]$ is equal to $\overline{x}$. In this case, we only take into account the most optimistic case, when the actual price is equal to its largest possible value $\overline{x}$.

When $\alpha=0$, this means that the fair price $f([\underline{x},\overline{x}])$ of an object with the price range $[\underline{x},\overline{x}]$ is equal to $\underline{x}$. In this case, we only take into account the most pessimistic case, when the actual price is equal to its smallest possible value $\underline{x}$.

For intermediate values α, we take into account both the optimistic and the pessimistic scenarios.

Proof.

1°. When we know the exact price x, i.e., when the interval has the degenerate form $[x,x]$, then the first property from Definition 1 implies that $x \le f([x,x]) \le x$, and thus, that $f([x,x]) = x$. In other words, in this case, the fair price is the actual (known) price x of the object.

2°. Let α denote the fair price $f([0,1])$ of the interval $[0,1]$. Due to the first property from Definition 1, we have $0 \le \alpha \le 1$.

3°. Let us now consider intervals of the type $[0,x]$, with $x > 0$. Let us denote the fair price of an object with the price range $[0,x]$ by $g(x) \stackrel{\text{def}}{=} f([0,x])$.

3.1°. For two such intervals $[0,x]$ and $[0,y]$, additivity (the second property from Definition 1) implies that $f([0,x+y]) = f([0,x]) + f([0,y])$, i.e., in terms of our new notation, that $g(x+y) = f(x) + g(y)$.

3.2°. For $x = y = 1$, we get $g(2) = g(1)+g(1) = \alpha+\alpha = 2\alpha$. For $x = 2$ and $y = 1$, we get $g(3) = g(2)+g(1) = 2\alpha+\alpha = 3\alpha$. By induction over n, we can now prove that for every positive integer n, we have $g(n) = n \cdot \alpha$.

3.3°. For each q, from $1 = \dfrac{1}{q} + \ldots + \dfrac{1}{q}$ (q times), we similarly conclude that $\alpha = g(1) = q \cdot g\left(\dfrac{1}{q}\right)$, and thus, that $g\left(\dfrac{1}{q}\right) = \dfrac{1}{q} \cdot \alpha$.

3.4°. From the above equality, for arbitrary p and q, from $\dfrac{p}{q} = \dfrac{1}{q} + \ldots + \dfrac{1}{q}$ (p times), we conclude that $g\left(\dfrac{p}{q}\right) = p \cdot g\left(\dfrac{1}{q}\right) = \dfrac{p}{q} \cdot \alpha$. Thus, for every positive rational number r, we have $g(r) = r \cdot \alpha$.

3.5°. For each $x \ge 0$, the first property implies that $g(x) = f([0,x]) \ge 0$. Thus, if $x \le y$, i.e., if $y - x \ge 0$, then $g(y) = g(x) + g(y-x)$ implies that $g(x) \le g(y)$.

3.6°. Every real number x can be approximated, with any given accuracy 2^{-n}, by rational numbers r_n and r'_n: $r_n \le x \le r'_n$, with $r_n \to x$ and $r'_n \to x$ as $n \to \infty$. From Part 3.5 of this proof, we thus conclude that $g(r_n) \le g(x) \le g(r'_n)$.

Since both r_n and r'_n are rational numbers, from Part 3.4, we conclude that $g(r_n) = r_n \cdot \alpha$ and $g(r'_n) = r'_n \cdot \alpha$. Hence, the above inequality takes the form $r_n \cdot \alpha \le g(x) \le r'_n \cdot \alpha$. In the limit when $r_n \to x$ and $r'_n \to x$, we get $x \cdot \alpha \le g(x) \le x \cdot \alpha$, therefore $g(x) = x \cdot \alpha$.

4°. A general interval $[\underline{x}, \overline{x}]$ can be represented as the sum of a degenerate interval $[\underline{x}, \underline{x}]$ and an interval $[0, \overline{x} - \underline{x}]$ with the zero left endpoint. Thus, due to additivity, we have

$$f([\underline{x}, \overline{x}]) = f([\underline{x}, \underline{x}]) + f([0, \overline{x} - \underline{x}]).$$

From Part 1, we know that $f([\underline{x}, \underline{x}]) = \underline{x}$. From Part 3, we know that

$$f([0, \overline{x} - \underline{x}]) = \alpha \cdot (\overline{x} - \underline{x}).$$

So, we get

$$f([\underline{x}, \overline{x}]) = \underline{x} + \alpha \cdot (\overline{x} - \underline{x}),$$

i.e.,

$$f([\underline{x}, \overline{x}]) = \alpha \cdot \overline{x} + (1 - \alpha) \cdot \underline{x}.$$

The proposition is proven.

5 An Argument in Favor of $\alpha = 0.5$

In principle, different values α are possible. We will show, however, that the most reasonable choice of α is $\alpha = 0.5$.

Indeed, the fact that we do not know the probabilities of different alternatives means that there is an event E whose probability we do not know. Any fixed amount of money x_0 can therefore be subdivided into two complementary bets:

- in the first bet, we give x_0 if the event E occurs and 0 otherwise;
- in the second bet, we give x_0 if the event E does not occur, and 0 otherwise.

For each of these two bets, the actual price can take any value from 0 to x_0, and we do not know the probability of different values. Thus, according to our result, the fair price of each bet is equal to $f([0, x_0]) = \alpha \cdot x_0 + (1 - \alpha) \cdot 0 = \alpha \cdot x_0$.

Thus, for the combination of these two bets, the fair price should be equal to the sum of their fair prices, i.e., to $\alpha \cdot x_0 + \alpha \cdot x_0 = 2 \cdot \alpha \cdot x_0$.

On the other hand, the combination of these two bets means getting the value x_0 without any uncertainty, so this combination has a fair price x_0. Thus, we conclude that $2 \cdot \alpha \cdot x_0 = x_0$, and hence, that $\alpha = 0.5$.

References

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