The Range of a Continuous Functional Under Set-Valued Uncertainty Is Always an Interval

Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu

Olga Kosheleva
University of Texas at El Paso, olgak@utep.edu

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Vladik Kreinovich and Olga Kosheleva
University of Texas at El Paso
500 W. UNiversity
El Paso, TX 79968, USA
vladik@utep.edu, olgak@utep.edu

Abstract

One of the main problems of interval computations is computing the range of a given function on a given multi-D interval (box). It is known that the range of a continuous function on a box is always an interval. However, if, instead of a box, we consider the range over a subset of this box, the range is, in general, no longer an interval. In some practical situations, we are interested in computing the range of a functional over a function defined with interval (or, more general, set-valued) uncertainty. At first glance, it may seem that under a non-interval set-valued uncertainty, the range of the functional may be different from an interval. However, somewhat surprisingly, we show that for continuous functionals, this range is always an interval.

1 Formulation of the Problem

Computing the range over a multi-D interval (box): reminder. In many practical situations, we know the dependence $y = f(x_1, \ldots, x_n)$ between the desired quantity y and the easy-to-measure quantities $x_1, \ldots, x_n$. In the ideal case, when we know the exact values $x_1, \ldots, x_n$ of the corresponding quantities, we can use this dependence to compute the value of the desired quantity y.

In practice, measurements are never absolutely exact; see, e.g., [3]: the measurement result $\widetilde{x}_i$ is, in general, somewhat different from the actual (unknown) value x_i of the corresponding quantity. In many cases, the only information that we have about the corresponding measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is the upper bound Δ_i on its absolute value: $|\Delta x_i| \le \Delta_i$. In such cases, once we get the measurement result $\widetilde{x}_i$, the only information that we have about the actual value x_i is that this value is somewhere on the interval $[\underline{x}_i, \overline{x}_i]$, where $\underline{x}_i = \widetilde{x}_i - \Delta_i$ and $\overline{x}_i = \widetilde{x}_i + \Delta_i$.

Different combinations of possible values $(x_1, \ldots, x_n)$ lead, in general, to different values of $y = f(x_1, \ldots, x_n)$. It is therefore desirable to find the range of such values of y, i.e., the set

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

Computing this range is one of the main problem of *interval computations*; see, e.g., [1, 2].

It is well known that for a continuous function $f(x_1, \ldots, x_n)$, the resulting range is always an interval.

From intervals to more general sets. Sometimes, in addition to knowing the bounds $\underline{x}_i$ and $\overline{x}_i$, we also know that some values from the corresponding interval $[\underline{x}_i, \overline{x}_i]$ are not possible. In such cases, the set X_i of all possible values of each quantity x_i is a proper subset of an interval – and often, a subset which is not connected.

For example, if we measure the kinetic energy of a particle moving in the x-direction, we then know the absolute value of its velocity, but not its direction. In this example, the set of all possible values of the velocity consists of two value $X = \{-v, v\}$. If we take into account that the energy – and thus, the absolute value of the velocity – can only be measured with some accuracy, then we get a more realistic set $X = [-\overline{v}, -\underline{v}] \cup [\underline{v}, \overline{v}]$.

In such cases, the range $\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}$ is not necessarily an interval. For example, if each set X_i consists of finitely many points, then we have finitely many possible tuples $(x_1, \ldots, x_n)$ and thus, only finitely many values $f(x_1, \ldots, x_n)$ corresponding to these tuples – so the range is a finite set and hence not an interval.

Continuous case. In some practical situations, the desired quantity y depends not on finitely many quantities $x_1, \ldots, x_n$, but on the whole signal $x(t)$, i.e., in effect, on infinitely many values $x(t)$ corresponding to all possible moments of time t from some interval $\left[\underline{T}, \overline{T}\right]$. In other words, we have a *functional* $y = f(x)$ that describes how the value y depends on the signal $x(t)$.

For example, one way to find the location on a submerged submarine is to measure its acceleration $x(t)$. If we know the initial velocity $v(\underline{T})$, then the velocity at each moment t can be found by integrating the acceleration, as $v(t) = v_0 + \int_{\underline{T}}^{t} x(s)\, ds$. Thus, once we know the initial coordinate y_0, we can find the coordinate y at the current moment $\overline{T}$ as an integral

$$y = y_0 + \int_{\underline{T}}^{\overline{T}} v(t)\, dt = y_0 + \int_{\underline{T}}^{\overline{T}} \left(v_0 + \int_{\underline{T}}^{t} x(s)\, ds \right) dt.$$

The values $x(t)$ can only be measured with some uncertainty. Thus, for each t, instead of the exact value $x(t)$, we only know the interval $[\underline{x}(t), \overline{x}(t)]$ that contains the actual (unknown) value $x(t)$. For different functions $x(t)$ from this interval, in general, we have different values of the function $f(x)$. It is therefore desirable to find the range of the functional $f(x)$ under this interval uncertainty, i.e., the range

$$\{f(x) : \underline{x}(t) \le x(t) \le \overline{x}(t) \text{ for all } t\}.$$

What if we have set uncertainty in the continuous case: formulation of the problem. What if for each t, in addition to the interval $[\underline{x}(t), \overline{x}(t)]$, we also know that the actual values $x(t)$ can only belong to an appropriate subset $X(t)$ of this interval? What can we then say about the range

$$\{f(x) : x(t) \in X(t) \text{ for all } t\}?$$

At first glance, it may seem that, similarly to the usual set-valued case, we can have a non-interval range. However, as we show in the paper, in the continuous case, the range is *always* an interval – even when we have set uncertainty with non-connected sets $X(t)$ instead of interval uncertainty.

2 Main Result

Proposition.

- *Let* $\left[\underline{T}, \overline{T}\,\right]$ *be an interval.*
- *Let* $\Delta > 0$ *be a real number.*
- *Let* X *be a mapping that maps each moment* $t \in \left[\underline{T}, \overline{T}\,\right]$ *into a subset* $X(t) \subseteq [-\Delta, \Delta]$.
- *Let* f *be a functional that maps every measurable function* $x(t)$ *for which* $|x(t)| \le \Delta$ *for all* t *into a real number.*
- *We also assume that* f *is continuous in terms of the* L^1*-distance*

$$d(x_1, x_2) = \int |x_1(t) - x_2(t)|\, dt.$$

Under these assumptions, the range $\{f(x) : x(t) \in X(t)$ *for all* $t\}$ *is a connected set.*

Comment. On the real line, the only connected sets are intervals – finite or infinite, open or closed or semi-open, degenerate or non-degenerate. Thus, the above results says that the range of a continuous functional under set-valued uncertainty is always an interval.

Proof. To prove connectedness, we must prove that for every two measurable functions $x_1(t)$ and $x_2(t)$, each real values y between $f(x_1)$ and $f(x_2)$ can also be represented as $f(x)$ for some measurable function $x(t)$ for which $x(t) \in X(t)$ for all t.

Indeed, let us consider, for each value $s \in \left[\underline{T}, \overline{T}\,\right]$, an auxiliary function $x_{(s)}(t)$ which is defined as follows:

- for $t \le s$, we have $x_{(s)}(t) = x_1(t)$; and

- for $t > s$, we have $x_{(s)}(t) = x_2(t)$.

It is easy to see that each of these auxiliary functions is also measurable.

For each t, the value $x_{(s)}(t)$ is equal to either $x_1(t)$ or $x_2(t)$. Both values are contained in the set $X(t)$, so we can conclude that $x_{(s)}(t) \in X(t)$ for all moments t.

From the above definition of the function $x_{(s)}$, it follows that:

- for $s = \overline{T}$, we have $x_{(s)} = x_1$, and
- for $s = \underline{T}$, we have $x_{(s)} = x_2$.

For every two numbers $s < s'$, the values of the functions $x_{(s)}(t)$ and $x_{(s')}(t)$ differ only for $t \in [s, s']$, where one of them is equal to $x_1(t)$ and another one to $x_2(t)$. Since the values of both functions $x_1(t)$ and $x_2(t)$ are located on the interval $[-\Delta, \Delta]$, the difference $|x_1(t) - x_2(t)|$ cannot exceed 2Δ. Thus, we have:

$$d\left(x_{(s)}, x_{(s')}\right) = \int_{\underline{T}}^{\overline{T}} \left|x_{(s)}(t) - x_{(s')}(t)\right| \, dt = \int_s^{s'} |x_1(t) - x_2(t)| \le (s' - s) \cdot 2\Delta.$$

As the difference $|s - s'|$ decreases, this distance tends to 0. Thus, the mapping $s \to x_{(s)}$ is continuous in the L^1-metric.

Since the functional $f(x)$ is continuous in the sense of this metric, we can therefore conclude that the mapping $s \to f\left(x_{(s)}\right)$ is also continuous. A continuous functions from real numbers to real numbers attains, with every two values, all intermediate values as well. Thus, for every real number y between the values $f(x_1) = f\left(x_{(\overline{T})}\right)$ and $f(x_2) = f\left(x_{(\underline{T})}\right)$, there exists a value s for which $f\left(x_{(s)}\right) = y$. Since we have shown that $x_{(s)}(t) \in X(t)$ for each t, this means that y indeed belongs to the desired range, thus the range is indeed connected.

The proposition is proven.

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