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The Range of a Continuous Functional Under Set-Valued Uncertainty Is Always an Interval

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Abstract

One of the main problems of interval computations is computing the range of a given function on a given multi-D interval (box). It is known that the range of a continuous function on a box is always an interval. However, if, instead of a box, we consider the range over a subset of this box, the range is, in general, no longer an interval. In some practical situations, we are interested in computing the range of a functional over a function defined with interval (or, more general, set-valued) uncertainty. At first glance, it may seem that under a non-interval set-valued uncertainty, the range of the functional may be different from an interval. However, somewhat surprisingly, we show that for continuous functionals, this range is always an interval.

1 Formulation of the Problem

Computing the range over a multi-D interval (box): reminder. In many practical situations, we know the dependence $y = f(x_1, \ldots, x_n)$ between the desired quantity $y$ and the easy-to-measure quantities $x_1, \ldots, x_n$. In the ideal case, when we know the exact values $x_1, \ldots, x_n$ of the corresponding quantities, we can use this dependence to compute the value of the desired quantity $y$.

In practice, measurements are never absolutely exact; see, e.g., [3]: the measurement result $\tilde{x}_i$ is, in general, somewhat different from the actual (unknown) value $x_i$ of the corresponding quantity. In many cases, the only information that we have about the corresponding measurement error $\Delta x_i \triangleq \tilde{x}_i - x_i$ is the upper bound $\Delta_i$ on its absolute value: $|\Delta x_i| \leq \Delta_i$. In such cases, once we get the measurement result $\tilde{x}_i$, the only information that we have about the actual value $x_i$ is that this value is somewhere on the interval $[\underline{x}_i, \overline{x}_i]$, where $\underline{x}_i = \tilde{x}_i - \Delta_i$ and $\overline{x}_i = \tilde{x}_i + \Delta_i$.

Different combinations of possible values $(x_1, \ldots, x_n)$ lead, in general, to different values of $y = f(x_1, \ldots, x_n)$. It is therefore desirable to find the range
of such values of $y$, i.e., the set
$$\{f(x_1, \ldots, x_n) : x_1 \in [x_{1L}, x_{1U}], \ldots, x_n \in [x_{nL}, x_{nU}]\}.$$ 
Computing this range is one of the main problems of interval computations; see, e.g., [1, 2].

It is well known that for a continuous function $f(x_1, \ldots, x_n)$, the resulting range is always an interval.

**From intervals to more general sets.** Sometimes, in addition to knowing the bounds $x_{1L}$ and $x_{1U}$, we also know that some values from the corresponding interval $[x_{1L}, x_{1U}]$ are not possible. In such cases, the set $X_i$ of all possible values of each quantity $x_i$ is a proper subset of an interval – and often, a subset which is not connected.

For example, if we measure the kinetic energy of a particle moving in the $x$-direction, we then know the absolute value of its velocity, but not its direction. In this example, the range $f(x_1, \ldots, x_n)$ corresponding to these tuples – so the range is a finite set and hence not an interval.

**Continuous case.** In some practical situations, the desired quantity $y$ depends not on finitely many quantities $x_1, \ldots, x_n$, but on the whole signal $x(t)$, i.e., in effect, on infinitely many values $x(t)$ corresponding to all possible moments of time $t$ from some interval $[T_L, T_U]$. In other words, we have a functional $y = f(x)$ that describes how the value $y$ depends on the signal $x(t)$.

For example, one way to find the location on a submerged submarine is to measure its acceleration $x(t)$. If we know the initial velocity $v(T_L)$, then the velocity at each moment $t$ can be found by integrating the acceleration, as $v(t) = v_0 + \int_{T_L}^{t} x(s) \, ds$. Thus, once we know the initial coordinate $y_0$, we can find the coordinate $y$ at the current moment $T_U$ as an integral
$$y = y_0 + \int_{T_L}^{T_U} v(t) \, dt = y_0 + \int_{T_L}^{T_U} \left( v_0 + \int_{T_L}^{t} x(s) \, ds \right) \, dt.$$

The values $x(t)$ can only be measured with some uncertainty. Thus, for each $t$, instead of the exact value $x(t)$, we only know the interval $[\bar{x}(t), \overline{x}(t)]$ that contains the actual (unknown) value $x(t)$. For different functions $x(t)$ from this interval, in general, we have different values of the function $f(x)$. It is therefore desirable to find the range of the functional $f(x)$ under this interval uncertainty, i.e., the range
$$\{f(x) : \bar{x}(t) \leq x(t) \leq \overline{x}(t) \text{ for all } t\}.$$
What if we have set uncertainty in the continuous case: formulation of the problem. What if for each $t$, in addition to the interval $[x(t), T(t)]$, we also know that the actual values $x(t)$ can only belong to an appropriate subset $X(t)$ of this interval? What can we then say about the range 

$$\{f(x) : x(t) \in X(t) \text{ for all } t\}?$$

At first glance, it may seem that, similarly to the usual set-valued case, we can have a non-interval range. However, as we show in the paper, in the continuous case, the range is always an interval – even when we have set uncertainty with non-connected sets $X(t)$ instead of interval uncertainty.

2 Main Result

Proposition.

- Let $[T, T]$ be an interval.
- Let $\Delta > 0$ be a real number.
- Let $X$ be a mapping that maps each moment $t \in [T, T]$ into a subset $X(t) \subseteq [-\Delta, \Delta]$.
- Let $f$ be a functional that maps every measurable function $x(t)$ for which $|x(t)| \leq \Delta$ for all $t$ into a real number.
- We also assume that $f$ is continuous in terms of the $L^1$-distance

$$d(x_1, x_2) = \int |x_1(t) - x_2(t)| dt.$$ 

Under these assumptions, the range $\{f(x) : x(t) \in X(t) \text{ for all } t\}$ is a connected set.

Comment. On the real line, the only connected sets are intervals – finite or infinite, open or closed or semi-open, degenerate or non-degenerate. Thus, the above results says that the range of a continuous functional under set-valued uncertainty is always an interval.

Proof. To prove connectedness, we must prove that for every two measurable functions $x_1(t)$ and $x_2(t)$, each real values $y$ between $f(x_1)$ and $f(x_2)$ can also be represented as $f(x)$ for some measurable function $x(t)$ for which $x(t) \in X(t)$ for all $t$.

Indeed, let us consider, for each value $s \in [T, T]$, an auxiliary function $x(s)(t)$ which is defined as follows:

- for $t \leq s$, we have $x(s)(t) = x_1(t)$; and
• for $t > s$, we have $x_{(s)}(t) = x_2(t)$.

It is easy to see that each of these auxiliary functions is also measurable.

For each $t$, the value $x_{(s)}(t)$ is equal to either $x_1(t)$ or $x_2(t)$. Both values are contained in the set $X(t)$, so we can conclude that $x_{(s)}(t) \in X(t)$ for all moments $t$.

From the above definition of the function $x_{(s)}$, it follows that:

• for $s = T^*$, we have $x_{(s)} = x_1$, and

• for $s = T$, we have $x_{(s)} = x_2$.

For every two numbers $s < s'$, the values of the functions $x_{(s)}(t)$ and $x_{(s')}(t)$ differ only for $t \in [s, s']$, where one of them is equal to $x_1(t)$ and another one to $x_2(t)$. Since the values of both functions $x_1(t)$ and $x_2(t)$ are located on the interval $[-\Delta, \Delta]$, the difference $|x_1(t) - x_2(t)|$ cannot exceed $2\Delta$. Thus, we have:

$$d(x_{(s)}, x_{(s')}) = \int_T^T |x_{(s)}(t) - x_{(s')}(t)| \, dt = \int_s^{s'} |x_1(t) - x_2(t)| \leq (s' - s) \cdot 2\Delta.$$  

As the difference $|s - s'|$ decreases, this distance tends to 0. Thus, the mapping $s \to x_{(s)}$ is continuous in the $L^1$-metric.

Since the functional $f(x)$ is continuous in the sense of this metric, we can therefore conclude that the mapping $s \to f(x_{(s)})$ is also continuous. A continuous functions from real numbers to real numbers attains, with every two values, all intermediate values as well. Thus, for every real number $y$ between the values $f(x_1) = f(x_{(T)})$ and $f(x_2) = f(x_{(T)})$, there exists a value $s$ for which $f(x_{(s)}) = y$. Since we have shown that $x_{(s)}(t) \in X(t)$ for each $t$, this means that $y$ indeed belongs to the desired range, thus the range is indeed connected.

The proposition is proven.

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