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Abstract: In this paper, we investigate the multi criteria group decision making with intuitionistic linguistic preference relation. The concept of Fuzzy Collective Solution (FCS) is used to evaluate and rank the candidate solution sets for modeling under linguistic assessments. Intuitionistic linguistic preference relation and associated aggregation procedures are then defined in a new concept of Fuzzy Pareto Solution. Numerical examples are presented to demonstrate computing procedures. The results affirm efficiency of the proposed method.

Keywords: Fuzzy Collective Solution; Fuzzy Pareto Solution; Group Decision Making; Intuitionistic Fuzzy Sets; Intuitionistic Linguistic Preference Relation.

1. Introduction

Group decision making (GDM) is useful to evaluate and judge the best solution among alternatives through a social group. GDM, in a fuzzy environment, contains four steps (Herrera, Herrera-Viedma & Verdegay, 1996): i) unifying evaluations from experts; ii) aggregating their opinions to a final score for each alternative represented by a linguistic label (Chen & Hwang, 1992; Cheng, 1999; Delgado, Verdegay & Vila, 1992; Fodor & Roubens, 2013; Herrera & Herrera-Viedma, 1997; Hwang & Lin, 2012); iii) ranking the labels; and iv) selecting the preferred alternative by group decision. It has been the fact that using linguistic labels makes experts' judgment more reliable and informative than using numeric scores. Fuzzy Collective Solution (FCS) is widely used to evaluate and rank the candidate solution for a model under linguistic assessments. The existing algorithms for FCS provided the tool for aggregation in GDM with intuitionistic preference relations.

Intuitionistic fuzzy set (IFS) is an extension of Zadeh's fuzzy set (1965) characterized by a membership and a non-membership function (Atanassov, 1986, 1999). In many complex decision making problems, the decision can be represented by IFS for better modelling of imprecise and uncertain data. Recently, some researchers applied IFS to GDM. Gau and Buehrer (1993) introduced the vague set, which is an equivalence of IFS. Hong and Choi (2000) developed approximate techniques for multi-attribute decision making using minimum and maximum operators. Szmidt and Kacpzyk (2002) proposed the intuitionistic fuzzy core and consensus winner in group decision making with intuitionistic (individual and social) fuzzy preference relations. Xu and Yager (2006) developed the intuitionistic fuzzy weighted geometric operator, fuzzy ordered and fuzzy hybrid geometric operator extending the traditional and ordered weighted geometric operator in the environment where given arguments are represented by IFSs.

Our contribution for GDM with intuitionistic preference relation is the new approach using the Pareto solution of the optimization theory and some aggregation procedures based on FCS to solve the model. This paper is organized in 8 sections with the first section is the introduction. Section 2 briefly reviews group decision making model and Low-operator. Section 3 introduces the concept of Fuzzy Collective Solution. Some computing algorithms for FCS of the multi-criteria problem are given in the Section 4. In section 5, intuitionistic linguistic preference relations are defined. Section 6 is devoted to a new concept of Fuzzy Pareto Solution (FPS) and some aggregation procedures for the FPS in the problems with intuitionistic linguistic preference relations. Section 7 shows an example to illustrate the approach. The final section is conclusion of this paper.

2. *A group decision making model*

2.1. *A GDM model under linguistic*

In this paper, let us consider the following model under linguistic assessments. Assume $X = \{x_1, x_2, ..., x_n\}$ is a finite set of alternatives and E ={e_1, …, e_m} is a finite set of experts. We assume that there exists a distinguished person, say the manager, who assigns an important degree $w(e_k) = w(k)$ for each expert such that $0 \le w(k) \le 1$, and $\sum_k w(k) = 1$.

Let $S = \{s_t, t = 1, ..., T\}$ be a finite and totally ordered linguistic-labels set. Assume that each expert $e_k \in E$ provides his/her opinions on X by mean of a linguistic preference relation $p_k : X \times X \to S$, where $p_k(i, j) = p_k(x_i, x_j) \in S,$ represents the linguistically assessed preference degree of alternative x_i over x_j. For example, we consider the following linguistic labels S [12]:

$$S = \{I, EU, VLC, SC, IM, MC, ML, EL, C\}$$

where they are trapezoidal fuzzy numbers on [0,1].

I	Impossible	(0, 0, 0, 0);
EU	Extremely-unlikely	(0.00, 0.01, 0.02, 0.07);
VLC	Very-low-chance	(0.04, 0.10, 0.18, 0.23);
SC	Small-chance	(0.17, 0.22, 0.36, 0.42);
IM	It- may	(0.32, 0.41, 0.58, 0.65);
MC	Meaningful-chance	(0.58, 0.63, 0.80, 0.86);
ML	Most-likely	(0.72, 0.78, 0.92, 0.97);
EL	Extremely-likely	(0.93, 0.98, 0.99, 1);
C	Certain	(1, 1, 1, 1);

2.2. *A modification of LOWA operator*

Yager (1998) defined the ordered weighted averaging (OWA) operator. The LOWA operator is based on OWA and the convex combination of linguistic labels (Llamazares, 2007). However, in many real cases, the weights of experts were not considered. Thus, we are concerning the aggregation problem in which the weights of experts are unknown.

Definition 1. Let $a = \{a_1, ..., a_m\}$ be a set of linguistic-labels to aggregate, and b is the associated ordered labels vector, i.e. $b = \{a_{im}, a_{i(m-1)}, \ldots, a_{i1}\}$, such that $a_{im} \geq a_{i(m-1)} \geq \ldots \geq a_{i1}$.

The *Low operator* is defined as

$$Low(a,w) = C\left\{(w_{im}, a_{im}), (1-w_{im}, Low(a', w'))\right\} \tag{1}$$

where $w = [w_1, ..., w_m]$, is a weight vector such that, $w_i \in [0,1]$ and $\Sigma_I w_i = 1$,

$$a' = \{a_{i(m-1)}, ..., a_{i1}\},\ w' = \{w'_{i(m-1)}, ..., w'_{i1}\},\ w'_j = w_j/(1-w_{im}). \tag{2}$$

C is the convex combination operator of 2 labels s_j, s_i, $j \geq i$ with $w_j>0$, $w_i>0$, $w_j+w_i=1$,

$$C\{(w_j,s_j).(w_i,s_i)\} = s_k, \tag{3}$$

where $k = i + \text{round}(w_j.(j-i))$, where round is the usual round operator.

2.3. *Multi-criteria group decision making model*

In our model, assume that there is a finite set of criteria: $C = \{C_1, C_2, ..., C_L\}$. Respect to each criterion C_l, each expert $e_k \in E$ provides his/her opinions on X by mean of a linguistic preference relation

$$p_{kl} : X \times X \to S, \quad \text{where} \quad p_{kl}(i,j) = p_{kl}(x_i, x_j) \in S \tag{4}$$

is the linguistically assessed preference degree of alternative x_i over x_j. Moreover, assume that there are important weights of criteria $\{\beta_l, l=1,2,..., L\}$, such that $0 \leq \beta_l \leq 1$, and $\Sigma_l \beta_l = 1$.

In our model we assume that $p_{kl} : X \times X \to S$ is *soft-recipodal* in the sense:

(i) $p_{kl}(i,i) = s_{T+1/2}$, for all i=1,…,n (5)

(ii) $p_{kl}(i,j) \geq s_{T+1/2}$, then $p_{kl}(j,i) \leq s_{T+1/2}$ (6)

Let $\{P^k, k=1,...,m\}$ be linguistic preference relations. For every $(i,j),\ i=1,...n, j=1,...,n$, and for each $s_t \in S,$ we define $W_{i,j}[s_t] = \sum_k \{w(k): p^k(i,j) = s_t\}$. (7)

This value is the sum of individual importance degrees of experts, who coincide to assign the linguistic label s_t as preference value of the alternative x_i over alternative x_j .

In the next section, we will define the concept of Fuzzy Collective Solution (FCS) and apply to evaluate and to rank the alternatives set.

3. *Fuzzy Collective Solution*

3.1. *Notions*

The concept of fuzzy collective solution is similar to the concept of aggregated dominance degrees of alternatives. For each pair (x_i,x_j), the *linguistic dominance degree* is defined by

$$E(x_i, x_j) = Low(S,U) \tag{8}$$

where $U = [u_T, ..., u_1]$, $u_t = W_{ij}[s_t]$ for t=1,...,T. (9)

Definition 2. The *fuzzy collective solution* (FCS) is a fuzzy set on the alternatives set X

$$FCS = \{ fcs(x_1)/x_1,\ fcs(x_2)/x_2, ...,\ fcs(x_n)/x_n \} \tag{10}$$

where the membership degree of the alternative x_i is calculated as follows:

$$fcs(x_i) = Low(S,V) \tag{11}$$

where $V = [v_T, ..., v_1]$, for t = 1,..., T, $v_t = \#\{ j : E(x_i, x_j) = s_t \} / n-1$. (12)

3.2. *Aggregated Fuzzy Collective Solution*

Now, we give a notion for aggregation of FCS in multi-criteria decision making problems. Suppose that for each criterion C_l, we obtained the corresponding fuzzy collective solution:

$$FCS_l = \{ fcs_l(x_1)/x_1, ..., fcs_l(x_n)/x_n \} \quad l = 1, 2, ..., L.$$

Definition 3. The *aggregated fuzzy collective solution* (aFCS) is a fuzzy set on the alternatives set X

$$aFCS = \{afcs(x_1)/x_1, afcs(x_2)/x_2, ..., afcs(x_n)/x_n\} \tag{13}$$

where the membership degree of the alternative x_i is calculated as

$$afcs(x_i) = Low(S, U_\beta), \quad i=1,\ldots, n \tag{14}$$

where $U_\beta = \left[u_{\beta_T}, \ldots, u_{\beta_1}\right]$, $U_{\beta_t} = \sum_l \{\beta_l : fcs(x_i) = s_t\}$, t=1, ...T (15)

Note that aFCS is a type-2 fuzzy set on X.

4. *Some computing procedures using FCS*

In the following computing procedures, we use the linguistic preference relations $\{p_{kl},\ k = 1,\ldots,m,\ l = 1,\ldots,L\}$, the experts' important weights $\{w(k) : e_k \in E\}$, such that $0 \le w(k) \le 1$, and $\sum_k w(k) = 1$ and the criteria's important weights $\{\beta_l, l = 1,\ldots,L\}$, such that $0 \le \beta_l \le 1$, and $\sum_l \beta_l = 1$. Now we consider some computing procedures using FCS for the model.

Algorithm 1:

Step 1.

1a. For each criterion C_l, using $\{p_{kl}, k = 1,\ldots,m\}$ and the weights $\{w(k) : e_k \in E\}$ calculate the *linguistic dominance degrees* $E_l = \left[E_l(i, j)\right] = \left[E(x_i, x_j)\right]$, i , j =1,…,n (16)

where $E_l(x_i, x_j) = Low(S, U_l)$, $U_l = \left[u_{lT}, \ldots, u_{l1}\right]$ (17)

where $u_{lt} = W_{ij}(s_t) = \sum\limits_k \{w(k) : p_{kl}(i, j) = s_t\}$, t=1,...,T (18)

1b. Using the $E_l = \left[E_l(x_i, x_j)\right]$, calculate the fuzzy collective solution

$$FCS_l = \{ fcs_l(x_1)/x_1,\ fcs_l(x_2)/x_2, \ldots,\ fcs_l(x_n)/x_n\} \tag{19}$$

as in the Definition 2.

Step 2. Using the $\{FCS_l, l = 1,\ldots,L\}$ and the important weights $\{\beta_l, l = 1,\ldots,L\}$, calculate the aggregated fuzzy collective solution (aFCS)

$$aFCS = \{afcs(x_1)/x_1, afcs(x_2)/x_2, \ldots, afcs(x_n)/x_n\}, \tag{20}$$

as in the Definition 3.

Step 3. Classify the set of alternatives X into subsets

$$X_t = \{x_i : \ afcs(x_i) = s_t\}, \quad \text{with } t = 1, \dots, T \tag{21}$$

Rank the non-empty sets $X_t \prec X_{t'}$ iff $t < t'$, and therefore, the solution is the set X_{t^*}, where $t^* = \max\{t : X_t \neq \emptyset, s_t \in S\}$.

Algorithm 2:

Step 1. For each criterion C_l, calculate the linguistic dominance degrees as in (16).

$$E_l = \left[E_l(x_i, x_j)\right], \ \text{i,j} = 1,\dots,n, \quad l = 1,\dots, L$$

Step 2.

2.a. Using $\{E_l, l = 1,\dots, L\}$ and the important weight $\{\beta_l, l = 1,\dots, L\}$, calculate the *totally social opinion relation:* $Q = [q(i, j)] = \left[q(x_i, x_j)\right]$, i , j = 1,…,n (22)

where $q(x_i, x_j) = Low(S, U_q)$, $U_q = \left[u_{qT}, \dots, u_{q1}\right]$ (23)

where $u_{qt} = W_{ij}^q(s_t) = \sum_l \{\beta_l : E_l(i, j) = s_t\}$, t=1,...,T (24)

2.b. Calculate the fuzzy collective solution according to the totally social opinion relation Q

$$FCS_Q = \left\{ fcs_Q(x_1)/x_1, fcs_Q(x_2)/x_2, \dots, fcs_Q(x_n)/x_n \right\} \tag{25}$$

as in Definition 2.

Step 3. Classify the alternative set X into subsets

$$X_t = \{x_i : \ fcs_Q(x_i) = s_t\}, \quad t = 1, \dots, T \tag{26}$$

and choose the solution as in the step 3 of the Algorithm 1.

Algorithm 3:

Step 1.

1a. For each expert e_k, using $\{p_{kl}, l = 1,\dots, L\}$ and the important weights $\{\beta_l, l = 1,\dots, L\}$, calculate the *relatively dominance degrees* F^k according to expert e_k as

$$F^k = \left[F^k(i, j)\right] = \left[F^k(x_i, x_j)\right], \qquad \text{i, j} = 1,\dots,n$$

where $F^k(x_i, x_j) = Low(S, U^k)$ (27)

where $U^k = [u_T^k, \dots, u_1^k]$ where $u_t^k = \sum_l \{\beta_l : p_{kl}(i, j) = s_t\}$, t=1,...,T (28)

1b. Using F^k, calculate *the fuzzy evaluation* FE^k according to the expert e_k

$$FE^k = \left\{ fe^k(x_1)/x_1,\ldots, fe^k(x_n)/x_n \right\} \quad (29)$$

with the membership degree of the alternative x_i is calculated as

$$fe^k(x_i) = Low(S, V_k), \quad i=1,\ldots, n \quad (30)$$

where $V_k = [v_T,\ldots,v_1]$, $v_t = \left| \left\{ j : F^k(x_i, x_j) = s_t,\ j \neq i \right\} \right| \big/ n-1$, t=1,...,T (31)

Step 2. Using the fuzzy evaluations $\left\{ FE^k, k=1,\ldots,m \right\}$ and the weights $\left\{ w(k) : e_k \in E \right\}$, calculate the *aggregated fuzzy evaluation* (aFE), which is a fuzzy set on the alternatives set X:

$$aFE = \left\{ afe(x_1)/x_1,\ldots, afe(x_n)/x_n \right\}, \quad (32)$$

as in Definition 3.

Step 3. Classify the set of alternatives X into subsets

$$X_t = \left\{ x_i :\ afe(x_i) = s_t \right\}, \quad t = 1, \ldots, T \quad (33)$$

and choose the solution as in the step 3 of the Algorithm 1.

5. Intuitionistic Linguistic Preference Relation

5.1. Intuitionistic Fuzzy Sets

Definition 4 (Atanassov, 1986). Let Y be a universe of discourse. Then an *intuitionistic fuzzy set* (IFS)

$$A = \left\{ ((\mu_A(y), \nu_A(y))/y) \middle| y \in Y \right\} \quad (34)$$

is characterized by a membership function $\mu_A : Y \to [0,1]$, and a non-membership function $\nu_A : Y \to [0,1]$, with the condition $0 \le \mu_A(y) + \nu_A(y) \le 1$ for all $y \in Y$, where the numbers $\mu_A(y)$ and $\nu_A(y)$ represent, respectively, the degree of membership and the degree of non-membership of the element y to the set *A*.

Definition 5 (Xu, 2007). Let $X = \left\{ x_1, x_2, \ldots, x_n \right\}$. An *intuitionistic preference relation* B on the set X is represented by a matrix:

$B = \left[b_{ij} \right]_{n \times n}$ where $b_{ij} = \left\{ (\mu(x_i, x_j), \nu(x_i, x_j))/(x_i, x_j) \right\}$, for all $i, j = 1, 2, \ldots, n$ (35)

where $\mu_{ij} = \mu(x_i, x_j) \in [0,1]$ is the preference degree of x_i in comparing with x_j and $\nu_{ij} = \nu(x_i, x_j) \in [0,1]$ is the non-preference degree of x_i in comparing with x_j.

In [33], Xu supposed that $\mu_{ij} = \mu(x_i, x_j)$ and $\nu_{ij} = \nu(x_i, x_j)$ satisfy the following condition:

$$0 \le \mu_{ij} + \nu_{ij} \le 1,\ \ \mu_{ji} = \nu_{ij},\ \ \nu_{ji} = \mu_{ij},\ \ \ \mu_{ii} = \nu_{ii} = 0.5\ ,\quad \text{for all } i, j = 1, 2, ..., n. \tag{36}$$

Next, we will define the intuitionistic linguistic preference relation and will present a new approach to solve the group decision problems with intuitionistic linguistic preference relations.

5.2. Intuitionistic linguistic preference relation

Definition 6. Let $X = \{x_1, x_2, ..., x_n\}$ be an alternatives set. Let S be a finite and totally ordered linguistic labels set: S= {s_t , t=1,..., T} where T is an odd number. An *intuitionistic linguistic preference relation* P on X is a matrix

$$P = \left[\, p_{ij} \,\right]_{n\times n} \text{ where } p_{ij} = \left\{(\mu(x_i, x_j), \nu(x_i, x_j)) / (x_i, x_j)\right\}, \text{for all } i, j = 1, 2, ..., n \tag{37}$$

where $\mu_{ij} = \mu(x_i, x_j) \in S$ is the linguistic preference degree of x_i in comparing with x_j and $\nu_{ij} = \nu(x_i, x_j) \in S$ is the linguistic non-preference degree of x_i in comparing with x_j.

Definition 7. (Soft-reciprocal condition) An intuitionistic linguistic preference relation P on the set X is said to be *soft-reciprocal* if

$$\mu_{ij} = \mu(x_i, x_j) \ge S_{\frac{T+1}{2}}, \text{ then } \mu_{ji} = \mu(x_j, x_i) < S_{\frac{T+1}{2}} \text{ and } \nu_{ij} = \nu(x_i, x_j) < S_{\frac{T+1}{2}}\ ,$$

$$\text{for all } i, j = 1, 2, ..., n \tag{38}$$

Note 1. This condition on preference relation is much weaker than other supposed conditions in the papers on decision making problems using preference relations. For example see (Xu, 2006).

6. Fuzzy Pareto Solution

In our approach, we need the Pareto Solution concept of Optimization Theory (Tuy, 1998).

6.1. *Pareto Solution in Optimization Theory*

Let $D \neq \varnothing$. Let $f : D \to R^n$ be a real function, i.e. for each $x \in D$,

We have $$f(x) = (f_1(x), f_2(x), \ldots, f_j(x), \ldots, f_n(x)). \tag{39}$$

Definition 8 *(Pareto Solution).* Consider the following optimization problem

f(x) $\to \max$

subject to $x \in D$.

A $x^* \in D$ is a *Pareto solution*, if there not exists $x' \in D$ such that

$$f(x^*) \leq f(x') \quad and \quad f(x^*) \neq f(x'), \text{ i.e.,} \tag{40}$$

for any $x' \in D$, if $f_j(x^*) < f_j(x')$ for some j, then there is $k \neq j$ such that $f_k(x') < f_k(x^*)$.

A generalization of this concept is the following.

6.2. *Generalized Pareto Solution*

Let $D \neq \varnothing$, S_i, $for\ i = 1, 2 \ldots, n$ be totally ordered sets, $S = S_1 \times S_2 \times \ldots \times S_n$ (41)

Let $g : D \to S$ be a map from D to S.

For each $x \in D$, $$g(x) = (g_1(x), \ldots, g_j(x), \ldots, g_n(x)). \tag{42}$$

Definition 9 . (Generalized Pareto Solution)

A $x^* \in D$ is a *generalized Pareto solution* if there not exists $x' \in D$ such that

$$g(x^*) \leq g(x') \quad and \quad g(x^*) \neq g(x'). \text{ i.e.,} \tag{43}$$

for any $x' \in D$, if $g_j(x^*) < g_j(x')$ for some j, then there is $k \neq j$ such that

$$g_k(x') < g_k(x^*). \tag{44}$$

We will see that this new definition is a useful concept in the group decision making problems with intuitionistic linguistic preference relations.

6.3. *Fuzzy Pareto Solution*

Definition 10. (Fuzzy Pareto Solution)

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an alternatives set. The evaluation information of the expert e_k about X is a IFS on X

$\{(p_M^k(x_1), p_V^k(x_1))/x_1, \ldots, (p_M^k(x_n), p_V^k(x_n)/x_n\}$, k=1, 2, ..., m

Let S_1, S_2 be totally ordered linguistic-label sets.

Suppose that there is an *aggregated evaluation map* $g : X \to S_1 \times S_2$

$$g(x) = \{(g_M(x_1), g_V(x_1))/x_1,\ \dots,\ (g_M(x_n), g_V(x_n)/x_n\}. \quad (45)$$

For each $x_i \in X$, i = 1,…,n

$g(x_i) = (g_M(x_i), g_V(x_i))$ is a linguistic IFS on X,

where $g_M(x_i) \in S_1$ is the first membership degree and $g_V(x_i) \in S_2$ is the second membership degree. A $x^* \in X$ is an *Fuzzy Pareto Solution (FPS)*, if there not exists $x' \in D$ such that:

$$g_M(x^*) < g_M(x') \quad and \quad g_V(x') \le g_V(x^*) \text{, or} \quad (46)$$

$$g_M(x^*) \le g_M(x') \quad and \quad g_V(x') < g_V(x^*). \quad (47)$$

In the case that $S_1 = S_2 = S$ is a linguistic labels set defined as in Section 1, FPS is a type-2 IFS.

6.4. Some aggregation procedures

We consider the following multi-criteria problem:

Given the alternatives set $X = \{x_1, x_2, ..., x_n\}$. Given the criteria set $\{C_1, C_2, ..., C_L\}$.

The expert set $E = \{e_1, ..., e_m\}$ with their importance $\{w(k) : e_k \in E\}$ such that $0 \le w(k) \le 1$,

and $\sum_k w(k) = 1$. Given criteria's important weights $\{\beta_1, \beta_2, ..., \beta_L\}$ such that $0 \le \beta_l \le 1$, and $\sum_l \beta_l = 1$.

The linguistic labels set:

$$S = \{s_1 = I, s_2 = EU, s_3 = VLC, s_4 = SC, s_5 = IM, s_6 = MC, s_7 = ML, s_8 = EL, s_9 = C\}$$

We assume that each expert $e_k \in E$ provides his/ her opinions on X by mean of intuitionistic linguistic preference relations

$$p_{kl} = (M_{kl}, V_{kl}), \quad k = 1, ..., m,\ l = 1, ..., L \quad (48)$$

satisfying the soft-reciprocal condition.

The linguistic preference relations are:

$$M_{kl} = \left[\mu_{kl}(x_i, x_j)\right],\ for\ i, j = 1, .., n, \quad \mu_{kl}(x_i, x_j) \in S \quad (49)$$

and the linguistic non-reference relations are

$$V_{kl} = \left[\nu_{kl}(x_i, x_j)\right],\ \ for\ i, j = 1, ..., n, \quad \nu_{kl}(x_i, x_j) \in S\ , \quad (50)$$

We have to evaluate and to choose the solution sets based on the given intuitionistic linguistic preference relations. Using the algorithms in Section 4, we present three aggregation procedures for FPS.

Aggregation procedure 1:

Step 1.

1.1.a. For each criterion C_l, using $p_{kl} = (M_{kl}, V_{kl}), k = 1,...,m$ and the weights $\{w(k) : e_k \in E\}$,

calculate the linguistic dominance degrees:

$$E_M^1(x_i, x_j) = Low(S, U_{M1}) \quad , \qquad i, j= 1,\dots,n \tag{51}$$

where U_{ML} is calculated as (18) with $p_{kl} = M_{kl}$.

1.1.b. For each criterion C_l , using the relation E_M^l , calculate the FCS according to each criterion C_l

$$FCS_M^l = \left\{ fcs_M^l(x_1)/x_1, ..., fcs_M^l(x_n)/x_n \right\}, \qquad l=1,\dots, L \tag{52}$$

as in Definition 2 .

1.2.a. For each criterion C_l , calculate the linguistic non-dominance degrees:

$$E_V^l(x_i, x_j) = Low(S, U_{Vl}) \quad , \qquad i, j= 1,\dots,n \tag{53}$$

where U_{Vl} is calculated as (18) with $p_{kl} = V_{kl}$.

1.2.b. For each criterion C_l , using the relation E_V^l , calculate the FCS according to each criterion C_l

$$FCS_V^l = \left\{ fcs_V^l(x_1)/x_1, ..., fcs_V^l(x_n)/x_n \right\}, \qquad l=1,\dots, L \tag{54}$$

as in Definition 2 .

Now we obtain Intuitionistic Fuzzy Evaluation

$$IFE^l = \left\{ (fcs_M^l(x_1), fcs_V^l(x_1))/x_1, .., (fcs_M^l(x_n), fcs_V^l(x_n))/x_n \right\} . \tag{55}$$

Step 2

Using the $\{(FCS_M^l, FCS_V^l), l = 1,..., L\}$ and the important weights $\{\beta_1, \beta_2, ..., \beta_L\}$, calculate the *aggregated fuzzy collective solutions* (aFCS):

$$aFCS_M = \{afcs_M(x_1)/x_1, ..., afcs_M(x_n)/x_n\} \quad \text{and} \tag{56}$$

$$aFCS_V = \{afcs_V(x_1)/x_1, ..., afcs_V(x_n)/x_n\} \tag{57}$$

as in Definition 3.

Finally, we obtain Aggregated Intuitionistic Fuzzy Evaluation (aIFE):

$$aIFE = \{(afcs_M(x_1)/x_1, afcs_V(x_1)) ..., (afcs_M(x_n)/x_n, afcs_V(x_n))\} \ . \tag{58}$$

Using aIFE, we choose *Aggregated Fuzzy Pareto Solution* (aFPS) of the problem.

Aggregation procedure 2:

Step 1.

For each criterion C_l, using $p_{kl} = (M_{kl}, V_{kl}),\quad k = 1,...,m$ and the weights $\{w(k) : e_k \in E\}$,

calculate the linguistic dominance degrees:

$$E_M^1(x_i, x_j) = Low(S, U_{M1}) ,\qquad \text{i, j= 1,…,n} \tag{59}$$

where U_{Ml} is calculated as (18) with $p_{kl} = M_{kl}$.

Calculate the linguistic non-dominance degrees:

$$E_V^1(x_i, x_j) = Low(S, U_{V1}) ,\qquad \text{i, j= 1,…,n} \tag{60}$$

where U_{Vl} is calculated as (18) with $p_{kl} = V_{kl}$.

Step 2.

2.a. Using the relations of linguistic dominance degrees $\{E_M^1, ..., E_M^L\}$ and the important weights $\beta = \{\beta_1, ..., \beta_L\}$, calculate the totally social opinion relation

$$Q_M = \left[q_M(x_i, x_j)\right],\quad i, j = 1,...,n \tag{61}$$

Where $q_M(x_i, x_j) = Low(S, U_{q_M})$, $\qquad U_{q_M} = \left[u_{qT}, ..., u_{q1}\right]$ (62)

where $u_{qt} = W_{ij}^q(s_t) = \sum_l \{\beta_l : E_M^l(i, j) = s_t\}$, $\qquad$ t=1,...,T . (63)

Using the relations of linguistic non-dominance degrees $\{E_V^1, ..., E_V^L\}$ and the important weights $\beta = \{\beta_1, ..., \beta_L\}$, calculate the totally social opinion relation

$$Q_V = \left[q_V(x_i, x_j)\right],\quad i, j = 1,...,n \tag{64}$$

where $q_V(x_i, x_j) = Low(S, U_{q_V})$, $\qquad U_{q_M} = \left[u_{qT}, ..., u_{q1}\right]$ (65)

where $u_{qt} = W_{ij}^q(s_t) = \sum_l \{\beta_l : E_V^l(i, j) = s_t\}$, $\qquad$ t=1,...,T . (66)

2.b. Using Q_M, Q_V, calculate the fuzzy collective solution according to the totally social opinion relations

$$FCS_{Q_M} = \left\{ fcs_{q_M}(x_1)/x_1, ..., fcs_{q_M}(x_n)/x_n \right\} \text{ and} \tag{67}$$

$$FCS_{Q_V} = \left\{ fcs_{q_V}(x_1)/x_1, ..., fcs_{q_V}(x_n)/x_n \right\} \tag{68}$$

as in Definition 2

From (67,68) we obtain Intuitionistic Fuzzy Evaluation

$$IFE_Q = \left\{ (fcs_{q_M}(x_1), fcs_{q_V}(x_1))/x_1, .., (fcs_{q_M}(x_n), fcs_{q_V}(x_n))/x_n \right\} \tag{69}$$

Using IFE_Q, we choose FPS of the problem.

Aggregation procedure 3:

Step 1. For each expert e_k, for k=1,...,m, use $p_{kl} = (M_{kl}, V_{kl}), \quad l = 1,...,L$, and the weights $\beta = \{\beta_1,...,\beta_L\}$, calculate $F_M^k(x_i, x_j) = Low(S, U_M^k)$ (70)

where $U_M^k = [u_T,...,u_1]$, where, $u_t = \sum_l \{\beta_l : M_{kl}(i,j) = s_t\}$ t=1,...,T, (71)

and the fuzzy evaluation $FE_M^k = \{ fe_M^k(x_1)/x_1,..., fe_M^k(x_n)/x_n \}$, k =1....,m

where $fe_M^k(x_i) = Low(S, V_M^k)$, (72)

where $V_M^k = [v_T,...,v_1]$, for t=1,...,T , $v_t = \left| \left\{ j : F_M^k(x_i, x_j) = s_t, j \neq i \right\} \right| \Big/ n-1$

(73)

as (31) with $F^k = F_M^k$.

Analogously, calculate $F_V^k(x_i, x_j) = Low(S, U_V^k)$, (74)

where $U_V^k = [u_T,...,u_1]$, where $u_t = \sum \{\beta_l : V_{kl}(i,j) = s_t\}$, t=1,...,T, (75)

and the fuzzy evaluation $FE_V^k = \{ fe_V^k(x_1)/x_1,..., fe_V^k(x_n)/x_n \}$, k=1,..,m (76)

where $fe_V^k(x_i) = Low(S, V^k)$, $V_V^k = [v_T,...,v_1]$,

$v_t = \left| \left\{ j : F_V^k(x_i, x_j) = s_t, j \neq i \right\} \right| \Big/ n-1$, t=1,…,T

as (31) with $F^k = F_V^k$.

Step 2. Using the fuzzy evaluation $\left\{ (FE_M^k, FE_V^k), \ k = 1,...,m \right\}$ and the weights {w(k): $e_k \in E$}, calculate the aggregated fuzzy evaluation (aFE):

$aFE_M = \{afe_M(x_1)/x_1,..., afce_M(x_n)/x_n\}$ and $aFE_V = \{afe_V(x_1)/x_1,..., afe_V(x_n)/x_n\}$ (77)

as in Definition 3.

Then we obtain *Aggregated Intuitionistic Fuzzy Evaluation* (aIFE):

$$aIFE = \left\{ (afe_M(x_1), afe_V(x_1))/x_1, ,..., (afe_M(x_n), afe_V(x_n))/x_n \right\}. \tag{78}$$

Using aIFE, we choose aFPS of the problem.

7. A numerical example

We consider the following multi criteria GDM problem with intuitionistic linguistic preference relations:

The alternatives set: $X = \{x_1, x_2, x_3, x_4\}$.

The expert set: $E = \{e_1, e_2, e_3\}$ with their importance w={0.2, 0.5, 0.3}.

The set of criteria: $C = \{C_1, C_2, C_3\}$ with their importance weighs $\beta = \{0.35, 0.4, 0.25\}$.

The linguistic labels set:

$$S = \{s_1 = I, s_2 = EU, s_3 = VLC, s_4 = SC, s_5 = IM, s_6 = MC, s_7 = ML, s_8 = EL, s_9 = C\}$$

.

The intuitionistic linguistic preference relations are relations

$$p_{kl} = (M_{kl}, V_{kl}), \quad k = 1,2,3, \; l = 1,2,3 \tag{79}$$

where

$$M_{kl} = \left[\mu_{kl}(x_i, x_j)\right], \quad i, j = 1,2,3,4 \tag{80}$$

$$V_{kl} = \left[\nu_{kl}(x_i, x_j)\right], \quad i, j = 1,2,3,4 \tag{81}$$

are given in the Appendix.

7.1. Computing with the aggregation procedure 1

Step1.

Step 1.1. Computing for criterion C_1

1.1.a. Use the linguistic preference relations

$$M_{21} = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ ML & IM & EL & IM \end{bmatrix}, \quad M_{31} = \begin{bmatrix} IM & MC & MC & EU \\ SC & IM & SC & EU \\ SC & MC & IM & EU \\ ML & IM & EL & IM \end{bmatrix}, \quad M_{31} = \begin{bmatrix} IM & MC & MC & EU \\ SC & IM & SC & EU \\ SC & MC & IM & EU \\ ML & IM & EL & IM \end{bmatrix}.$$

Calculate the linguistic dominance degrees:

$E_M^1(x_i, x_j) = Low(S, U_{M1})$, i,j = 1,2,3, 4 as in (51).

$$E_M^1(x_1, x_1) = Low((s_5),(1)) = s_5 = IM.$$

$$E_M^1(x_1, x_2) = Low((s_6, s_7, s_6),(0.2,0.5,0.3)) = C\{(0.5, s_7),(0.5, Low((s_6),(1))\}$$
$$= C\{(0.5, s_7),(0.5, s_6)\} = s_{k(1,2)},$$
$$k(1,2) = 6 + round((0.5).(7-6)) = 6+1 = 7 \quad \Rightarrow s_{k(1,2)} = s_7 = ML.$$

$$E_M^1(x_1, x_3) = Low((s_7, s_6, s_6),(0.2,0.5,0.3)) = C\{(0.2, s_7),(0.8, Low((s_6),(1))\}$$
$$= C\{(0.2, s_7),(0.8, s_6)\} = s_{k(1,3)},$$
$$k(1,3) = 6 + round((0.2).(8-6)) = 6+0 = 6 \quad \Rightarrow s_{k(1,3)} = s_6 = MC.$$

$$E_M^1(x_1,x_4) = Low((s_2,s_2,s_2),(0.2,0.5,0.3)) = C\{(0.2,s_2),(0.8,Low((s_2),(1))\}$$
$$= C\{(0.2,s_2),(0.8,s_2)\} = s_2 = EU.$$
$$E_M^1(x_2,x_1) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,s_4)\} = s_4 = SC.$$
$$E_M^1(x_2,x_2) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\{(0.2,s_5),(0.8,s_5)\} = s_5 = IM.$$
$$E_M^1(x_2,x_3) = Low((s_4,s_6,s_4),(0.2,0.5,0.3)) = C\{(0.5,s_6),(0.5,Low((s_4,s_4),1))\}$$
$$= C\{(0.5,s_6),(0.5,s_4)\} = s_{k(2,3)},$$
$$k(2,3) = 4 + round((0.5).(6-4)) = 4+1 = 5 \quad \Rightarrow s_{k(2,3)} = s_5 = IM.$$
$$E_M^1(x_2,x_4) = Low((s_3,s_2,s_2),(0.2,0.5,0.3)) = C\{(0.2,s_3),(0.8,Low((s_2,s_2),(1)\}$$
$$= C\{(0.2,s_3),(0.8,s_2)\} = s_{k(2,4)},$$
$$k(2,4) = 2 + round((0.2).(3-1)) = 2+0 = 2 \quad \Rightarrow s_{k(2,4)} = s_2 = EU\ .$$
$$E_M^1(x_3,x_1) = Low((s_3,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.5,s_4),(0.5,Low((s_4,s_3),(3/5,2/5))\}$$
$$= C\{(0.5,s_4),(0.5,s_4)\} = s_4 = SC.$$
$$E_M^1(x_3,x_2) = Low((s_6,s_4,s_6),(0.2,0.5,0.3)) = C\{(0.2,s_6),(0.8,Low((s_6,s_4),(3/8,5/8))\}$$
$$= C\{(0.2,s_6),(0.8,s_5)\} = s_{k(3,2)} = s_5 = IM.$$
since
$$Low((s_6,s_4),(3/8,5/8)) = s_k, \quad k = 4 + round((3/8).(6-4)) = 4+1 = 5 \quad \Rightarrow s_k = s_5$$
$$E_M^1(x_3,x_3) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\{(0.2,s_5),(0.8,s_5)\} = s_5 = IM\ .$$
$$E_M^1(x_3,x_4) = Low((s_2,s_2,s_2),(0.2,0.5,0.3)) = C\{(0.2,s_2),(0.8,Low((s_2),(1))\}$$
$$= C\{(0.2,s_2),(0.8,s_2)\} = s_2 = EU.$$
$$E_M^1(x_4,x_1) = Low((s_8,s_7,s_7),(0.2,0.5,0.3)) = C\{(0.2,s_8),(0.8,Low((s_7),(1))\}$$
$$= C\{(0.2,s_8),(0.8,s_7)\} = s_{k(4,1)},$$
$$k(4,1) = 7 + round((0.2).(8-7)) = 7 \quad \Rightarrow s_{k(4,1)} = s_7 = ML\cdot$$
$$E_M^1(x_4,x_2) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\{(0.2,s_5),(0.8,Low((s_5),(1))\}$$
$$= C\{(0.2,s_5),(0.8,s_5)\} = s_5 = IM.$$
$$E_M^1(x_4,x_3) = Low((s_7,s_8,s_9),(0.2,0.5,0.3)) = C\{(0.3,s_9),(0.7,Low((s_8,s_7),(5/8,2/8))\}$$
$$= C\{(0.3,s_9),(0.7,s_8)\} = s_{k(4,1)},$$
since
$$Low((s_8,s_7),(5/7,2/7)) = s_k, \quad k = 7 + round((5/7).(8-7)) = 7+1 = 8 \quad \Rightarrow s_k = s_8$$
$$k(4,3) = 8 + round((0.3).(9-8)) = 8 \quad \Rightarrow s_{k(4,3)} = s_8 = EL\ .$$

$E_M^1(x_4, x_4) = Low((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C\{(0.2, s_5), (0.8, s_5)\} = s_5 = IM$ We obtain the linguistic dominance degrees for criterion C_1 :

$$E_M^1 = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & IM & EU \\ SC & IM & IM & EU \\ ML & IM & EL & IM \end{bmatrix}. \quad (82)$$

Calculate the FCS by Definition 2.

$fcs_M^1(x_1) = Low((s_7, s_6, s_2), (1/3, 1/3, 1/3)) = C\{(1/3, s_7), (2/3, Low((s_6, s_2), (0.5, 0.5))\}$
$= C\{(1/3, s_7), (2/3, s_4)\} = s_k$,

since

$Low((s_6, s_2), (0.5, 0.5)) = s_{k'}, k' = 2 + round((0.5).(6-2)) = 2 + 2 = 4 \Rightarrow s_{k'} = s_4$,

$k = 4 + round((1/3).(7-4)) = 4 + 1 = 5 \quad \Rightarrow s_k = s_5 = IM$.

$fcs_M^1(x_2) = Low((s_5, s_4, s_2), (1/3, 1/3, 1/3)) = C\{(1/3, s_5), (2/3), Low((s_4, s_2), (1/2, 1/2))\}$
$= C\{(1/3, s_5), (2/3, s_3)\} = s_k$,

since

$Low((s_4, s_2), (0.5, 0.5)) = s_{k'}, \quad k' = 2 + round((0.5).(4-2)) = 2 + 1 = 3 \quad \Rightarrow s_{k'} = s_3$

$k = 3 + round((1/3).(5-3)) = 3 + 1 = 4 \quad \Rightarrow s_k = s_4 = SC$.

$fcs_M^1(x_3) = Low((s_5, s_4, s_2), (1/3, 1/3, 1/3)) = C\{(1/3, s_5), (2/3, Low((s_4, s_2), (1/2, 1/2))\}$
$= C\{(1/3, s_5), (2/3, s_3)\} = s_4 = SC.$

$fcs_M^1(x_4) = Low((s_7, s_5, s_8), (1/3, 1/3, 1/3)) = C\{(1/3), s_8, 2/3.Low((s_7, s_5), (0.5, 0.5)\}$
$= C\{(1/3, s_8), (2/3, s_6)\} = s_k$,

since

$Low((s_7, s_5), (0.5, 0.5)) = s_{k'}, \quad k' = 5 + round((0.5).(7-5)) = 5 + 1 = 6 \quad \Rightarrow s_{k'} = s_6$

$k = 6 + round((1/3).(8-6)) = 6 + 1 = 7 \quad \Rightarrow s_k = s_7 = ML$.

We obtain FCS for criterion C_1.

$$FCS_M^1 = \{IM / x_1, SC / x_2, SC / x_3, ML / x_4\}. \quad (83)$$

1.1.b. Use the linguistic non-preference relations

$$V_{11} = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix}, \quad V_{21} = \begin{bmatrix} SC & EU & VLC & SC \\ EU & SC & VLC & EU \\ VLC & MC & SC & VLC \\ EU & EU & VLC & SC \end{bmatrix}, \quad V_{31} = \begin{bmatrix} SC & SC & SC & SC \\ EU & SC & SC & EU \\ EU & SC & SC & I \\ EU & SC & I & SC \end{bmatrix}$$

Calculate the linguistic dominance degrees using non-preference relations:

$E_V^1(x_i, x_j) = Low(S, U_{V1})$, i, j = 1,2,3,4 as in (53).

$$E_V^1(x_1, x_1) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C\{(0.2, s_4), (0.8, Low((s_4), (1)\}$$
$$= C\{(0.2, s_4), (0.8, s_4)\} = s_4 = SC.$$

$$E_V^1(x_1, x_2) = Low((s_4, s_2, s_4), (0.2, 0.3, 0.5)) = C\{(0.2, s_4), (0.8Low((s_4, s_2), (3/8, 5/8))\}$$
$$= C\{(0.2, s_4), (0.8, s_3)\} = s_{k(1,2)},$$

since

$$Low((s_4, s_2), (3/8, 5/8)) = s_{k'}, \quad k' = 2 + round((3/8).(4-2)) = 2 + 1 = 3 \quad \Rightarrow s_{k'} = s_3$$
$$k(1,2) = 3 + round((0.2).(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(1,2)} = s_3 = VLC$$

$$E_V^1(x_1, x_3) = Low((s_3, s_3, s_4), (0.3, 0.5, 0.2)) = C\{(0.3, s_3), (0.7Low((s_4, s_3), (2/7, 5/7))\}$$
$$= C\{(0.3, s_3), (0.7, s_3)\} = s_3 = VLC,$$

since

$$Low((s_4, s_3), (2/7, 5/7)) = s_{k'}, \quad k' = 3 + round((2/7)(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_{k'} = s_3$$

$$E_V^1(x_1, x_4) = Low((s_3, s_4, s_4), (0.2, 0.5, 0.3)) = C\{(0.5, s_4), (0.5Low((s_4, s_3), (3/5, 2/5))\}$$
$$= C\{(0.5, s_4), (0.5, s_4)\} = s_4 = SC,$$

since

$$Low((s_4, s_3), (3/5, 2/5)) = s_{k'}, \quad k' = 3 + round((3/5).(4-3)) = 3 + 1 = 4 \quad \Rightarrow s_{k'} = s_4$$

$$E_V^1(x_2, x_1) = Low((s_2, s_2, s_2), (0.3, 0.5, 0.2)) = s_2 = EU$$

$$E_V^1(x_2, x_2) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C\{(0.2, s_4), (0.8, Low((s_4), (1))\}$$
$$= C\{(0.2, s_4), (0.8, s_4)\} = s_4 = SC.$$

$$E_V^1(x_2, x_3) = Low((s_3, s_3, s_4), (0.2, 0.5, 0.3)) = C\{(0.3, s_4), (0.7Low((s_3), (1))\}$$
$$= C\{(0.3, s_4), (0.7, s_3)\} = s_{k(2,3)},$$
$$k(2,3) = 3 + round((0.3).(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(2,3)} = s_3 = VLC.$$

$$E_V^1(x_2, x_4) = Low((s_2, s_2, s_2), (0.3, 0.5, 0.2)) = s_2 = EU$$

$$E_V^1(x_3, x_1) = Low((s_5, s_3, s_2), (0.2, 0.5, 0.3)) = C\{(0.2, s_5), (0.8Low((s_3, s_2), (5/8, 3/8))\}$$
$$= C\{(0.2, s_4), (0.8, s_3)\} = s_{k(3,1)},$$

since

$$Low((s_3,s_2),(5/8,3/8)) = s_{k'}, \quad k' = 2 + round((5/8).(3-2)) = 2+1 = 3 \quad \Rightarrow s_{k'} = s_3$$

..

$$k(3,1) = 3 + round((0.2).(5-3)) = 3+0 = 3 \quad \Rightarrow s_{k(3,1)} = s_3 = VLC.$$

$$E_V^1(x_3,x_2) = Low((s_4,s_6,s_4),(0.2,0.5,0.3)) = C\{(0.5,s_6),(0.5Low((s_4),(1))\}$$
$$= C\{(0.5,s_6),(0.5,s_4)\} = s_{k(3,2)},$$
$$k(3,2) = 4 + round((0.5).(6-4)) = 4+1 = 5 \quad \Rightarrow s_{k(3,2)} = s_5 = IM.$$

$$E_V^1(x_3,x_3) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_4),(1))\}$$
$$= C\{(0.2,s_4),(0.8,s_4)\} = s_4 = SC.$$

$$E_V^1(x_3,x_4) = Low((s_4,s_3,s_1),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8Low((s_3,s_1),(5/8,3/8))\}$$
$$= C\{(0.2,s_4),(0.8,s_2)\} = s_{k(3,4)},$$

since

$$Low((s_3,s_1),(5/8,3/8)) = s_{k'}, \quad k' = 1 + round((5/8).(3-1)) = 1+1 = 2 \quad \Rightarrow s_{k'} = s_2$$
$$k(3,4) = 2 + round((0.2).(4-2)) = 2+0 = 2 \quad \Rightarrow s_{k(3,4)} = s_2 = EU.$$

$$E_V^1(x_4,x_1) = Low((s_2,s_2,s_2),(0.3,0.5,0.2)) = s_2 = EU.$$

$$E_V^1(x_4,x_2) = Low((s_4,s_2,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8Low((s_4,s_2),(3/8,5/8))\}$$
$$= C\{(0.2,s_4),(0.8,s_3)\} = s_{k(4,2)},$$

since

$$Low((s_4,s_2),(3/8,5/8)) = s_{k'}, \quad k' = 2 + round((3/8).(4-2)) = 2+1 = 3 \quad \Rightarrow s_{k'} = s_3$$
$$k(4,2) = 3 + round((0.2).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4,2)} = s_3 = VLC.$$

$$E_V^1(x_4,x_3) = Low((s_3,s_3,s_1),(0.2,0.5,0.3)) = C\{(0.7,s_3),(0.3,s_1)\} = s_{k(4,3)},$$
$$k(4,3) = 1 + round((0.7).(3-1)) = 1+1 = 2 \quad \Rightarrow s_{k(4,3)} = s_2 = EU.$$

$$E_V^1(x_4,x_4) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_4),(1))\}$$
$$= C\{(0.2,s_4),(0.8,s_4)\} = s_4 = SC.$$

We obtain the linguistic dominance degrees using non-preference relations for criterion C_1.

$$E_V^1 = \begin{bmatrix} SC & VLC & VLC & SC \\ EU & SC & VLC & EU \\ VLC & IM & SC & EU \\ EU & VLC & EU & SC \end{bmatrix} \tag{84}$$

Calculate the FCS as in (54).

$$fcs_V^1(x_1) = Low((s_4, s_3), (1/3, 2/3)) = C\{(1/3, s_4), (2/3, s_3)\} = s_k \ ,$$

$$k = 3 + round((1/3).(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_k = s_3 = VLC \ .$$

$$fcs_V^1(x_2) = Low((s_3, s_2), (1/3, 2/3)) = C\{(1/3, s_3), (2/3), s_2)\} = s_k \ ,$$

$$k = 2 + round((1/3).(3-2)) = 2 + 0 = 2 \quad \Rightarrow s_k = s_2 = EU \ .$$

$$fcs_V^1(x_3) = Low((s_3, s_5, s_2), (1/3, 1/3, 1/3)) = C\{(1/3, s_5), (2/3, Low((s_3, s_2), (1/2, 1/2))\}$$

$$= C\{(1/3, s_5), (2/3, s_3)\} = s_k ,$$

since

$$Low((s_3, s_2), (0.5, 0.5)) = s_{k'}, k' = 2 + round((0.5).(3-2)) = 2 + 1 = 3 \Rightarrow s_{k'} = s_3$$

$$k = 3 + round((1/3).(5-3)) = 3 + 1 = 4 \quad \Rightarrow s_k = s_4 = SC \ .$$

$$fcs_V^1(x_4) = Low((s_3, s_2), (1/3, 2/3)) = C\{(1/3, s_3), (2/3, s_2)\} = s_k ,$$

$$k = 2 + round((1/3).(3-2)) = 2 + 0 = 2 \quad \Rightarrow s_k = s_2 = EU \ .$$

We obtain FCS using non-preference relations for the criterion C_1.

$$FCS_V^1 = \{VLC/x_1, EU/x_2, SC/x_3, EU/x_4\} \ . \tag{85}$$

Finally, we obtain IFE for the criterion C_1.

$$IFE^1 = \{(IM, VLC)/x_1, (SC, EU)/x_2, (SC, SC)/x_3, (ML, EU)/x_4\} , \tag{86}$$

and Fuzzy Pareto Solution (FPS) for criterion C_1 is x_4.

Step 1.2. Computing for criterion C_2.

1.2.a. Use the linguistic preference relations

$$M_{12} = \begin{bmatrix} IM & MC & ML & IM \\ VLC & IM & SC & SC \\ VLC & SC & IM & EU \\ SC & IM & MC & IM \end{bmatrix}, M_{22} = \begin{bmatrix} IM & ML & MC & MC \\ SC & IM & IM & SC \\ VLC & SC & IM & SC \\ SC & MC & MC & IM \end{bmatrix}, M_{32} = \begin{bmatrix} IM & IM & MC & MC \\ SC & IM & MC & IM \\ VLC & SC & IM & VLC \\ SC & SC & IM & IM \end{bmatrix}$$

Calculate the linguistic dominance degrees:

$$E_M^2(x_i, x_j) = Low(S, U_{M2}) \quad , \qquad \text{i ,j =1,2,3,4} \tag{87}$$

$$U_{M2} = \left[u_{2T}, ..., u_{21}\right], \quad \text{with} \qquad u_{2t} = W_{ij}(s_t) = \sum_k \left\{w(k) : M_{k2}(i, j) = s_t\right\},$$

t=1,...,T. (88)

$$E_M^2(x_1, x_1) = Low((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C\left\{(0.2, s_5), (0.8, Low((s_5, 1)\right\} = s_5 = IM.$$

$$E_M^2(x_1, x_2) = Low((s_7, s_6, s_5), (0.5, 0.2, 0.3)) = C\left\{(0.5, s_7), (0.5, Low((s_6, s_5), (2/5, 3/5))\right\}$$
$$= C\left\{(0.5, s_7), (0.5, s_5)\right\} = s_{k(1,2)},$$

since

$$Low((s_6, s_5), (2/5, 3/5)) = s_{k'}, k' = 5 + round((2/5).(6-5)) = 5 + 0 = 5 \Rightarrow s_{k'} = s_5$$

.

$$k(1,2) = 5 + round((0.5).(7-5)) = 5 + 1 = 6 \quad \Rightarrow s_{k(1,2)} = s_6 = MC \, .$$

$$E_M^2(x_1, x_3) = Low((s_7, s_6, s_6), (0.2, 0.5, 0.3)) = C\left\{(0.2, s_7), (0.8, Low((s_6), (1))\right\}$$
$$= C\left\{(0.2, s_7), (0.8, s_6)\right\} = s_{k(1,3)},$$

$$k(1,3) = 6 + round((0.2).(7-6)) = 6 + 0 = 6 \quad \Rightarrow s_{k(1,3)} = s_6 = MC \, .$$

$$E_M^2(x_1, x_4) = Low((s_5, s_6, s_6), (0.2, 0.5, 0.2)) = C\left\{(0.8, s_6), (0.2, s_5)\right\} = s_{k(1,4)}$$

$$k(1,4) = 5 + round((0.8).(6-5)) = 5 + 1 = 6 \Rightarrow s_{k(1,4)} = s_6 = MC \, .$$

$$E_M^2(x_2, x_1) = Low((s_3, s_4, s_4), (0.2, 0.5, 0.3)) = C\left\{(0.8, s_4), (0.2, s_3)\right\} = s_{k(2,1)},$$

$$k(2,1) = 3 + round((0.8).(4-3)) = 3 + 1 = 4 \Rightarrow s_{k(2,1)} = s_4 = SC$$

$$E_M^2(x_2, x_2) = Low((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C\left\{(0.2, s_5), (0.8, Low((s_5, 1)\right\} = s_5 = IM.$$

$$E_M^2(x_2, x_3) = Low((s_4, s_5, s_6), (0.2, 0.5, 0.3)) = C\left\{(0.3, s_6), (0.7, Low((s_5, s_4), (5/7, 2/7))\right\}$$
$$= C\left\{(0.3, s_6), (0.7, s_5)\right\} = s_{k(2,3)},$$

since

$$Low((s_5, s_4), (5/7, 2/7)) = s_{k'}, k' = 4 + round((5/7).(5-4)) = 4 + 1 = 5$$

$$k(2,3) = 5 + round((0.3).(6-5)) = 5 + 0 = 5 \quad \Rightarrow s_{k(2,3)} = s_5 = IM \, .$$

$$E_M^2(x_2,x_4) = Low((s_4,s_4,s_5),(0.2,0.5,0.3)) = C\left\{(0.3,s_5),(0.7,Low((s_4),(1))\right\}$$
$$= C\left\{(0.3,s_5),(0.7,s_4)\right\} = s_{k(2,4)}.$$

$$k(2,4) = 4 + round((0.3).(5-4)) = 4+0 = 4 \quad \Rightarrow s_{k(2,3)} = s_4 = SC.$$

$$E_M^2(x_3,x_1) = Low((s_3,s_3,s_3),(0.2,0.5,0.3)) = C\left\{(s_3),(1)\right\} = s_3 = VLC.$$
$$E_M^2(x_3,x_2) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\left\{(s_4),(1)\right\} = s_4 = SC.$$
$$E_M^2(x_3,x_3) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\left\{(0.2,s_5),(0.8,Low((s_5,1)\right\} = s_5 = IM.$$

$$E_M^2(x_3,x_4) = Low((s_2,s_4,s_3),(0.2,0.5,0.3)) = C\left\{(0.5,s_4),(0.5)Low((s_3,s_2)(3/5,2/5))\right\}$$
$$= C\left\{(0.5,s_4),(0.5,s_3)\right\} = s_{k(3,4)}.$$

since

$$Low((s_3,s_2),(3/5,2/5)) = s_{k'},\ k' = 2 + round((3/5).(3-2)) = 2+1 = 3$$
$$k(3,4) = 3 + round((0.5).(4-3)) = 3+1 = 4 \quad \Rightarrow s_{k(3,4)} = s_4 = SC.$$
$$E_M^2(x_4,x_1) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\left\{(s_4),(1)\right\} = s_4 = SC.$$
$$E_M^2(x_4,x_2) = Low((s_5,s_6,s_4),(0.2,0.5,0.3)) = C\left\{(0.5,s_6),(0.5)Low((s_5,s_4)(2/5,3/5))\right\}$$
$$= C\left\{(0.5,s_6),(0.5,s_4)\right\} = s_{k(4,2)}.$$

since

$$Low((s_5,s_4),(2/5,3/5)) = s_{k'},\ k' = 4 + round((2/5).(5-4)) = 4+0 = 4$$
$$k(4,2) = 4 + round((0.5).(6-4)) = 1+1 = 5 \quad \Rightarrow s_{k(4,2)} = s_5 = IM.$$
$$E_M^2(x_4,x_3) = Low((s_6,s_6,s_5),(0.2,0.5,0.3)) = C\left\{(0.7,s_6),(0.3,s_5)\right\} = s_6 = MC.$$

$$E_M^2(x_4,x_4) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\left\{(0.2,s_5),(0.8,Low((s_5,1)\right\} = s_5 = IM.$$

We obtain the linguistic dominance degrees for criterion C_2 :

$$E_M^2 = \begin{bmatrix} IM & MC & MC & MC \\ SC & IM & IM & SC \\ VLC & SC & IM & SC \\ SC & SC & MC & IM \end{bmatrix}. \tag{89}$$

Calculate the FCS

$$fcs_M^2(x_1) = Low((s_6),(1)) = s_6 = MC.$$

$$fcs_M^2(x_2) = Low((s_5, s_4),(1/3, 2/3)) = C((1/3, s_5),(2/3, s_4)\} = s_k,$$

$$k = 4 + round((1/3).(5-4)) = 4 + 0 = 4 \quad \Rightarrow s_k = s_4 = SC$$

$$fcs_M^2(x_3) = Low((s_4, s_3),(2/3,1/3)) = C\{(2/3, s_4),(1/3, s_3)\} = s_k,$$

$$k = 3 + round((2/3).(4-3)) = 3 + 1 = 4 \quad \Rightarrow s_k = s_4 = SC$$

$$fcs_M^2(x_4) = Low((s_6, s_4),(1/3, 2/3)) = C\{(1/3, s_6),(2/3, s_4)\} = s_k,$$

$$k = 4 + round((1/3).(6-4)) = 4 + 1 = 5 \quad \Rightarrow s_k = s_5 = IM$$

and the FCS $FCS_M^2 = \{MC / x_1, SC / x_2, SC / x_3, IM / x_4\}$. (90)

1.2.b. Use the linguistic non-preference relations

$$V_{12} = \begin{bmatrix} SC & VLC & SC & EU \\ SC & SC & IM & VLC \\ MC & SC & SC & MC \\ MC & MC & VLC & SC \end{bmatrix}, \quad V_{22} = \begin{bmatrix} SC & EU & VLC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & VLC \\ SC & VLC & VLC & SC \end{bmatrix}, \quad V_{32} = \begin{bmatrix} SC & SC & VLC & EU \\ IM & SC & VLC & SC \\ SC & SC & SC & SC \\ MC & SC & SC & SC \end{bmatrix}$$

Calculate the linguistic non-dominance degrees:

$E_V^2(x_i, x_j) = Low(S, U_{V2})$, for i,j =1,…,4 (91)

$U_{V2} = [u_{2T}, ..., u_{21}]$,with $u_{2t} = W_{ij}(s_t) = \sum_k \{w(k) : V_{k2}(i, j) = s_t\}$,for each t=1,...,T. (92)

$$E_V^2(x_1, x_1) = Low((s_4, s_4, s_4),(0.2, 0.5, 0.3)) = C\{(0.2, s_4),(0.8, Low((s_4),(1))\}$$
$$= C\{(0.2, s_4),(0.8, s_4)\} = s_4 = SC.$$

$$E_V^2(x_1, x_2) = Low((s_3, s_2, s_4),(0.2, 0.5, 0.3)) = C\{(0.3, s_4),(0.7, Low((s_3, s_2),(2/7, 5/7)\}$$
$$= C\{(0.3, s_4),(0.7, s_2)\} = s_{k(2,1)},$$

since

$$Low((s_3, s_2),(2/7, 5/7)) = s_{k'}, \; k' = 2 + round((2/7).(3-2)) = 2 + 0 = 2,$$

$$k(1,2) = 2 + round((0.3).(4-2)) = 2 + 1 = 3 \quad \Rightarrow s_{k(2,1)} = s_3 = VLC.$$

$$E_V^2(x_1,x_3)=Low((s_4,s_3,s_3),(0.2,0.5,0.3))=C\{(0.3,s_4),(0.7)Low((s_3)(1))\}$$
$$=C\{(0.3,s_4),(0.7,s_3)\}=s_{k(1,3)}.$$

$$k(1,3)=3+round((0.3).(4-3))=3+0=3 \quad \Rightarrow s_{k(1,3)}=s_3=VLC.$$

$$E_V^2(x_1,x_4)=Low((s_2,s_3,s_2),(0.2,0.5,0.3))=C\{(0.5,s_3),(0.5)Low((s_2)(1))\}$$
$$=C\{(0.5,s_3),(0.5,s_2)\}=s_{k(1,4)}.$$

$$k(1,4)=2+round((0.5).(3-2))=2+1=3 \quad \Rightarrow s_{k(1,4)}=s_3=VLC.$$

$$E_V^2(x_2,x_1)=Low((s_4,s_5,s_5),(0.2,0.5,0.3))=C\{(0.5,s_5),(0.5)Low((s_5,s_4)(3/5,2/5))\}$$
$$=C\{(0.5,s_5),(0.5,s_5)\}=s_5=IM,$$

since

$$Low((s_5,s_4),(3/5,2/5))=s_{k'},\ k'=4+round((3/5).(5-4))=4+1=5.$$

$$E_V^2(x_2,x_2)=Low((s_4,s_4,s_4),(0.2,0.5,0.3))=C\{(0.2,s_4),(0.8,Low((s_4),(1))\}$$
$$=C\{(0.2,s_4),(0.8,s_4)\}=s_4=SC.$$

$$E_V^2(x_2,x_3)=Low((s_5,s_3,s_3),(0.2,0.5,0.3))=C\{(0.2,s_5),(0.8)Low((s_3)(1))\}$$
$$=C\{(0.2,s_5),(0.8,s_3)\}=s_{k(2,3)},$$

$$k(2,3)=3+round((0.2).(5-3))=3+0=3 \quad \Rightarrow s_{k(2,3)}=s_3=VLC.$$

$$E_V^2(x_2,x_4)=Low((s_3,s_4,s_4),(0.2,0.5,0.3))=C\{(0.5,s_4),(0.5)Low((s_4,s_3),(3/5,2/5))\}$$
$$=C\{(0.5,s_4),(0.5,s_4)\}=s_4=SC,$$

since

$$Low((s_4,s_3),(3/5,2/5))=s_{k'},\ k'=3+round((3/5).(4-3))=3+1=4.$$

$$E_V^2(x_3,x_1)=Low((s_6,s_5,s_4),(0.2,0.5,0.3))=C\{(0.2,s_6),(0.8)Low((s_5,s_4)(5/8,3/8))\}$$
$$=C\{(0.2,s_6),(0.8,s_5)\}=s_{k(3,1)}.$$

Since

$Low((s_5,s_4),(5/8,3/8)) = s_{k'},\ \ k' = 4 + round((5/8).(5-4)) = 4+1 = 5,$

$k(3,1) = 5 + round((0.2).(6-5)) = 5+0 = 5 \quad \Rightarrow s_{k(3,1)} = s_5 = IM.$

$E_V^2(x_3,x_2) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_4),(1))\}$

$= C\{(0.2,s_4),(0.8,s_4)\} = s_4 = SC.$

$E_V^2(x_3,x_3) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_4),(1))\}$

$= C\{(0.2,s_4),(0.8,s_4)\} = s_4 = SC.$

$E_V^2(x_3,x_4) = Low((s_6,s_3,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_6),(0.8)Low((s_4,s_3)(3/8,5/8))\}$

$= C\{(0.2,s_6),(0.8,s_3)\} = s_{k(3,4)}.$

since

$Low((s_4,s_3),(3/8,5/8)) = s_{k'},\ \ k' = 3 + round((3/8).(4-3)) = 3+0 = 3,$

$E_V^2(x_4,x_1) = Low((s_4,s_4,s_6),(0.2,0.5,0.3)) = C\{(0.3,s_6),(0.7)Low((s_4)(1))\}$

$= C\{(0.3,s_6),(0.7,s_4)\} = s_{k(4,1)}.$

$k(4,1) = 4 + round((0.3).(6-4)) = 4+1 = 5 \quad \Rightarrow s_{k(4,1)} = s_5 = IM.$

$E_V^2(x_4,x_2) = Low((s_4,s_3,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8)Low((s_4,s_3)(3/8,5/8))\}$

$= C\{(0.2,s_4),(0.8,s_3)\} = s_{k(4,2)}.$

since

$Low((s_4,s_3),(3/8,5/8)) = s_{k'},\ \ k' = 3 + round((3/8).(5-4)) = 3+0 = 3,$

$k(4,2) = 3 + round((0.2).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4,2)} = s_3 = VLC.$

$E_V^2(x_4,x_3) = Low((s_3,s_3,s_4),(0.2,0.5,0.3)) = C\{(0.3,s_4),(0.7)Low((s_3)(1))\}$

$= C\{(0.3,s_4),(0.7,s_3)\} = s_{k(4,3)}.$

$k(4,3) = 3 + round((0.3).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4,3)} = s_3 = VLC.$

$$E_V^2(x_4,x_4) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_4),(1))\}$$
$$= C\{(0.2,s_4),(0.8,s_4)\} = s_4 = SC.$$

We obtain the linguistic non-dominance degrees using $\{V_{12}, V_{22}, V_{32}\}$ for criterion C_2:

$$E_V^2 = \begin{bmatrix} SC & VLC & VLC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \\ IM & VLC & VLC & SC \end{bmatrix}. \qquad (93)$$

Calculate FCS

$$fcs_V^2(x_1) = Low((s_3),(1)) = s_3 = VLC$$

$$fcs_V^2(x_2) = Low((s_5,s_4,s_3),(1/3,2/3)) = C\{(1/3,s_5),(2/3,Low((s_4,s_3),(0.5,0.5))\}$$
$$= C\{(1/3,s_5),(2/3,s_4)\} = s_k,$$

since

$$Low((s_4,s_3),(0.5,0.5)) = s_{k'}, k' = 3 + round((0.5).(4-3)) = 3+1 = 4 \Rightarrow s_{k'} = s_4$$

..

$$k = 4 + round((1/3).(5-4)) = 4+0 = 4 \quad \Rightarrow s_k = s_4 = SC.$$

$$fcs_V^2(x_3) = Low((s_5,s_4),(1/3,2/3)) = C\{(1/3,s_5),(2/3),s_4)\} = s_{k,}$$

$$k = 4 + round((1/3).(5-4)) = 4+0 = 4 \quad \Rightarrow s_k = s_4 = SC.$$

$$fcs_V^2(x_4) = Low((s_5,s_4,s_3),(1/3,2/3)) = C\{(1/3,s_5),(2/3,Low((s_4,s_3),(0.5,0.5))\}$$
$$= C\{(1/3,s_5),(2/3,s_4)\} = s_k,$$

since

$$Low((s_4,s_3),(0.5,0.5)) = s_{k'}, k' = 3 + round((0.5).(4-3)) = 3+1 = 4 \Rightarrow s_{k'} = s_4$$

$$k = 4 + round((1/3).(5-4)) = 4+0 = 4 \quad \Rightarrow s_k = s_4 = SC.$$

We obtain FCS using non-preference relations for criterion C_2

$$FCS_V^2 = \{VLC/x_1, SC/x_2, SC/x_3, SC/x_4\}. \qquad (94)$$

Finally, we obtain IFE for criterion C_2

$$IFE^2 = \{(MC, VLC)/x_1, (SC, SC)/x_2, (SC, SC)/x_3, (IM, SC)/x_4\}. \quad (95)$$

and FPS for criterion C_2 is x_1 .

Step 1.3. Computing for criterion C_3.

1.3.a. Use the linguistic preference relations

$$M_{13} = \begin{bmatrix} IM & MC & ML & IM \\ SC & IM & MC & SC \\ VLC & SC & IM & IM \\ SC & MC & SC & IM \end{bmatrix}, \quad M_{23} = \begin{bmatrix} IM & ML & IM & ML \\ VLC & IM & IM & IM \\ SC & MC & SC & SC \\ VLC & SC & SC & IM \end{bmatrix}, \quad M_{33} = \begin{bmatrix} IM & ML & SC & MC \\ SC & IM & MC & IM \\ IM & SC & IM & SC \\ VLC & SC & IM & IM \end{bmatrix}$$

Calculate the linguistic dominance degrees

$$E_M^3(x_i, x_j) = Low(S, U_{M3}) \;, \quad \text{i, j =1,2,3,4} \quad (96)$$

$$U_{M3} = [u_{3T}, ..., u_{31}], \quad \text{with} \quad u_{3t} = W_{ij}(s_t) = \sum_k \{w(k) : M_{k3}(i, j) = s_t\},$$

t=1,...,T. (97)

$$E_M^3(x_1, x_1) = Low((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C\{(0.2, s_5), (0.8, Low((s_5, 1)\} = s_5 = IM.$$

$$E_M^3(x_1, x_2) = Low((s_6, s_7, s_7), (0.2, 0.5, 0.3)) = C\{(0.5, s_7), (0.5, Low((s_7, s_6), (3/5, 2/5))\}$$
$$= C\{(0.5, s_7), (0.5, s_7)\} = s_7 = ML,$$

since

$$Low((s_7, s_6), (3/5, 2/5)) = s_{k'}, k' = 6 + round((3/5).(7-6)) = 6 + 1 = 7 \Rightarrow s_{k'} = s_7$$

$$E_M^3(x_1, x_3) = Low((s_7, s_5, s_4), (0.2, 0.5, 0.3)) = C\{(0.2, s_7), (0.8, Low((s_5, s_4)(5/8, 3/8))\}$$
$$= C\{(0.2, s_7), (0.8, s_5)\} = s_{k(1,3)},$$

since

$$Low((s_5, s_4), (5/8, 3/5)) = s_{k'}, k' = 4 + round((5/8).(5-4)) = 4 + 1 = 5 \Rightarrow s_{k'} = s_5,$$

$$k(1,3) = 5 + round((0.2).(7-5)) = 5 + 0 = 5 \quad \Rightarrow s_{k(1,3)} = s_5 = IM \; .$$

$$E_M^3(x_1,x_4) = Low((s_5,s_7,s_7),(0.2,0.5,0.3)) = C\left\{(0.5,s_7),(0.5,Low((s_7,s_5),(3/5,2/5))\right\}$$
$$= C\left\{(0.5,s_7),(0.5,s_6)\right\} = s_{k(1,4)},$$

since

$$Low((s_7,s_5),(3/5,2/5)) = s_{k'}, k' = 5 + round((3/5).(7-5)) = 5+1 = 6 \Rightarrow s_{k'} = s_6,$$

$$k(1,4) = 6 + round((0.5).(7-6)) = 6+1 = 7 \quad \Rightarrow s_{k(1,4)} = s_7 = ML.$$

$$E_M^3(x_2,x_1) = Low((s_4,s_3,s_4),(0.2,0.5,0.3)) = C\left\{(0.2,s_4),(0.8,Low((s_4,s_3)(3/8,5/8))\right\}$$
$$= C\left\{(0.2,s_4),(0.8,s_3)\right\} = s_{k(2,1)},$$

since

$$Low((s_4,s_3),(3/8,5/8)) = s_{k'}, k' = 3 + round((3/8).(4-3)) = 3+0 = 3 \Rightarrow s_{k'} = s_3,$$

$$k(2,1) = 3 + round((0.2).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(2,1)} = s_3 = VLC.$$

$$E_M^3(x_2,x_2) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\left\{(0.2,s_5),(0.8,Low((s_5,1)\right\} = s_5 = IM.$$

$$E_M^3(x_2,x_3) = Low((s_6,s_5,s_6),(0.2,0.5,0.3)) = C\left\{(0.2,s_6),(0.8,Low((s_6,s_5),(3/8,5/8))\right\}$$
$$= C\left\{(0.2,s_6),(0.8,s_5)\right\} = s_{k(2,3)},$$

since

$$Low((s_6,s_5),(3/8,5/8)) = s_{k'}, k' = 5 + round((3/8).(6-5)) = 5+0 = 5 \Rightarrow s_{k'} = s_5$$

$$k(2,3) = 5 + round((0.2).(6-5)) = 5+0 = 5 \quad \Rightarrow s_{k(2,3)} = s_5 = IM.$$

$$E_M^3(x_2,x_4) = Low((s_4,s_5,s_5),(0.2,0.5,0.3)) = C\left\{(0.5,s_5),(0.5,Low((s_5,s_4)(3/5,2/5))\right\}$$
$$= C\left\{(0.5,s_5),(0.5,s_5)\right\} = s_5 = IM,$$

since

$$Low((s_5,s_4),(3/5,2/5)) = s_{k'}, k' = 4 + round((3/5).(5-4)) = 4+1 = 5 \Rightarrow s_{k'} = s_5.$$

$$E_M^3(x_3,x_1) = Low((s_3,s_4,s_5),(0.2,0.5,0.3)) = C\{(0.3,s_5),(0.7,Low((s_4,s_3)(5/7,2/7))\}$$
$$= C\{(0.3,s_5),(0.7,s_4)\} = s_{k(3,1)},$$

since

$$Low((s_4,s_3),(5/7,2/7)) = s_{k'}, k' = 3 + round((5/7).(4-3)) = 3+1 = 4 \Rightarrow s_{k'} = s_4,$$

$$k(3,1) = 4 + round((0.3).(5-4)) = 4+0 = 4 \quad \Rightarrow s_{k(3,1)} = s_4 = SC.$$
$$E_M^3(x_3,x_2) = Low((s_4,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_4,1)\} = s_4 = SC.$$

$$E_M^3(x_3,x_3) = Low((s_5,s_5,s_5),(0.2,0.5,0.3)) = C\{(0.2,s_5),(0.8,Low((s_5,1)\} = s_5 = IM.$$

$$E_M^2(x_3,x_4) = Low((s_5,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_5),(0.8,Low((s_4)(1))\}$$
$$= C\{(0.2,s_5),(0.8,s_4)\} = s_{k(3,4)},$$

$$k(3,4) = 4 + round((0.2).(5-4)) = 4+0 = 4 \quad \Rightarrow s_{k(3,4)} = s_4 = SC.$$
$$E_M^3(x_4,x_1) = Low((s_4,s_3,s_3),(0.2,0.5,0.3)) = C\{(0.2,s_4),(0.8,Low((s_3)(1))\}$$
$$= C\{(0.2,s_5),(0.8,s_3)\} = s_{k(4,1)},$$

$$k(4,1) = 3 + round((0.2).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4,1)} = s_3 = VLC.$$
$$E_M^3(x_4,x_2) = Low((s_6,s_4,s_4),(0.2,0.5,0.3)) = C\{(0.2,s_6),(0.8,Low((s_4)(1))\}$$
$$= C\{(0.2,s_6),(0.8,s_4)\} = s_{k(4,2)},$$

$$k(4,2) = 4 + round((0.2).(6-4)) = 4+0 = 4 \quad \Rightarrow s_{k(4,2)} = s_4 = SC.$$
$$E_M^3(x_4,x_3) = Low((s_4,s_4,s_5),(0.2,0.5,0.3)) = C\{(0.3,s_5),(0.7,Low((s_4)(1))\}$$
$$= C\{(0.3,s_5),(0.7,s_4)\} = s_{k(4,3)},$$

$$k(4,3) = 4 + round((0.3).(5-4)) = 4 + 0 = 4 \quad \Rightarrow s_{k(4,3)} = s_4 = SC.$$

$$E_M^3(x_4, x_4) = Low((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C\{(0.2, s_5), (0.8, Low((s_5, 1)\} = s_5 = IM.$$

We obtain the linguistic dominance degrees

$$E_M^3 = \begin{bmatrix} IM & ML & IM & ML \\ VLC & IM & IM & IM \\ SC & SC & IM & SC \\ VLC & IM & SC & IM \end{bmatrix}. \quad (98)$$

Calculate FCS by Definition 2.

$$fcs_M^3(x_1) = Low((s_7, s_5), (2/3, 1/3)) = C\{(2/3, s_7), (1/3, s_5)\} = s_k$$

$$k = 5 + round((2/3).(7-5)) = 5 + 1 = 6 \quad \Rightarrow s_k = s_6 = MC.$$

$$fcs_M^3(x_2) = Low((s_5, s_3), (2/3, 1/3)) = C\{(2/3, s_5), (1/3, s_3)\} = s_k,$$

$$k = 3 + round((2/3).(5-2)) = 3 + 1 = 4 \quad \Rightarrow s_k = s_4 = SC.$$

$$fcs_M^3(x_3) = Low((s_4), (1)) = s_4 = SC.$$

$$fcs_M^3(x_4) = Low((s_5, s_4, s_3), (1/3, 1/3, 1/3)) = C\{(1/3, s_5), (2/3, Low((s_4, s_3), (0.5, 0.5))\} =$$

$$= C\{(1/3, s_5), (2/3, s_4)\} = s_k,$$

since

$$Low((s_4, s_3), (0.5, 0.5)) = s_{k'}, k' = 3 + round((0.5).(4-3)) = 3 + 1 = 4 \Rightarrow s_{k'} = s_4,$$

$$k = 4 + round((1/3).(5-4)) = 4 + 0 = 4 \quad \Rightarrow s_k = s_4 = SC.$$

We obtain $$FCS_M^3 = \{MC/x_1, SC/x_2, SC/x_3, SC/x_4\}. \quad (99)$$

1.3.b. Use the linguistic non-preference relations

$$V_{13} = \begin{bmatrix} SC & SC & VLC & SC \\ SC & SC & SC & SC \\ IM & VLC & SC & SC \\ IM & SC & SC & SC \end{bmatrix}, \quad V_{23} = \begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & VLC & VLC \\ IM & IM & SC & SC \\ MC & VLC & VLC & SC \end{bmatrix}, \quad V_{33} = \begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & SC & VLC \\ SC & SC & SC & IM \\ SC & SC & SC & SC \end{bmatrix}$$

Calculate the linguistic non-dominance degrees:

$$E_V^3(x_i, x_j) = Low(S, U_{V3}), \text{ i,j } =1,..,4. \quad (100)$$

$U_{V3} = [u_{3T}, ..., u_{31}]$, with $u_{3t} = W_{ij}(s_t) = \sum_k \{w(k) : V_{k3}(i, j) = s_t\}$,

t=1,..., T. (101)

$$E_V^3(x_1, x_1) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C\{(0.2, s_4), (0.8, Low((s_4), (1))\}$$
$$= C\{(0.2, s_4), (0.8, s_4)\} = s_4 = SC.$$

$$E_V^3(x_1, x_2) = Low((s_4, s_3, s_3), (0.2, 0.5, 0.3)) = C\{(0.2, s_4), (0.8)Low((s_3)(1))\}$$
$$= C\{(0.2, s_4), (0.8, s_3)\} = s_{k(1,2)}.$$

$$k(1,2) = 3 + round((0.2).(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(2,1)} = s_3 = VLC.$$

$$E_V^3(x_1, x_3) = Low((s_3, s_4, s_4), (0.2, 0.5, 0.3)) = C\{(0.5, s_4), (0.5)Low((s_4, s_3)(3/5, 2/5))\}$$
$$= C\{(0.5, s_4), (0.5, s_4)\} = s_4 = SC,$$

since

$$Low((s_4, s_3), (3/5, 2/5)) = s_{k'}, k' = 3 + round((3/5).(4-3)) = 3 + 1 = 4 \Rightarrow s_{k'} = s_4.$$

$$E_V^3(x_1, x_4) = Low((s_4, s_3, s_3), (0.2, 0.5, 0.3)) = C\{(0.2, s_4), (0.8)Low((s_3)(1))\}$$
$$= C\{(0.2, s_4), (0.8, s_3)\} = s_{k(1,4)}.$$

$$k(1,4) = 3 + round((0.2).(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(1,4)} = s_3 = VLC.$$

$$E_V^3(x_2, x_1) = Low((s_4, s_5, s_5), (0.2, 0.5, 0.3)) = C\{(0.5, s_5), (0.5)Low((s_5, s_4)(3/5, 2/5))\}$$
$$= C\{(0.5, s_5), (0.5, s_5)\} = s_5 = IM,$$

since

$$Low((s_5, s_4), (3/5, 2/5)) = s_{k'}, k' = 4 + round((3/5).(5-4)) = 4 + 1 = 5.$$

$$E_V^3(x_2, x_2) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C\{(0.2, s_4), (0.8, Low((s_4), (1))\}$$
$$= C\{(0.2, s_4), (0.8, s_4)\} = s_4 = SC.$$

$$E_V^3(x_2,x_3)=Low((s_4,s_3,s_4),(0.2,0.5,0.3))=C\left\{(0.5,s_4),(0.5,s_3)\right\}=s_{k(2,3)},$$
$$k(2,3)=3+round((0.5).(4-3))=3+1=4,\quad \Rightarrow s_{k(2,3)}=s_4=SC.$$
$$E_V^3(x_2,x_4)=Low((s_4,s_3,s_3),(0.2,0.5,0.3))=C\left\{(0.2,s_4),(0.8)Low((s_3)(1))\right\}$$
$$=C\left\{(0.2,s_4),(0.8,s_3)\right\}=s_{k(2,4)}.$$

$$k(2,4)=3+round((0.2).(4-3))=3+0=3\quad \Rightarrow s_{k(2,4)}=s_3=VLC.$$
$$E_V^3(x_3,x_1)=Low((s_5,s_5,s_4),(0.2,0.5,0.3))=C\left\{(0.2,s_5),(0.8)Low((s_5,s_4)(5/8,3/8))\right\}$$
$$=C\left\{(0.2,s_5),(0.8,s_5)\right\}=s_{k(3,1)}.$$

since
$$Low((s_5,s_4),(5/8,3/8))=s_{k'},\ k'=4+round((5/8).(5-4))=4+1=5,$$
$$k(3,1)=5+round((0.2).(5-5))=5+0=5\quad \Rightarrow s_{k(3,1)}=s_5=IM.$$
$$E_V^3(x_3,x_2)=Low((s_3,s_5,s_4),(0.2,0.5,0.3))=C\left\{(0.2,s_3),(0.8,Low((s_5,s_4),(5/8,3/8))\right\}$$
$$=C\left\{(0.2,s_4),(0.8,s_5)\right\}=s_{k(3,2)},$$

since
$$Low((s_5,s_4),(5/8,3/8))=s_{k'},\ k'=4+round((5/8).(5-4))=4+1=5,$$
$$k(3,2)=4+round((0.8).(5-4))=4+1=5\quad \Rightarrow s_{k(3,2)}=s_5=IM.$$
$$E_V^3(x_3,x_3)=Low((s_4,s_4,s_4),(0.2,0.5,0.3))=C\left\{(0.2,s_4),(0.8,Low((s_4),(1))\right\}$$
$$=C\left\{(0.2,s_4),(0.8,s_4)\right\}=s_4=SC.$$

$$E_V^3(x_3,x_4)=Low((s_4,s_4,s_5),(0.2,0.5,0.3))=C\left\{(0.3,s_5),(0.8)Low((s_4)(1))\right\}$$
$$=C\left\{(0.2,s_5),(0.8,s_4)\right\}=s_{k(3,4)}.$$

$$k(3,4)=4+round((0.2).(5-4))=4+0=4\quad \Rightarrow s_{k(3,4)}=s_4=SC.$$
$$E_V^3(x_4,x_1)=Low((s_5,s_6,s_4),(0.2,0.5,0.3))=C\left\{(0.5,s_6),(0.5)Low((s_5,s_4)(2/5,3/5))\right\}$$
$$=C\left\{(0.5,s_6),(0.5,s_4)\right\}=s_{k(4,1)}.$$

since

$$Low((s_5, s_4),(2/5,3/5)) = s_{k'},\ \ k' = 4 + round((2/5).(5-4)) = 4+0 = 4,$$

$$k(4,1) = 4 + round((0.5).(6-4)) = 4+1 = 5 \quad \Rightarrow s_{k(4,1)} = s_5 = IM.$$

$$E_V^3(x_4, x_2) = Low((s_4, s_3, s_4),(0.2,0.5,0.3)) = C\left\{(0.2, s_4),(0.8)Low((s_4, s_3)(3/8,5/8))\right\}$$

$$= C\left\{(0.2, s_4),(0.8, s_3)\right\} = s_{k(4,2)}.$$

since

$$Low((s_4, s_3),(3/8,5/8)) = s_{k'},\ \ k' = 3 + round((3/8).(5-4)) = 3+0 = 3,$$

$$k(4,2) = 3 + round((0.2).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4,2)} = s_3 = VLC.$$

$$E_V^3(x_4, x_3) = Low((s_4, s_3, s_4),(0.2,0.5,0.3)) = C\left\{(0.2, s_4),(0.8)Low((s_4, s_3)(3/8,5/8))\right\}$$

$$= C\left\{(0.2, s_4),(0.8, s_3)\right\} = s_{k(4,3)}.$$

since

$$Low((s_4, s_3),(3/8,5/8)) = s_{k'},\ \ k' = 3 + round((3/8).(5-4)) = 3+0 = 3,$$

$$k(4,2) = 3 + round((0.2).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4,2)} = s_3 = VLC.$$

$$E_V^2(x_4, x_4) = Low((s_4, s_4, s_4),(0.2,0.5,0.3)) = C\left\{(0.2, s_4),(0.8, Low((s_4),(1))\right\}$$

$$= C\left\{(0.2, s_4),(0.8, s_4)\right\} = s_4 = SC.$$

We obtain the linguistic non-dominance degree

$$E_V^3 = \begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & VLC & VLC \\ IM & SC & SC & SC \\ IM & VLC & VLC & SC \end{bmatrix}. \qquad (102)$$

Calculate FCS by Definition 2.

$$fcs_V^3(x_1) = Low((s_4, s_3),(1/3,2/3)) = C\left\{(1/3, s_4),(2/3, s_3)\right\} = s_k,$$

$$k = 3 + round((1/3).(4-3)) = 3+0 = 3 \quad \Rightarrow s_k = s_3 = SC \Rightarrow\ fcs_V^3(x_1) = SC.$$

$$fcs_V^3(x_2) = Low((s_5, s_4, s_3), (1/3, 2/3)) = C\{(1/3, s_5), (2/3, Low((s_4, s_3), (0.5, 0.5))\}$$
$$= C\{(1/3, s_5), (2/3, s_4)\} = s_k,$$

since

$$Low((s_4, s_3), (0.5, 0.5)) = s_{k'}, k' = 3 + round((0.5).(4-3)) = 3+1 = 4 \Rightarrow s_{k'} = s_4$$

$$k = 4 + round((1/3).(5-4)) = 4+0 = 4 \quad \Rightarrow s_k = s_4 = SC.$$
$$fcs_V^3(x_3) = Low((s_5, s_4), (2/3, 1/3)) = C\{(2/3, s_5), (1/3), s_4)\} = s_k,$$
$$k = 4 + round((2/3).(5-4)) = 4+1 = 5 \quad \Rightarrow s_k = s_5 = IM \Rightarrow fcs_V^3(x) = IM.$$

$$fcs_V^3(x_4) = Low((s_5, s_3), (1/3, 2/3)) = C\{(1/3, s_5), (2/3, s_3)\} = s_k,$$
$$k = 3 + round((1/3).(5-4)) = 3+0 = 3 \quad \Rightarrow s_k = s_3 = VLC \Rightarrow fcs_V^3(x_4) = VLC.$$

We obtain $$FCS_V^3 = \{VLC/x_1, SC/x_2, IM/x_3, VLC/x_4\} \tag{103}$$

Finally, we obtain IFE for criterion C_3

$$IFE^3 = \{(MC, VLC)/x_1, (SC, SC)/x_2, (SC, IM)/x_3, (SC, VLC)/x_4\} \tag{104}$$

and the FPS for criterion C_3 is x_1.

Step2. Use $\{(FCS_M^l, FCS_V^l), l = 1, 2, 3\}$ and the criteria's important weights $\beta = \{0.35, 0.4, 0.25\}$, calculate the *aggregated fuzzy collective solutions* (aFCS) as in Definition 3.

$$aFCS = \{afcs(x_1)/x_1, ..., afcs(x_n)/x_n\}, \tag{105}$$

where $$afcs_M(x_i) = Low(S, U_{M\beta}), \quad i=1,\dots,n \tag{106}$$

where $U_{M\beta} = \left[u_{M\beta_T}, ..., u_{M\beta_1}\right]$, t=1, ...T, $U_{M\beta_t} = \sum_l \left\{\beta_l : fcs_M^l(x_i) = s_t\right\}$ (107)

Calculate $afcs_M(x_1)$. Because $fcs_M^1(x_1) = s_5$, $fcs_M^2(x_1) = s_6$, $fcs_M^3(x_1) = s_6$,

$afcs_M(x_1) = Low((s_6, s_5), (0.65, 0.35) = C\{(0.65, s_6), (0.35, s_5)\} = s_k,$
$k = 5 + round((0.65).(6-5)) = 5+1 = 6 \quad \Rightarrow s_k = s_6 = MC \Rightarrow afcs_M(x_1) = MC.$

Calculate $afcs_M(x_2).$ Because
$fcs_M^1(x_2) = s_4,\ fcs_M^2(x_2) = s_4,\ fcs_M^3(x_2) = s_4,$
$afcs_M(x_2) = Low((s_4), (1) = s_4 = SC.$

Calculate $afcs_M(x_3).$ Because
$fcs_M^1(x_3) = s_4,\ fcs_M^2(x_3) = s_4,\ fcs_M^3(x_3) = s_4,$
$afcs_M(x_3) = Low((s_4), (1) = s_4 = SC.$

Calculate $afcs_M(x_4).$ Because
$fcs_M^1(x_4) = s_7,\ fcs_M^2(x_4) = s_5,\ fcs_M^3(x_4) = s_4,$

$afcs_M(x_4) = Low((s_7, s_5, s_4), (0.35, 0.4, 0.25) =$
$= C\{(0.35, s_7), (0.65, Low((s_5, s_4), (0.4/0.65, 0.25/0.65))\} = C\{(0.35, s_7), (0.65, s_5)\} = s_k,$
since
$Low((s_5, s_4), (0.4/0.65, 0.25/0.65)) = s_{k'}, k' = 4 + round((0.4/0.65).(5-4)) = 4+1 = 5 \Rightarrow s_{k'} = s_5,$

$k = 5 + round((0.35).(7-5)) = 5+1 = 6 \quad \Rightarrow s_k = s_6 = MC \Rightarrow afcs_M(x_4) = MC.$ Fina
lly, we obtain

$$aFCS_M = \{MC/x_1, SC/x_2, SC/x_3, MC/x_4\}. \tag{108}$$

Calculate

$$aFCS_V = \{afcs_V(x_1)/x_1, ..., afcs_V(x_4)/x_4\}, \tag{109}$$

where $afcs_V(x_i) = Low(S, U_{V\beta})$, i=1,2,3,4 (110)

where $U_{V\beta} = \left[u_{V\beta_T}, ..., u_{V\beta_1}\right]$, t=1, ...T,

$$U_{V\beta_t} = \sum_l \{\beta_l : fcs_V^l(x_i) = s_t\}\ . \tag{111}$$

Calculate $afcs_V(x_1).$ Because $fcs_M^1(x_1) = s_3,\ fcs_M^2(x_1) = s_3,\ fcs_M^3(x_1) = s_3,$
$afcs_V(x_1) = Low((s_3), (1)) = s_3 = VLC.$

Calculate $afcs_V(x_2).$ Because $fcs_V^1(x_2) = s_4,\ fcs_V^2(x_2) = s_4,\ fcs_V^3(x_2) = s_3,$
$afcs_M(x_2) = Low((s_4, s_2), (0.65, 0.35) = s_k,$

$k = 2 + round((0.65).(4-2)) = 2+1=3 \quad \Rightarrow s_k = s_3 = VLC \Rightarrow afcs_V(x_2) = VLC.$

Calculate $afcs_V(x_3)$. Because $fcs_V^1(x_3) = s_4,\ fcs_V^2(x_3) = s_4,\ fcs_V^3(x_3) = s_5,$

$afcs_V(x_3) = Low((s_5, s_4), (0.25, 0.75) = C\{(0.25, s_5), (0.75, s_4)\}\, s_k,$

$k = 4 + round((0.25).(5-4)) = 4+0=4 \quad \Rightarrow s_k = s_4 = SC \Rightarrow afcs_V(x_3) = SC.$

Calculate $afcs_V(x_4)$. Because $fcs_V^1(x_4) = s_2,\ fcs_V^2(x_4) = s_4,\ fcs_V^3(x_4) = s_3,$

$afcs_V(x_4) = Low((s_4, s_3, s_2), (0.35, 0.4, 0.25) =$

$= C\{(0.35, s_4), (0.65, Low((s_3, s_2), (0.4/0.65, 0.25/0.65))\} = C\{(0.35, s_4), (0.65, s_3)\} = s_k,$

since

$Low((s_3, s_2), (0.4/0.65, 0.25/0.65)) = s_{k'}, k' = 2 + round((0.4/0.65).(3-2)) = 2+1=3 \Rightarrow s_{k'} = s_3,$

$k = 3 + round((0.35).(4-3)) = 3+0=3 \quad \Rightarrow s_k = s_3 = VLC \Rightarrow afcs_V(x_4) = VLC.$

We obtain $aFCS_V = \{VLC/x_1, VLC/x_2, SC/x_3, VLC/x_4\}.$ (112)

Finally, we obtain Aggregated Intuitionistic Fuzzy Evaluation (aIFE)

$$aIFE = \{(MC, VLC)/x_1, (SC, VLC)/x_2, (SC, SC)/x_3, (MC, VLC)/x_4\}. \tag{113}$$

The *Aggregated Fuzzy Pareto Solution* (aFPS) are $\{x_1, x_4\}$.

7.2. Computing with the aggregation procedure 2

Step 1.

For each criterion C^l , l=1,2,3 calculate the linguistic dominance degrees as in (51, 53) we obtain:

$$E_M^1 = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & IM & EU \\ SC & IM & IM & EU \\ ML & IM & EL & IM \end{bmatrix}, \quad E_M^2 = \begin{bmatrix} IM & ML & MC & MC \\ SC & IM & IM & SC \\ VLC & IM & IM & SC \\ SC & SC & MC & IM \end{bmatrix}, \quad E_M^3 = \begin{bmatrix} IM & ML & IM & ML \\ VLC & IM & IM & IM \\ SC & SC & IM & SC \\ VLC & IM & SC & IM \end{bmatrix}$$

and

$$E_V^1 = \begin{bmatrix} SC & VLC & VLC & SC \\ EU & SC & VLC & EU \\ VLC & IM & SC & EU \\ EU & VLC & EU & SC \end{bmatrix}, E_V^2 = \begin{bmatrix} SC & VLC & VLC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \\ IM & SC & VLC & SC \end{bmatrix}, E_V^3 = \begin{bmatrix} SC & ML & IM & ML \\ VLC & SC & IM & IM \\ SC & SC & SC & SC \\ VLC & IM & SC & SC \end{bmatrix}$$

Step 2**.**

2.a1. Using these relations of linguistic relative dominance degrees $\{E_M^1, E_M^2, E_M^3\}$ and the important weights $\beta = \{0.35, 0.4, 0.25\}$, calculate the totally social opinion relation:

$$Q_M = \left[q_M(x_i, x_j) \right], i = 1,...,4, \; j = 1,...,4. \tag{114}$$

where $q_M(x_i, x_j) = Low(S, U_{q_M})$, $U_{q_M} = \left[u_{qT},...,u_{q1} \right]$, (115)

where $u_{qt} = W_{ij}^q(s_t) = \sum_l \left\{ \beta_l : E_M^l(i,j) = s_t \right\}$, t=1,...,T . (116)

$q_M(x_1, x_1) = Low((s_5, s_5, s_5), (0.35, 0.4, 0.25)) = Low((s_5), (1)) = s_5 = IM.$

$q_M(x_1, x_2) = Low((s_7, s_6, s_7), (0.35, 0.4, 0.25)) = Low((s_7, 0.6), (s_6, 0.4)) = s_{k(1,2)},$

$k(1,2) = 6 + round((0.6).(7-6)) = 6 + 1 = 7 \quad \Rightarrow s_{k(1,2)} = s_7 = ML \Rightarrow q_M(x_1, x_2) = ML.$

.

$q_M(x_1, x_3) = Low((s_6, s_6, s_5), (0.2, 0.5, 0.3)) = C\left\{(0.7, s_6), (0.3, s_5)\right\} = s_{k(1,3)},$

$k(1,3) = 5 + round((0.7).(6-5)) = 5 + 1 = 6 \quad \Rightarrow s_{k(1,3)} = s_6 = MC.$

$q_M(x_1, x_4) = Low((s_2, s_6, s_7), (0.35, 0.4, 0.25)) = C\left\{(0.25, s_7), (0.75, Low((s_6, s_2), (40/75, 35/75))\right\}$

$= C\left\{(0.25, s_7), (0.75, s_4)\right\} = s_{k(1,4)},$

since

$Low((s_6, s_2), (40/75, 25/75)) = s_{k'}, k' = 2 + round((40/75).(6-2)) = 2 + 2 = 4 \Rightarrow s_{k'} = s_4,$

$k(1,4) = 4 + round((0.25).(7-4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(1,4)} = s_5 = IM.$

$q_M(x_2, x_1) = Low((s_4, s_4, s_3), (0.35, 0.4, 0.25)) = C\left\{(0.75, s_4), (0.25, s_3)\right\} = s_{k(2,1)},$

$k(2,1) = 3 + round((0.75).(4-3)) = 3 + 1 = 4 \quad \Rightarrow s_{k(2,1)} = s_4 = SC.$

$q_M(x_2, x_2) = Low((s_5, s_5, s_5), (0.35, 0.4, 0.25)) = Low((s_5), (1)) = s_5 = IM.$

$q_M(x_2, x_3) = Low((s_5, s_5, s_5), (0.35, 0.4, 0.25)) = Low((s_5), (1)) = s_5 = IM.$

$q_M(x_2, x_4) = Low((s_2, s_4, s_5), (0.35, 0.4, 0.25)) = C\left\{(0.25, s_5), (0.75, Low((s_4, s_2), (40/75, 35/75))\right\}$

$= C\left\{(0.25, s_5), (0.75, s_3)\right\} = s_{k(2,4)},$

since

$Low((s_4, s_2), (40/75, 25/75)) = s_{k'}, k' = 2 + round((40/75).(4-2)) = 2 + 1 = 3 \Rightarrow s_{k'} = s_3,$

$k(2,4) = 3 + round((0.25).(5-3)) = 3 + 1 = 4 \quad \Rightarrow s_{k(2,4)} = s_4 = SC.$

$q_M(x_3, x_1) = Low((s_4, s_3, s_4), (0.35, 0.4, 0.25)) = C\left\{(0.6, s_4), (0.4, s_3)\right\} = s_{k(3,1)},$

$$k(3,1)=3+round((0.6).(4-3))=3+1=4 \quad \Rightarrow s_{k(3,1)}=s_4=SC.$$

$$q_M(x_3,x_2)=Low((s_5,s_4,s_4),(0.35,0.4,0.25))=C\{(0.35,s_5),(0.65,s_4)\}=s_{k(3,2)},$$

$$k(3,2)=4+round((0.35).(5-4))=4+0=4 \quad \Rightarrow s_{k(3,2)}=s_4=SC.$$

$$q_M(x_3,x_3)=Low((s_5,s_5,s_5),(0.35,0.4,0.25))=Low((s_5),(1))=s_5=IM.$$

$$q_M(x_3,x_4)=Low((s_2,s_4,s_4),(0.35,0.4,0.25))=C\{(0.65,s_4),(0.35,s_2)\}=s_{k(3,4)},$$

$$k(3,4)=2+round((0.65).(4-2))=2+1=3 \quad \Rightarrow s_{k(3,4)}=s_3=VLC.$$

$$q_M(x_4,x_1)=Low((s_7,s_4,s_3),(0.35,0.4,0.25))=C\{(0.25,s_7),(0.75,Low((s_4,s_3),(40/75,35/75))\}$$
$$=C\{(0.25,s_7),(0.75,s_4)\}=s_{k(4,1)},$$

since

$$Low((s_4,s_3),(40/75,25/75))=s_{k'},\ k'=3+round((40/75).(4-3))=3+1=4 \Rightarrow s_{k'}=s_4,$$

$$k(4,1)=4+round((0.25).(7-4))=4+1=5 \quad \Rightarrow s_{k(1,4)}=s_5=IM.$$

$$q_M(x_4,x_2)=Low((s_5,s_4,s_5),(0.35,0.4,0.25))=C\{(0.6,s_5),(0.4,s_4)\}=s_{k(4,2)},$$

$$k(4,2)=4+round((0.6).(5-4))=4+1=5 \quad \Rightarrow s_{k(4,2)}=s_5=IM.$$

$$q_M(x_4,x_3)=Low((s_8,s_6,s_4),(0.35,0.4,0.25))=C\{(0.35,s_8),(0.65,Low((s_6,s_4),(40/65,25/65))\}$$
$$=C\{(0.35,s_8),(0.65,s_5)\}=s_{k(4,3)},$$

$$Low((s_6,s_4),(40/65,25/65))=s_{k'},\ k'=4+round((40/75).(6-4))=4+1=5 \Rightarrow s_{k'}=s_5,$$

$$k(4,3)=5+round((0.35).(8-5))=5+1=6 \Rightarrow s_{k(4,3)}=s_6=MC.$$

$$q_M(x_4,x_4)=Low((s_5,s_5,s_5),(0.35,0.4,0.25))=Low((s_5),(1))=s_5=IM.$$

We obtain the totally social opinion relation

$$Q_M=\begin{bmatrix} IM & ML & MC & IM \\ SC & IM & IM & SC \\ SC & IM & IM & VLC \\ IM & IM & MC & IM \end{bmatrix}. \qquad (117)$$

2.a2. Using these relations of linguistic non-dominance degrees $\{E_V^1, E_V^2, E_V^3\}$ and the important weights $\beta=\{0.35,0.4,0.25\}$, calculate the totally social opinion relation

$$Q_V = \left[q_V(x_i, x_j) \right], i = 1, ..., 4, j = 1, ..., 4. \quad (118)$$

where $q_V(x_i, x_j) = Low(S, U_{q_V})$, $\quad U_{q_M} = \left[u_{qT}, ..., u_{q1} \right]$ (119)

where $u_{qt} = W_{ij}^q(s_t) = \sum_l \left\{ \beta_l : E_V^l(i, j) = s_t \right\}$, t=1,...,T . (120)

$$q_V(x_1, x_1) = Low((s_4, s_4, s_4), (0.35, 0.4, 0.25)) = Low((s_4), (1)) = s_4 = SC.$$

$$q_V(x_1, x_2) = Low((s_3, s_3, s_3), (0.35, 0.4, 0.25)) = Low((s_3), (1)) = s_3 = VLC.$$

$$q_V(x_1, x_3) = Low((s_3, s_3, s_4), (0.35, 0.4, 0.25)) = C\left\{(0.25, s_4), (0.75, s_3)\right\} = s_{k(1,3)},$$

$$k(1,3) = 3 + round((0.25).(4-3)) = 3 + 0 = 3 \Rightarrow s_{k(1,3)} = s_3 = VLC.$$

$$q_V(x_1, x_4) = Low((s_4, s_3, s_3), (0.35, 0.4, 0.25)) = C\left\{(0.35, s_4), (0.65, s_3)\right\} = s_{k(1,4)},$$

$$s_{k(1,4)} = 3 + round((0.35).(4-3)) = 3 + 0 = 3 \Rightarrow \quad s_{k(1,4)} = s_3 = VLC.$$

$$q_V(x_2, x_1) = Low((s_2, s_5, s_5), (0.35, 0.4, 0.25)) = C\left\{(0.75, s_5), (0.25, s_2)\right\} = s_{k(2,1)},$$

$$k(2,1) = 2 + round((0.75).(5-2)) = 2 + 2 = 4, \quad \Rightarrow s_{k(2,1)} = s_4 = SC.$$

$$q_V(x_2, x_2) = Low((s_4, s_4, s_4), (0.35, 0.4, 0.25)) = Low((s_4), (1)) = s_4 = SC.$$

$$q_V(x_2, x_3) = Low((s_3, s_3, s_3), (0.35, 0.4, 0.25)) = Low((s_3), (1)) = s_3 = VLC.$$

$$q_V(x_2, x_4) = Low((s_2, s_4, s_3), (0.35, 0.4, 0.25)) = C\left\{(0.4, s_4), (0.6, Low((s_3, s_2), (25/60, 35/60))\right\}$$

$$= C\left\{(0.25, s_5), (0.6, s_2)\right\} = s_{k(2,4)},$$

since

$$Low((s_3, s_2), (25/60, 35/60)) = s_{k'}, k' = 2 + round((25/60).(3-2)) = 2 + 0 = 2 \Rightarrow s_{k'} = s_2,$$

$$k(2,4) = 2 + round((0.25).(5-2)) = 2 + 1 = 3 \quad \Rightarrow s_{k(2,4)} = s_3 = VLC.$$

$$q_V(x_3, x_1) = Low((s_3, s_5, s_5), (0.35, 0.4, 0.25)) = C\left\{(0.65, s_5), (0.35, s_3)\right\} = s_{k(3,1)},$$

$$k(3,1) = 3 + round((0.65)(5-3)) = 3 + 1 = 4, \Rightarrow \quad s_{k(3.1)} = s_4 = SC.$$

$$q_V(x_3, x_2) = Low((s_5, s_4, s_4), (0.35, 0.4, 0.25)) = C\left\{(0.35, s_5), (0.65, s_4)\right\} = s_{k(3,2)},$$

$$k(3,2) = 4 + round((0.35).(5-4)) = 4 + 0 = 4 \quad \Rightarrow s_{k(3,2)} = s_4 = SC.$$

$$q_V(x_3, x_3) = Low((s_4, s_4, s_4), (0.35, 0.4, 0.25)) = Low((s_4), (1)) = s_4 = SC.$$

$$q_V(x_3,x_4)=Low((s_2,s_4,s_4),(0.35,0.4,0.25))=C\{(0.65,s_4),(0.35,s_2)\}=s_{k(3,4)},$$
$$k(3,4)=2+round((0.65).(4-2))=2+1=3,\quad \Rightarrow s_{k(3,4)}=s_3=VLC.$$
$$q_V(x_4,x_1)=Low((s_2,s_5,s_5),(0.35,0.4,0.25))=C\{(0.65,s_5),(0.35,s_2)\}=s_{k(4,1)},$$
$$k(4,1)=2+round((0.65).(5-2))=2+2=4,\quad \Rightarrow s_{k(4,1)}=s_4=SC.$$
$$q_V(x_4,x_2)=Low((s_3,s_3,s_3),(0.35,0.4,0.25))=Low((s_3),(1))=s_3=VLC.$$
$$q_V(x_4,x_3)=Low((s_2,s_3,s_3),(0.35,0.4,0.25))=C\{(0.65,s_3),(0.35,s_2)\}=s_{k(4,3)},$$
$$k(4,3)=2+round((0.65).(3-2))=2+1=3,\quad \Rightarrow s_{k(4,3)}=s_3=VLC.$$
$$q_V(x_4,x_4)=Low((s_4,s_4,s_4),(0.35,0.4,0.25))=Low((s_4),(1))=s_4=SC.$$

We obtain the totally social opinion relation using intuitionistic linguistic non-preference relations:

$$Q_V=\begin{bmatrix} SC & SC & SC & SC \\ VLC & SC & SC & SC \\ SC & SC & SC & VLC \\ VLC & SC & VLC & SC \end{bmatrix}. \tag{121}$$

2.b. Using Q_M , Q_V ,calculate the *fuzzy collective solution* , calculate

$$FCS_{Q_M}=\left\{ fcs_{QM}(x_1)/x_1,\ fcs_{QM}(x_2)/x_2,\ fcs_{QM}(x_3)/x_3,\ fcs_{QM}(x_4)/x_4\right\}, \tag{122}$$

$$fcs_{QM}(x_1)=Low((s_7,s_6,s_5),(1/3,1/3,1/3))=C\left\{(1/3,s_7),(2/3,Low((s_6,s_5),(0.5,0.5))\right\}$$
$$=C\left\{(1/3,s_7),(2/3,s_6)\right\}=k(1),$$

since

$$Low((s_6,s_5),(0.5,0.5))=C((0.5,s_6),(0.5,s_5)\}=s_{k'},k'=5+round((0.5).(6-5))=5+1=6,$$
$$k(1)=6+round((1/3).(7-6))=6+0=6\quad \Rightarrow s_{k(1)}=s_6=MC.$$
$$fcs_{QM}(x_2)=Low((s_5,s_4),(1/3,2/3))=C((1/3,s_5),(2/3,s_4)\}=s_{k(2)},$$
$$k(2)=4+round((1/3).(5-4))=4+0=4,\quad \Rightarrow s_{k(2)}=s_4=SC.$$

$$fcs_{QM}(x_3)=Low((s_5,s_4,s_3),(1/3,1/3,1/3))=C\{(1/3,s_5),(2/3,Low((s_4,s_3),(0.5,0.5))\}$$
$$=C\left\{(1/3,s_5),(2/3,s_4)\right\}=s_k,\quad k=4+round((1/3).(5-4))=4+0=4,s_k=s_4=SC.$$
$$fcs_{QM}(x_4)=Low((s_6,s_5),(1/3,2/3))=C\{(1/3,s_6),(2/3,s_5)\}=s_k,$$
$$k=5+round((1/3).(6-5))=5+0=5,\quad \Rightarrow s_k=s_5=IM.$$

and $FCS_{QM} = \{MC/x_1, SC/x_2, SC/x_3, IM/x_4\}$.

Calculate the

$$FCS_{QV} = \{ fcs_{QV}(x_1)/x_1, fcs_{QV}(x_2)/x_2, fcs_{QV}(x_3)/x_3, fcs_{QV}(x_4)/x_4 \},$$

. $fcs_{QV}(x_1) = Low((s_3),(1))) = s_3 = VLC.$

$$fcs_{QV}(x_2) = Low((s_4, s_3),(2/3,1/3)) = C((2/3, s_4),(1/3, s_3)\} = s_{k(2)},$$
$$k(2) = 3 + round((2/3).(4-3)) = 3+1 = 4, \quad \Rightarrow s_{k(2)} = s_4 = SC.$$

$$fcs_{QV}(x_3) = Low((s_4, s_3),(2/3,1/3)) = C\{(2/3, s_4),(1/3, s_3)\} = s_{k(3)},$$

$$k(3) = 3 + round((2/3).(4-3)) = 3+1 = 4 \quad \Rightarrow s_{k(3)} = s_4 = SC.$$

$$fcs_{QV}(x_4) = Low((s_4, s_3),(1/3,2/3)) = C\{(1/3, s_4),(2/3, s_3)\} = s_{k(4)},$$
$$k(4) = 3 + round((1/3).(4-3)) = 3+0 = 3 \quad \Rightarrow s_{k(4)} = s_3 = VLC.$$

We obtain $FCS_{QV} = \{VLC/x_1, SC/x_2, SC/x_3, VLC/x_4\}$.

Finally, we obtain Intuitionistic Fuzzy Evaluation:

$$IFE_Q = \{(MC,VLC)/x_1,(SC,SC)/x_2,(SC,SC)/x_3,(IM,VLC)/x_4\}.$$

From IFE_Q we choose the FPS of the problem. The Fuzzy Pareto Solution of the problem is $\{x_1\}$.

7.3. Computing with the Aggregation procedure 3

Step 1.1. Computing for expert e_1

Use

$$M_{11} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & SC & VLC \\ VLC & MC & IM & EU \\ EL & IM & ML & IM \end{bmatrix}, M_{12} = \begin{bmatrix} IM & MC & ML & IM \\ VLC & IM & SC & SC \\ VLC & SC & IM & EU \\ SC & IM & MC & IM \end{bmatrix}, M_{13} = \begin{bmatrix} IM & MC & ML & IM \\ SC & IM & MC & SC \\ VLC & SC & IM & IM \\ SC & MC & SC & IM \end{bmatrix}$$

with the weights $\beta = (0.35, 0.4, 0.25)$, calculate $F_M^k(x_i, x_j) = Low(S, U_M^k)$, i ,j=1,2,3,4

using (70, 71), we obtain the relative dominance degrees according to e_1

$$F_M^1 = \begin{bmatrix} IM & MC & ML & SC \\ SC & IM & SC & SC \\ VLC & IM & IM & VLC \\ IM & IM & MC & IM \end{bmatrix},$$

and using (72, 73) calculate the fuzzy evaluation FE_M^1

$$FE_M^1 = \{MC/x_1, SC/x_2, SC/x_3, IM/x_4\}.$$

Use

$$V_{11} = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix}, V_{12} = \begin{bmatrix} SC & VLC & SC & EU \\ SC & SC & IM & VLC \\ MC & SC & SC & MC \\ MC & MC & VLC & SC \end{bmatrix}, V_{13} = \begin{bmatrix} SC & SC & VLC & SC \\ SC & SC & SC & SC \\ IM & VLC & SC & SC \\ IM & SC & SC & SC \end{bmatrix}$$

with the weights $\beta = (0.35, 0.4, 0.25)$. Calculate $F_V^k(x_i, x_j) = Low(S, U_V^k),$ i ,j=1,2,3,4 .

Using (74, 75), we obtain the relative non-dominance degree according to e_1

$$F_V^1 = \begin{bmatrix} SC & SC & VLC & VLC \\ VLC & SC & SC & VLC \\ IM & SC & SC & IM \\ SC & SC & VLC & SC \end{bmatrix},$$

and using (76) calculate the fuzzy evaluation FE_V^1

$$FE_V^1 = \{VLC/x_1, VLC/x_2, IM/x_3, SC/x_4\}.$$

Finally, we obtain IFE according to e_1

$$\{(MC, VLC)/x_1, (SC, VLC)/x_2, (SC, IM)/x_3, (IM, SC)/x_4\},$$

and the FPS according to e_1 is x_1 .

Step 1.2. Computing for expert e_2

Use

$$M_{21} = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ ML & IM & EL & IM \end{bmatrix}, M_{22} = \begin{bmatrix} IM & ML & MC & MC \\ SC & IM & IM & SC \\ VLC & SC & IM & SC \\ SC & MC & MC & IM \end{bmatrix}, M_{23} = \begin{bmatrix} IM & ML & IM & ML \\ VLC & IM & IM & IM \\ SC & MC & SC & SC \\ VLC & SC & SC & IM \end{bmatrix}.$$

using (70, 71), we obtain the relative dominance degrees according to e_2

$$F_M^2 = \begin{bmatrix} IM & ML & MC & SC \\ SC & IM & IM & SC \\ SC & IM & IM & VLC \\ IM & IM & MC & IM \end{bmatrix},$$

and the fuzzy evaluation FE_M^2 is

$$FE_M^2 = \{MC/x_1, SC/x_2, SC/x_3, IM/x_4\}.$$

Use

$$V_{21} = \begin{bmatrix} SC & EU & VLC & SC \\ EU & SC & VLC & EU \\ VLC & MC & SC & VLC \\ EU & EU & VLC & SC \end{bmatrix}, V_{22} = \begin{bmatrix} SC & EU & VLC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & VLC \\ SC & VLC & VLC & SC \end{bmatrix}, V_{23} = \begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & VLC & VLC \\ IM & IM & SC & SC \\ MC & VLC & VLC & SC \end{bmatrix}.$$

Using (74, 75), we obtain the relative non-dominance degree according to e_2

$$F_V^2 = \begin{bmatrix} SC & EU & VLC & VLC \\ SC & SC & VLC & VLC \\ SC & IM & SC & VLC \\ SC & VLC & VLC & SC \end{bmatrix},$$

and the fuzzy evaluation FE_V^2

$$FE_V^2 = \{VLC/x_1, VLC/x_2, SC/x_3, VLC/x_4\}.$$

Finally, we obtain IFE according to e_2

$$\{(MC,VLC)/x_1, (SC,VLC)/x_2, (SC,SC)/x_3, (IM,VLC)/x_4\}$$

and using (76), the FPS according to e_2 is x_1.

Step 1.3. Computing for expert e_3

Use

$$M_{31} = \begin{bmatrix} IM & MC & MC & EU \\ SC & IM & SC & EU \\ SC & MC & IM & EU \\ ML & IM & EL & IM \end{bmatrix}, M_{32} = \begin{bmatrix} IM & IM & MC & MC \\ SC & IM & MC & IM \\ VLC & SC & IM & VLC \\ SC & SC & IM & IM \end{bmatrix}, M_{33} = \begin{bmatrix} IM & ML & SC & MC \\ SC & IM & MC & IM \\ IM & SC & IM & SC \\ VLC & SC & IM & IM \end{bmatrix}.$$

using (70, 71), we obtain the relative dominance degrees according to e_3

$$F_M^3 = \begin{bmatrix} IM & MC & MC & IM \\ SC & IM & IM & SC \\ SC & IM & IM & VLC \\ IM & SC & MC & IM \end{bmatrix},$$

and the fuzzy evaluation FE_M^3

$$FE_M^3 = \{MC/x_1, SC/x_2, SC/x_3, IM/x_4\}.$$

Use

$$V_{31} = \begin{bmatrix} SC & SC & SC & SC \\ EU & SC & SC & EU \\ EU & SC & SC & I \\ EU & SC & I & SC \end{bmatrix}, V_{32} = \begin{bmatrix} SC & SC & VLC & EU \\ IM & SC & VLC & SC \\ SC & SC & SC & SC \\ MC & SC & SC & SC \end{bmatrix}, V_{33} = \begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & SC & VLC \\ SC & SC & SC & IM \\ SC & SC & SC & SC \end{bmatrix}.$$

Using (74, 75), we obtain the relative non-dominance degree according to e_3

$$F_V^3 = \begin{bmatrix} SC & SC & SC & VLC \\ SC & SC & SC & VLC \\ VLC & SC & SC & VLC \\ SC & SC & VLC & SC \end{bmatrix},$$

and the fuzzy evaluation FE_V^3 is $FE_V^3 = \{SC/x_1, SC/x_2, VLC/x_3, SC/x_4\}$.

Finally, we obtain IFE according to e_3

$$\{(MC, SC)/x_1, (SC, SC)/x_2, (SC, VLC)/x_3, (IM, SC)/x_4\},$$

and the FPS according to e_3 is $\{x_1, x_3\}$.

Step 2. Use the *fuzzy evaluation* $\{(FE_M^k, FE_V^k), k = 1,2,3\}$ and the weights {w(k): $e_k \in E$}, calculate the *aggregated fuzzy evaluation* (aFE).

$$aFE_M = \{MC/x_1, SC/x_2, SC/x_3, IM/x_4\},$$

$$aFE_V = \{VLC/x_1, VLC/x_2, SC/x_3, SC/x_4\}.$$

We obtain the aggregated intuitionistic fuzzy evaluation (aIFE).

$$aIFE = \{(MC, VLC)/x_1, (SC, VLC)/x_2, (SC, SC)/x_3, (IM, SC)/x_4\}.$$

Finally, the aFPS of the problem is x_1.

8. Conclusions

This paper focuses on multi criteria group decision making problem under linguistic assessments. The linguistic preference relations model forms a useful tool in representing decision makers' choices. Some aggregation operators and computing processes using Fuzzy Collective Solution were given in Sections 3 and 4. Then, we proposed a new approach based on the new concept of Fuzzy Pareto Solution for the group decision making based on intuitionistic linguistic preference relations. Some aggregation procedures for the FPS and a computing example were also proposed. In the future, we will further investigate various aggregation procedures in the situations with type-2 intuitionistic fuzzy preference information.

APPENDIX

$$M_{11}=\begin{bmatrix} IM & MC & ML & EU \\ SC & IM & SC & VLC \\ VLC & MC & IM & EU \\ EL & IM & ML & IM \end{bmatrix},\ M_{21}=\begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ ML & IM & EL & IM \end{bmatrix},\ M_{31}=\begin{bmatrix} IM & MC & MC & EU \\ SC & IM & SC & EU \\ SC & MC & IM & EU \\ ML & IM & EL & IM \end{bmatrix},$$

$$M_{12}=\begin{bmatrix} IM & MC & ML & IM \\ VLC & IM & SC & SC \\ VLC & SC & IM & EU \\ SC & IM & MC & IM \end{bmatrix},\ M_{22}=\begin{bmatrix} IM & ML & MC & MC \\ SC & IM & IM & SC \\ VLC & SC & IM & SC \\ SC & MC & MC & IM \end{bmatrix},\ M_{32}=\begin{bmatrix} IM & IM & MC & MC \\ SC & IM & MC & IM \\ VLC & SC & IM & VLC \\ SC & SC & IM & IM \end{bmatrix},$$

$$M_{13}=\begin{bmatrix} IM & MC & ML & IM \\ SC & IM & MC & SC \\ VLC & SC & IM & IM \\ SC & MC & SC & IM \end{bmatrix},\ M_{23}=\begin{bmatrix} IM & ML & IM & ML \\ VLC & IM & IM & IM \\ SC & MC & SC & SC \\ VLC & SC & SC & IM \end{bmatrix},\ M_{33}=\begin{bmatrix} IM & ML & SC & MC \\ SC & IM & MC & IM \\ IM & SC & IM & SC \\ VLC & SC & IM & IM \end{bmatrix},$$

$$V_{11}=\begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix},\ V_{21}=\begin{bmatrix} SC & EU & VLC & SC \\ EU & SC & VLC & EU \\ VLC & MC & SC & VLC \\ EU & EU & VLC & SC \end{bmatrix},\ V_{31}=\begin{bmatrix} SC & SC & SC & SC \\ EU & SC & SC & EU \\ EU & SC & SC & I \\ EU & SC & I & SC \end{bmatrix},$$

$$V_{12}=\begin{bmatrix} SC & VLC & SC & EU \\ SC & SC & IM & VLC \\ MC & SC & SC & MC \\ MC & MC & VLC & SC \end{bmatrix},\ V_{22}=\begin{bmatrix} SC & EU & VLC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & VLC \\ SC & VLC & VLC & SC \end{bmatrix},\ V_{32}=\begin{bmatrix} SC & SC & VLC & EU \\ IM & SC & VLC & SC \\ SC & SC & SC & SC \\ MC & SC & SC & SC \end{bmatrix},$$

$$V_{13}=\begin{bmatrix} SC & SC & VLC & SC \\ SC & SC & SC & SC \\ IM & VLC & SC & SC \\ IM & SC & SC & SC \end{bmatrix},\ V_{23}=\begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & VLC & VLC \\ IM & IM & SC & SC \\ MC & VLC & VLC & SC \end{bmatrix},\ V_{33}=\begin{bmatrix} SC & VLC & SC & VLC \\ IM & SC & SC & VLC \\ SC & SC & SC & IM \\ SC & SC & SC & SC \end{bmatrix}.$$

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