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Fuzzy Xor Classes from Quantum Computing*

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Abstract. By making use of quantum parallelism, quantum processes provide parallel modelling for fuzzy connectives and the corresponding computations of quantum states can be simultaneously performed, based on the superposition of membership degrees of an element with respect to the different fuzzy sets. Such description and modelling is mainly focussed on representable fuzzy Xor connectives and their dual constructions. So, via quantum computing not only the interpretation based on traditional quantum circuit is considered, but also the notion of quantum process in the qGM model is applied, proving an evaluation of a corresponding simulation by considering graphical interfaces of the VPE-qGM programming environment. The quantum interpretations come from measurement operations performed on the corresponding quantum states.

1 Introduction

Fuzzy logic (FL) and quantum computing (CQ) are relevant research areas consolidating the analysis and the search for new solutions for difficult problems faster than the classical logical approach or conventional computing. Similarities between these areas in the representation and modelling of uncertainty have been explored in [1],[2], [3], [4] and [5].

The former expresses the uncertainty of human being's reasoning by making use of the Fuzzy Sets Theory (FST), as a mathematical model inheriting the imprecision of natural language and determining the membership degree of an element in a fuzzy set. Based on such theory, fuzzy techniques will help physicists and mathematicians to transform their imprecise ideas into new computational programs [6]. The latter approach models the uncertainty of the real world by making use of properties (superposition and entanglement) of quantum mechanics, suggesting an improvement in the efficiency regarding complex tasks. Thus, simulations using classical computers allow the development and validation of basic quantum algorithms (QAs), anticipating the knowledge related to their behaviors when executed in a quantum hardware. In this scenario, the *VPE-qGM* (Visual Programming Environment for the Quantum Geometric Machine

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Model), previously described in [7] and [8], is a quantum simulator that adds the following advantages: the visual modelling and the parallel or distributed simulation of QAs. Additionally, it is possible to show the application and evolution of quantum states through integrated graphical interfaces [9].

So, it would be interesting to investigate new methods dealing with quantum fuzzy applications. Extending previous works in [10] and [11], the aim of this work is mainly related to: (i) the description via QC of representable fuzzy Xor connectives and dual constructions, by using the traditional quantum circuits (qCs); (ii) the modelling of fuzzy X(N)or connectives based on quantum processes in the qGM model and corresponding simulation, by considering graphical interfaces of the VPE-qGM.

The class of exclusive or is actively used in commonsense and expert reasoning, justifying a practical need for a fuzzy version. Additionally, the novelty and interest of the discussion and new results about the fuzzy Xor can also be applied in computer design and in quantum computing algorithms.

In this context, this paper considers the uncertainty described by such connectives which can be modelled by quantum transformations and related computations, by quantum states. Thus, it contributes to develop quantum algorithms representing fuzzy X(N)or operations.

This paper is organized as follows: Sect. 2 presents the fundamental concepts of fuzzy logic. FSs can be obtained by fuzzy X(N)or operators as presented in Section 2. Moreover, Section 3 brings the main concepts of QC connected with FSs resulting from X(N)or-connectives. In Section 4, the approach for describing FSs using the QC is depicted. Sect. 5 presents the operations on FSs modelled from quantum transformations, considering the fuzzy X(N)operations. Finally, conclusions and further studies are discussed in Section 7.

2 Preliminary on Fuzzy Logic

Fuzzy sets (FSs) aim to overcome the limitations when the transitions from one class to another are carried out smoothly. Properties and operations of FSs are obtained from the generalization of the classical approach.. A **membership function** $f_A(x) : \mathcal{X} \rightarrow [0,1]$ determines the membership degree (MD) of the element $x \in \mathcal{X}$ to the set A, such that $0 \leq f_A(x) \leq 1$. Thus, a **fuzzy set A** related to a set $\mathcal{X} \neq \emptyset$ is given by the expression: $A = \{(x, f_A(x)) : x \in \mathcal{X}\}$.

A function $N : [0,1] \rightarrow [0,1]$ is a **fuzzy negation** (FN) when the following conditions hold:

N1 $N(0) = 1$ and $N(1) = 0$;
N2 If $x \leq y$ then $N(x) \geq N(y)$, for all $x, y \in [0,1]$;
N3 $N(N(x)) = x$, for all $x \in [0,1]$.

Fuzzy negations verifying the involutive property in N3 are called strong fuzzy negations. See the standard negation: $N_S(x) = 1 - x$.

When N is a FN, the N-*dual function* of $f : [0,1]^n \rightarrow [0,1]$ is given by

$$f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n))). \quad (1)$$

If N is involutive, $(f_N)_N = f$, that is the N-dual function of f_N coincides with f. In addition, if $f = f_N$ then it is clear that f is a self-dual function.

Fuzzy connectives can be represented by aggregation functions. Herein, we consider triangular norms (t-norms) and triangular conorms (t-conorms).

Definition 1. *A **triangular (co)norm** is an operation $(S)T : [0,1]^2 \to [0,1]$ such that, for all $x, y, z \in [0,1]$, the following properties hold:*

***T1**: $T(x,y) = T(y,x)$;* ***S1**: $S(x,y) = S(y,x)$;*
***T2**: $T(T(x,y),z) = T(x,T(y,z))$;* ***S2**: $S(S(x,y),z) = S(x,S(y,z))$;*
***T3**: if $x \leq z$ then $T(x,y) \leq T(z,y)$;* ***S3**: if $x \leq z$ then $S(x,y) \leq S(z,y)$*
***T4**: $T(x,0) = 0$ and $T(x,1) = x$;* ***S4**: $S(x,1) = 1$ and $S(x,0) = x$*

Among different definitions of t-norms and t-conorms [12], in this work we consider the *Algebraic Product* and *Algebraic Sum*, respectively given as:

$$T_P(x,y) = x \cdot y; \quad \text{and} \quad S_P(x,y) = x + y - x \cdot y,\ \forall x, y \in [0,1]. \tag{2}$$

In the following, a fuzzy eXclusive or (Xor) operator $E : [0,1]^2 \to [0,1]$ and its dual construction, a fuzzy eXclusive Not or (XNor) connective $E : [0,1]^2 \to [0,1]$ are both defined via axiomatization:

A function $\mathsf{E}(\mathsf{D}) : [0,1]^2 \to [0,1]$ is a **fuzzy exclusive (not) or**, called X(N)or, if it satisfies the following properties, for all $x, y \in [0,1]$:

E0: $\mathsf{E}(1,1) = \mathsf{E}(0,0) = 0$ and $\mathsf{E}(1,0) = 1$; **D0**: $\mathsf{D}(1,1) = \mathsf{D}(0,0)=1$ and $\mathsf{D}(0,1)=0$;
E1: $\mathsf{E}(x,y) = \mathsf{E}(y,x)$; **D1**: $\mathsf{D}(x,y) = \mathsf{D}(y,x)$;
E2: If $x \leq y$ then $\mathsf{E}(0,x) \leq \mathsf{E}(0,y)$; If $x \leq y$ then $\mathsf{E}(1,x) \geq \mathsf{E}(1,y)$. **D2**: If $x \leq y$ then $\mathsf{D}(0,x) \geq \mathsf{D}(0,y)$; If $x \leq y$ then $\mathsf{D}(1,x) \leq \mathsf{D}(y,1)$.

This paper considers the class of representable fuzzy X(N)or connectives meaning that they can be obtained by compositions performed over aggregation functions (t-norms and t-conorms) and fuzzy negations. In particular, a fuzzy X(N)or operator obtained via a defining standard over the Algebraic Product and Algebraic Sum, respectively given by Eq.(2)a and Eq.(2)b, together with standard fuzzy negation is defined in the following.

2.1 Operations over Fuzzy Sets

Let A, B be FSs based on the complement, intersection and union operations.

Definition 2. *Let N be a fuzzy negation. The **complement of** A with respect to $\mathcal{X}$, is a FS $A' = \{(x, f_{A'}) : x \in \mathcal{X}\}$, whose MF $f_{A'} : \mathcal{X} \to [0,1]$ is given by $f_{A'}(x) = N(f_A(x))$, for all $x \in \mathcal{X}$.*

So, a membership degree related to A' is given by the following expression $f_{A'}(x) = N_S(f_A(x)) = 1 - f_A(x)$, for all $x \in \mathcal{X}$.

Definition 3. *Let $T, S : [0,1]^2 \to [0,1]$ be a t-norm and a t-conorm. The **intersection** and **union** between the FSs A and B, both defined with respect to X, results in the corresponding fuzzy sets*

$$A \cap B = \{(x, f_{A\cap B}(x)) : x \in \mathcal{X} \text{ and } f_{A\cap B}(x) = T(f_A(x), f_B(x))\}; \tag{3}$$

$$A \cup B = \{(x, f_{A\cup B}(x)) : x \in \mathcal{X} \text{ and } f_{A\cup B}(x) = S(f_A(x), f_B(x))\}. \tag{4}$$

In this paper, the MFs related to an intersection $A \cap B$ and an union $A \cup B$ are obtained by applying the product t-norm, the algebraic sum and standard negation to the MDs $f_A(x)$ and $f_B(x)$ respectively given as:

$$f_{A\cap B}(x) = f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}; \tag{5}$$

$$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \tag{6}$$

3 FSs Resulting from X(N)or-connectives

The representable class of fuzzy Xor connective obtained by aggregation functions T, S and a FN N, denoted as $\mathsf{E}_{S,T,N}\colon [0,1]^2 \to [0,1]$, is based on the classical logical equivalence $\alpha \bigcirc \beta \equiv (\neg\alpha \wedge \beta) \vee \neg(\alpha \wedge \neg\beta)$.

Definition 4. *Let S be a t-conorm, T be a t-norm and N be a strong FN, A and B be FS related to $\mathcal{X}$, a non- empty set. The fuzzy X(N)or operator $\mathsf{E}_{S,T,N}(\mathsf{D}_{T,S,N})\colon [0,1]^2 \to [0,1]$ results in a FS*

$$A \bigcirc B = \{(x, f_{A\bigcirc B}(x))\colon x \in \mathcal{X}\} \quad \textit{and} \quad A\Box B = \{(x, f_{A\Box B}(x))\colon x \in \mathcal{X}\}$$

whose corresponding membership function $f_{A\bigcirc B}, (f_{A\Box B})\colon \chi \to [0,1]$ is given as

$$f_{A\bigcirc B}(x) = \mathsf{E}(f_A(x), f_B(x)) \quad \textit{and} \quad f_{A\Box B}(x) = \mathsf{D}(f_A(x), f_B(x)). \tag{7}$$

Representable fuzzy X(N)ors obtained by compositions performed on the product t-norm T_P, the probabilistic sum S_P and the standard negation N_S are expressed as:

$$\mathsf{E}_{\oplus} \equiv \mathsf{E}_{S_P,T_P,N_S} \qquad \mathsf{D}_{\boxplus} \equiv \mathsf{D}_{T_P,S_P,N_S}$$

Proposition 1. *Let S_P be the probabilistic sum t-conorm, T_P be the produc t-norm, N_S be the standard fuzzy negation and the related $\mathsf{E}_{\oplus}$ ($\mathsf{D}_{\boxplus}$) fuzzy X(N)or. Consider A and B as FS related to $\mathcal{X} \neq \emptyset$.*

(i) *The FS **obtained by the fuzzy Xor operator** $\mathsf{E}_{\oplus}$, denoted by $A \oplus B$ and whose MF $f_{A\oplus B}\colon \chi \to [0,1]$ provides, for all $x \in \chi$, a MD given as*

$$f_{A\oplus B}(x)=(f_B(x) + f_A(x) - f_A(x)f_B(x))(1 - f_A(x)f_B(x)) - 2f_A(x)f_B(x). \tag{8}$$

(ii) *The FS **obtained by the fuzzy XNor operator** $\mathsf{E}_{\boxplus}$, denoted by $A \boxplus B$ and whose MF $f_{A\boxplus B}\colon \chi \to [0,1]$ provides, for all $x \in \chi$, a MD given as*

$$f_{A\boxplus B}(x) = 1 - (1 - f_A(x)f_B(x))(f_A(x) + f_B(x) - f_A(x)f_B(x)). \tag{9}$$

In a dual construction, we have the following:

Definition 5. *Let S be a t-conorm, T be a t-norm and N be a strong fuzzy negation, A and B be FS related to $\mathcal{X}$. The dual construction of a fuzzy X(N)or operator $\mathsf{E}(\mathsf{D})\colon [0,1]^2 \to [0,1]$ results in a FS*

$$(A\bigcirc B)_N = \{(x, f_{(A\bigcirc B)_N}(x))\colon x \in \mathcal{X}\} \textit{ and } (A\Box B)_N = \{(x, f_{(A\Box B)_N}(x))\colon x \in \mathcal{X}\}.$$

Proposition 2. *The MFs* $(f_{A\bigcirc B})_N, (f_{A\Box B})_N : \chi \rightarrow [0,1]$ *in Def. 5 are given as*

$$(f_{A\bigcirc B})_N(x) = (f_{A\bigcirc_N B})(x) = f_{A\Box B}(x);\;\; (f_{A\Box B})_N(x) = (f_{A\Box_N B})(x) = f_{A\bigcirc B}(x) \quad (10)$$

Proof. Based on Eq. (7) in Definition 4, for all $x \in \chi$, it holds that:

$$(f_{A\bigcirc B})_N(x) = N\left(\mathsf{E}(N(f_A(x)), N(f_B(x)))\right) = \mathsf{D}(f_A(x), f_B(x)) = f_{A\Box B}(x);$$
$$(f_{A\Box B})_N(x) = N\left(\mathsf{D}(N(f_A(x)), N(f_B(x)))\right) = \mathsf{E}(f_A(x), f_B(x)) = f_{A\bigcirc B}(x).$$

Theorem 1. *Let* $(\mathsf{E}_\oplus, \mathsf{D}_\boxplus)$ *be a pair of mutual* N_S*-dual fuzzy Xor-connectives. Then* $(f_{A\oplus B}, f_{A\boxplus B})$ *is also a pair of mutual* N_S*-dual MFs.*

Proof. It follows from Eq. (10) and the expressions below:

$$\begin{aligned}(f_{A\oplus B})_{N_S}(x) &= \mathsf{E}_{\oplus N_S}(f_A(x), f_B(x)) \quad \text{by Eq. (7)}\\ &= N_S\left(\mathsf{E}_\oplus(N_S(f_A(x)), N_S(f_B(x)))\right) \quad \text{by Eq. (1)}\\ &= 1 - ((1 - f_B(x) + 1 - f_A(x) - (1 - f_A(x))(1 - f_B(x)))\\ &\quad (1 - (1 - f_A(x))(1 - f_B(x))) - 2(1 - f_A(x))(1 - f_B(x)) \text{by Eq.(8)}\\ &= 1 - (1 - f_A(x) f_B(x))(f_A(x) + f_B(x) - f_A(x) f_B(x)) = f_{A\boxtimes B}(x) \text{by Eq. (9).}\end{aligned}$$

4 Modelling Fuzzy Sets through Quantum Computing

In *QC*, the qubit is the basic information unit, being the simplest quantum system, defined by a unitary and bi-dimensional state vector. Qubits are generally described, in Dirac's notation [13], by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

The coefficients α and β are complex numbers for the amplitudes of the corresponding states in the computational basis (state space), respecting the condition $|\alpha|^2 + |\beta|^2 = 1$, which guarantees the unitarity of the state vectors of the quantum system, represented by $(\alpha, \beta)^t$.

The state space of a quantum system with multiple *qubits* is obtained by the tensor product of the space states of its subsystems. Considering a quantum system with two *qubits*, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$, the state space comprehends the tensor product $|\psi\rangle \otimes |\varphi\rangle = \alpha\cdot\gamma|00\rangle + \alpha\cdot\delta|01\rangle + \beta\cdot\gamma|10\rangle + \beta\cdot\delta|11\rangle$.

The state transition of a quantum system is performed by controlled and unitary transformations associated with orthogonal matrices of order 2^N, with N being the number of *qubits* within the system, preserving norms, and thus, probability amplitudes.

For instance, the definition of the *Pauly X* transformation and its application over a one-dimensional and two-dimensional quantum systems are presented in the Fig. 1. Furthermore, a Toffoli transformation is also shown in order to describe a controlled operation for a 3 *qubits* system. In this case, the *NOT* operator (*Pauly X*) is applied to the *qubit* $|\sigma\rangle$ when the current states of the first two *qubits* $|\psi\rangle$ and $|\varphi\rangle$ are both $|1\rangle$.

In order to obtain information from a quantum system, it is necessary to apply measurement operators, defined by a set of linear operators M_m, called projections. The index M refers to the possible measurement results. If the state

1 and 2-qubit Pauly X transformations	Toffoli transformation
$X\|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ $X^{\otimes 2}\|\Pi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \delta \\ \gamma \\ \beta \\ \alpha \end{pmatrix}$	$T = \begin{pmatrix} Id & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & X \end{pmatrix}$

Fig. 1. Examples of quantum transformations

of a quantum system is $|\psi\rangle$ immediately before the measurement, the probability of an outcome occurrence is given by $p(|\psi\rangle) = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$.

When measuring a *qubit* $|\psi\rangle$ with $\alpha, \beta \neq 0$, the probability of observing $|0\rangle$ and $|1\rangle$ are, respectively, given by the following expressions:

$$p(0)=\langle\phi|M_0^\dagger M_0|\phi\rangle=\langle\phi|M_0|\phi\rangle=|\alpha|^2 \text{ and } p(1)=\langle\phi|M_1^\dagger M_1|\phi\rangle=\langle\phi|M_1|\phi\rangle=|\beta|^2.$$

After the measuring process, the quantum state $|\psi\rangle$ has $|\alpha|^2$ as the probability to be in the state $|0\rangle$ and $|\beta|^2$ as the probability to be in the state $|1\rangle$.

4.1 Describing Fuzzy Sets through Quantum States

The description of FSs from the QC viewpoint considers a FS A, which is given by the membership function $f_A(x)$. Without losing generality, let $\mathcal{X}$ be a finite subset with cardinality N ($|\mathcal{X}| = N$). Thus, the definitions can be extended to infinite sets, by considering a quantum computer with an infinite quantum register [13].

As stated in [14], consider $\mathcal{X} \neq \emptyset, |\mathcal{X}| = N$, $i \in \mathbb{N}_N = \{1, 2, ..., N\}$ and a membership function, $f_A : \mathcal{X} \to [0, 1]$. A **classical fuzzy state**(CFS) of N-*qubits* is an N-dimensional quantum state, given by

$$|s_f\rangle = \bigotimes_{1 \leq i \leq N} [\sqrt{1 - f_A(x_i)}|0\rangle + \sqrt{f_A(x_i)}|1\rangle]. \tag{11}$$

When $f(1) = a$, $f(2) = b$ and $a, b \in]0, 1[$, superpositions of quantum states corresponding to FSs are obtained and expressed as

$$\begin{aligned} |s_f\rangle &= (\sqrt{a}|1\rangle + \sqrt{1-a}|0\rangle) \otimes (\sqrt{b}|1\rangle + \sqrt{1-b}|0\rangle) \\ &= \sqrt{(1-a)(1-b)}|00\rangle + \sqrt{b(1-a)}|01\rangle + \sqrt{a(1-b)}|10\rangle + \sqrt{ab}|11\rangle. \end{aligned} \tag{12}$$

As it can be seen, the application of a membership function f to each element in the image-set $f[\mathcal{X}]$ defines a quantum state. In other words, a canonical orthonormal basis in $\otimes^N \mathcal{C}$ denotes a classical quantum register of N-*qubits*. Thus, one can describe the classical state of the register $|1100\ldots0\rangle$ of N qubits when $f(1) = f(2) = 1$ and $f(i) = 0$ when $i \in \{\mathbb{N}_N - \{1, 2\}\}$.

The generalized expression, described in [14], states a CFS of $N-$*qubits* as:

$$\begin{aligned} |s_f\rangle &= (1-f(1))^{\frac{1}{2}}(1-f(2))^{\frac{1}{2}}\ldots(1-f(n))^{\frac{1}{2}}|00\ldots00\rangle + \\ &\quad f(1)^{\frac{1}{2}}(1-f(2))^{\frac{1}{2}}\ldots(1-f(n))^{\frac{1}{2}}|10\ldots00\rangle + f(1)^{\frac{1}{2}}f(2)^{\frac{1}{2}}\ldots f(n)^{\frac{1}{2}}|11\ldots11\rangle. \end{aligned} \tag{13}$$

From the perspective of QC, a FS is a superposition of crisp sets. Each $|s_f\rangle$ is a quantum state described as a superposition of crisp sets and generated by the tensor product of non-entangled quantum registers [13].

A linear combination of membership functions representing the fuzzy classical states formalizes the notion of a fuzzy quantum state [14]. So, a **quantum fuzzy set** (QFS) is conceived as quantum superposition of FSs, simultaneously.

In Eq (13), a quantum state $|s_f\rangle$ in $\mathcal{C}^{2^N}$ is characterized as a N-dimensional orthonormal set in $\mathcal{C}^{2^N}$, see more details in [13] and [15].

Definition 6. *Consider $f_i : X \rightarrow [0,1]$, $i \in \{1,...,k\}$, as a collection of MFs generating fuzzy subsets A_i and $\{|s_{f_1}\rangle, \ldots, |s_{f_k}\rangle\} \subseteq [CFS]$, such that their components are two by two orthonormal vectors. When $\{c_1, \ldots, c_k\} \subseteq \mathcal{C}$, the linear combination $|s\rangle = c_1|s_{f_1}\rangle + \ldots + c_k|s_{f_k}\rangle$ defines a **quantum** FS (QFS).*

By Def. 6, an N-dimensional quantum fuzzy state can be entangled or not, depending on the family of classical fuzzy states.

5 FS Operations from Quantum Transformations

According to [14], FSs can be obtained by quantum superposition of CFSs associated with a quantum state. Additionally, interpretations for fuzzy operations such as complement, intersection and union are obtained from the NOT, AND and OR quantum transformations.

Let $f, g : \mathcal{X} \rightarrow [0,1]$ be MFs related to FSs A and B. For $x \in \mathcal{X}$, the corresponding pair $(|s_{f(x)}\rangle, |s_{g(x)}\rangle)$ of CFSs is given as:

$$|S_f(x)\rangle = \sqrt{f(x)}|1\rangle + \sqrt{1-f(x)}|0\rangle \text{ and } |S_g(x)\rangle = \sqrt{g(x)}|1\rangle + \sqrt{1-g(x)}|0\rangle \quad (14)$$

In order to simplify the paper notation, the MD defined by $f_A(x)$, which is related to an element $x \in \mathcal{X}$ in the FS A, will be denoted by f_A.

5.1 Fuzzy Complement Operator

The complement of a FS is performed by the standard negation, which is obtained by the NOT operator, defined as

$$NOT(|S_{f_A}\rangle) = \sqrt{1-f_A}|1\rangle + \sqrt{f_A}|0\rangle \quad (15)$$

The complement operator NOT^N can be applied to the state $|s_f\rangle = \otimes_{1 \leq i \leq N}|s_{f_i}\rangle$, resulting in an N-dimensional quantum superposition of 1-*qubit* states, described as $\mathcal{C}^{2^N}$ in the computational basis, represented by $NOT^N|s_f\rangle$ and expressed as

$$NOT^N(|S_{f_A}\rangle) = NOT(\otimes_{1 \leq i \leq N}(f_A(i)^{\frac{1}{2}}|1\rangle(1-f_A(i))^{\frac{1}{2}}|0\rangle)) \quad (16)$$

Now, Eqs. (17) and (18) describe other applications related to the NOT transformation used to describe other fuzzy Xor operators, which act on the 2nd e 3rd-*qubits* of a quantum system, respectively:

$$NOT_2(|S_{f_1}\rangle|s_{f_2}\rangle) = |S_{f_1}\rangle \otimes NOT|s_{f_2}\rangle \quad (17)$$

$$NOT_{2,3}(|S_{f_1}\rangle|S_{f_2}\rangle|s_{f_3}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle \otimes NOT|s_{f_3}\rangle. \quad (18)$$

5.2 Modelling of Fuzzy Intersection and Union Operators

The fuzzy intersection operator is modelled by the ***AND* operator** expressed through the *Toffoli* quantum transformation as

$$AND(|s_{f_i}\rangle, |s_{g_i}\rangle) = T(|s_{f_i}\rangle, |s_{g_i}\rangle, |0\rangle). \quad (19)$$

So, we obtain the quantum state $|S_2\rangle$ given by the following expression:

$$|S_2\rangle = \sqrt{f_A f_B}|111\rangle + \sqrt{f_A(1-f_B)}|100\rangle + \sqrt{(1-f_A)f_B}|010\rangle + \sqrt{(1-f_A)(1-f_B)}|000\rangle. \quad (20)$$

Thus, a measurement performed over the third *qubit* ($|1\rangle$) in the quantum state expressed by Eq. (20), provides the following output:

$$\bullet|S_{f_0}\rangle = |111\rangle, \text{ with probability } p(1) = f_A \cdot f_B.$$

Then, for all $x \in X$,let $f_A(x)$ and $f_B(x)$ be the MD of $x \in \mathcal{X}$ in the FS defined by MF $f_A(x) : \mathcal{X} \to U$ and $g_A(x) : \mathcal{X} \to U$, respectively. Then, $f_A(x) \cdot f_B(x)$ indicates the MD of x in the intersection of such FSs A, B. Analogously, a measurement of third *qubit* ($|0\rangle$) in Eq. (20), returns an output state given as:

$$\bullet|S_{f_1}\rangle = \frac{1}{\sqrt{(1-f_A)f_B}}(\sqrt{f_A(1-f_B)}|100\rangle + \sqrt{(1-f_A)f_B}|010\rangle + \sqrt{(1-f_A)(1-f_B)}|000\rangle),$$

with probability $p(0) = 1 - f_A(x) \cdot f_B(x)$. In this case, an expression of the complement of the intersection between FSs A and B is given by $1 - p(0) = f_A(x) \cdot f_B(x)$. This probability indicates the non-MD of x is in the FS $A \cap B$. We also conclude that, by Eq. (20), it corresponds to the standard negation of product t-norm [12].

Let $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ be quantum states given by Eqs. (14)a and (14)b, respectively. The union of FSs is modelled by the ***OR* operator** as the complement of AND operator, and therefore it is given as:

$$OR(|S_f\rangle, |S_g\rangle) = NOT^3(T(NOT|S_f\rangle, NOT|S_g\rangle, |0\rangle)). \quad (21)$$

In the following, by applying the NOT^3 and Toffoli operators we have that:

$$|S_4\rangle = \sqrt{(1-f_A)(1-f_B)}|000\rangle + \sqrt{(1-f_A)f_B}|011\rangle + \sqrt{f_A(1-f_B)}|101\rangle + \sqrt{f_A f_B}|111\rangle. \quad (22)$$

Observe that, a measure performed on third *qubit* ($|1\rangle$) of quantum state in Eq. (22) results in the final state:

$$\bullet|S_{f_1}\rangle = \frac{1}{\sqrt{f_B(1-f_A)+f_A)}}(\sqrt{(1-f_A)f_B}|011\rangle + \sqrt{f_A(1-f_B)}|101\rangle + \sqrt{f_A f_B}|111\rangle),$$

with corresponding probability $p = f_A + f_B - f_A \cdot f_B$ of $x_i \in \mathcal{X}$ is in both FSs A e B. The OR operator, expressed by Eq. (22), is therefore defined by the t-conorm product [12]. Additionally, a measure also performed in the third *qubit* (but related to state $|0\rangle$) returns

$$\bullet|S_{f_0}\rangle = |000\rangle, \text{ with probability } p(0) = (1-f_A) \cdot (1-f_B),$$

indicating that $x \in \mathcal{X}$ does not belong to $A \cup B$ (neither A nor B).

6 Modelling and Simulation of a Fuzzy Xor Connective $\mathsf{E}_{\oplus}$

A representable fuzzy X(N)or can be obtained by a composition of quantum operations $(NOT, T, CNOT, \ldots)$ and other controlled ones (AND, OR, NOT_N) previously discussed in Sections 5.1 and 5.2. Extending this approach, this section introduces the expressions modelling the quantum operators of fuzzy X(N)or and simulating them in the VPE-qGM based on Eqs. (23)a and (23)b, respectively given as

$$|S_f\rangle = \frac{\sqrt{2}}{2}|1\rangle + \frac{\sqrt{2}}{2}|0\rangle \quad \text{and} \quad |S_g\rangle = \frac{\sqrt{3}}{3}|1\rangle + \frac{\sqrt{6}}{3}|0\rangle. \tag{23}$$

Let $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ be quantum states in Eqs. (14)a and (14)b, respectively. The fuzzy Xor $\mathsf{E}_{\oplus}$ is modelled by the **quantum operator** $XOR_{\oplus}$ given by:

$$XOR_{\oplus}(|S_f\rangle, |S_g\rangle) = OR(AND(NOT|S_f\rangle, |S_g\rangle), AND(|S_f\rangle, NOT|S_g\rangle)) \tag{24}$$

By applying the NOT_6 and AND operators, we obtain the quantum state

$$|S_5\rangle = NOT_7(T_{3,6,7}(NOT_3(T_{1,2,3}(|S_f\rangle, |S_g\rangle, |0\rangle), NOT_6(T_{4,5,6}(NOT|S_f\rangle, NOT|S_g\rangle, |0\rangle), |0\rangle)))$$

The initial state $|s_0\rangle = (|f_A\rangle \otimes |f_B\rangle \otimes |0\rangle)^2 \otimes |0\rangle$ graphically presented in the quantum circuit of Figure 2(a) is extended in Eq. (25) below,

$$\begin{aligned}|s_0\rangle = (((&\sqrt{1-f_A}|0\rangle + \sqrt{f_A}|1\rangle) \otimes (\sqrt{1-f_B}|0\rangle + \sqrt{f_B}|1\rangle) \otimes |0\rangle)) \otimes \\ &((\sqrt{1-f_A}|0\rangle + \sqrt{f_A}|1\rangle) \otimes (\sqrt{1-f_B}|0\rangle + \sqrt{f_B}|1\rangle) \otimes |0\rangle))) \otimes |0\rangle.\end{aligned} \tag{25}$$

Thus, according with column 5 related to Table 1, presenting the non zero coefficients of quantum states in a temporal evolution of computations related to the fuzzy Xor $E_{\oplus}$, we obtain the quantum state in the following Eq.(26):

$$\begin{aligned}|S_5\rangle_{\mathsf{E}_{\oplus}} = &\sqrt{f_A f_B(1-f_A)(1-f_B)}|0010010\rangle + (1-f_B)\sqrt{f_A(1-f_A)}|0010110\rangle + \\ &f_A\sqrt{f_B(1-f_B)}|0011010\rangle + f_B\sqrt{f_A(1-f_A)}|0110010\rangle + \\ &\sqrt{f_A f_B(1-f_A)(1-f_B)}|0110110\rangle + f_A f_B|0111010\rangle + \\ &(1-f_A)\sqrt{f_B(1-f_B)}|1010010\rangle + (1-f_A)(1-f_B)|1010110\rangle + \\ &\sqrt{f_A f_B(1-f_A)(1-f_B)}|1011010\rangle + f_A(1-f_B)|0011101\rangle + \\ &f_A\sqrt{f_B(1-f_B)}|0110001\rangle + (1-f_B)\sqrt{f_A(1-f_A)}|1011101\rangle + \\ &f_B(1-f_A)|1100011\rangle + (1-f_A)\sqrt{f_B(1-f_B)}|1100111\rangle + \\ &f_B\sqrt{f_A(1-f_A)}|1101011\rangle + \sqrt{f_A f_B(1-f_A)(1-f_B)}|1101101\rangle\end{aligned} \tag{26}$$

Additionally, a measure performed on the 7th *qubit* of quantum state described by Eq. (26) results in the final state:

$$\begin{aligned}\bullet|S'_{f_0}\rangle = &\frac{1}{\sqrt{f_A + f_B - 3f_A f_B + f_A f_B^2 + f_A^2 f_B - f_A^2 f_B^2}}(f_A(1-f_B)|0011101\rangle + \\ &f_A\sqrt{f_B(1-f_B)}|0110001\rangle + (1-f_B)\sqrt{f_A(1-f_A)}|1011101\rangle + f_B(1-f_A)|1100011\rangle + \\ &(1-f_A)\sqrt{f_B(1-f_B)}|1100111\rangle + f_B\sqrt{f_A(1-f_A)}|1101011\rangle + \\ &\sqrt{f_A(1-f_A)(1-f_B)}|1101101\rangle).\end{aligned}$$

Table 1. Temporal evolution related to computation of the fuzzy Xor $E_\oplus$

	T0	T1	T2	T3	T4	T5
$(1-f_A)(1-f_B)$	0000000	1000100	1000100	1010110	1010111	1010110
$(1-f_A)\sqrt{f_B(1-f_B)}$	0000100	1000000	1000000	1010010	1010011	1010010
$(1-f_B)\sqrt{f_A(1-f_A)}$	0001000	1001100	1001110	1011100	1011100	1011101
$\sqrt{f_Af_B(1-f_A)(1-f_B)}$	0001100	1001000	1001000	1011010	1011011	1011010
$(1-f_A)\sqrt{f_B(1-f_B)}$	0100000	1100100	1110100	1100110	1100110	1100111
$f_B(1-f_A)$	0100100	1100000	1110000	1100010	1100010	1100011
$\sqrt{f_Af_B(1-f_A)(1-f_B)}$	0101000	1101100	1111110	1101100	1101100	1101101
$f_B\sqrt{f_A(1-f_A)}$	0101100	1101000	1111000	1101010	1101010	1101011
$(1-f_B)\sqrt{f_A(1-f_A)}$	1000000	0000100	0000100	0010110	0010111	0010110
$\sqrt{f_Af_B(1-f_A)(1-f_B)}$	1000100	0000000	0000000	0010010	0010011	0010010
$f_A(1-f_B)$	1001000	0001100	0001110	0011100	0011100	0011101
$f_A\sqrt{f_B(1-f_B)}$	1001100	0001000	0001000	0011010	0011011	0011010
$\sqrt{f_Af_B(1-f_A)(1-f_B)}$	1100000	0100100	0100100	0110110	0110111	0110110
$f_B\sqrt{f_A(1-f_A)}$	1100100	0100000	0100000	0110010	0110011	0110010
$f_A\sqrt{f_B(1-f_B)}$	1101000	0101100	0101110	0110000	0110000	0110001
f_Af_B	1101100	0101000	0101000	0111010	0111011	0111010

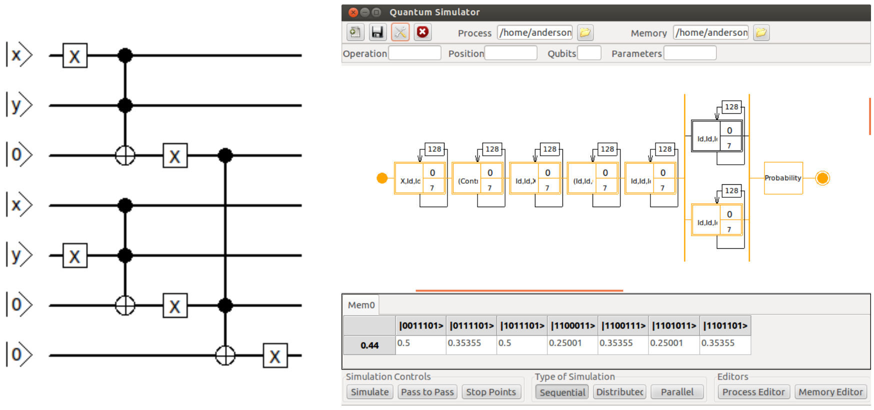
with corresponding probability $p(1) = f_A + f_B - 3f_Af_B + f_Af_B^2 + f_A^2f_B - f_A^2f_B^2$ indicating the MD of an element $x \in \mathcal{X}$ in the $FS\ A \oplus B$ obtained by applying the fuzzy Xor connective $\mathsf{E}_\oplus$ and taking $f_A(x), f_B(x)$ as the arguments of the related MF. So, a measure also performed in the 7th *qubit* (but related to state $|0\rangle$) returns

$$\bullet|S_{f_1}\rangle = \frac{1}{\sqrt{1-(f_A+f_B-3f_Af_B+f_Af_B^2+f_A^2f_B-f_A^2f_B^2)}}(\sqrt{f_Af_B(1-f_A)(1-f_B)}|0010010\rangle$$
$$+(1-f_B)\sqrt{f_A(1-f_A)}|0010110\rangle + f_A\sqrt{f_B(1-f_B)}|0011010\rangle +$$
$$f_B\sqrt{f_A(1-f_A)}|0110010\rangle + \sqrt{f_Af_B(1-f_A)(1-f_B)}|0110110\rangle + f_Af_B|0111010\rangle +$$
$$(1-f_A)\sqrt{f_B(1-f_B)}|1010010\rangle + (1-f_A)(1-f_B)|1010110\rangle +$$
$$\sqrt{f_Af_B(1-f_A)(1-f_B)}|1011010\rangle),$$

with $p(0) = 1 - (f_A + f_B - 3f_Af_B + f_Af_B^2 + f_A^2f_B - f_A^2f_B^2)$.

See in Figure 2(b) that the simulation in *VPE-qGM* is consistent with Eq. (26) by taking initial states of Eq. (23)a and Eq. (23)b. After a measurement, one of the following states is reached:

- $|S_5^{'}\rangle = \frac{\sqrt{324}}{\sqrt{144}}(\frac{1}{3}|0011101\rangle + \frac{\sqrt{18}}{18}|0110001\rangle + \frac{1}{9}|1011101\rangle + \frac{1}{6}|1100011\rangle + \frac{\sqrt{18}}{18}|1100111\rangle + \frac{1}{6}|1101011\rangle + \frac{\sqrt{18}}{18}|1101101\rangle)$, with probability $p(1) = 44\%$
- $|S_5^{''}\rangle = \frac{\sqrt{324}}{\sqrt{180}}(\frac{\sqrt{18}}{18}|0010010\rangle + \frac{1}{3}|0010110\rangle + \frac{\sqrt{18}}{18}|0011010\rangle + \frac{1}{6}|0110010\rangle + \frac{\sqrt{18}}{18}|0110110\rangle + \frac{1}{6}|0111010\rangle + \frac{\sqrt{18}}{18}|1010010\rangle + \frac{1}{3}|1010110\rangle + \frac{\sqrt{18}}{18}|1011010\rangle)$, with probability $p(0) = 56\%$



(a) Xor $\mathsf{E}_\oplus$ (b) VPE $\mathsf{E}_\oplus$

Fig. 2. Modelling and simulating fuzzy Xor $\mathsf{E}_\oplus$ operator in the VPE-qGM

Analogously, in order to model and simulate the fuzzy XNor $\mathsf{D}_\boxplus$, consider the quantum operator $XNOR_\boxplus$ given as:

$$XNOR_\boxplus(|s_{f_i}\rangle, |s_{g_i}\rangle) = AND(OR(NOT|s_{f_i}\rangle, |s_{g_i}\rangle), OR(|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) \qquad (27)$$

Therefore, based on the AND, OR and NOT transformation, we obtain that

$$|S_5\rangle = T_{3,6,7}(NOT_3(T_{1,2,3}(|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |0\rangle)), NOT_6(T_{4,5,6}(NOT|s_{f_i}\rangle, |s_{g_i}\rangle, |0\rangle)), |0\rangle)$$

Analogously, it can be developed for simulation in the $VPE\text{-}qGM$ of Eq. (27).

7 Conclusion and Final Remarks

The visual approach of the VPE-qGM environment enables the implementation and validation of fuzzy X(N)or operations using QC. The description of these operations is based on compositions of controlled and unitary quantum transformations, and the corresponding interpretation of fuzzy operations is obtained by applying operators of projective measurements.

Further work aims to consolidate this specification including not only other fuzzy connectives, constructors (e.i. automorphisms and reductions) and the corresponding extension of (de)fuzzyfication methodology from formal structures provided by QC. Finally, it may also contribute to designing new algorithms based on considering the abstractions provided by quantum FSs and related interpretation of fuzzy logic concepts.

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