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Why Product "And"-Operation Is Often Efficient: One More Argument

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Abstract

It is an empirical fact that the algebraic product is one the most efficient "and"-operations in fuzzy logic. In this paper, we provide one of the possible explanations of this empirical phenomenon.

1 Formulation of the Problem

Fuzzy logic and "and"-operations (t-norms): a brief reminder. To describe the experts' uncertainty in their statements, Lotfi Zadeh proposed a special formalism of *fuzzy logic*, in which for each statement, the expert's degree of certainty in this statement is described by a number from the interval $[0, 1]$. In this description, 1 means that the expert is absolutely sure that the corresponding statement is true, 0 means that the expert is absolutely sure that the statement is false, and values between 0 and 1 correspond to intermediate degree of confidence; see, e.g., [1, 2, 3].

To make a decision, an expert often uses several statements. For example, he or she may use a rule according to which a certain action need to be taken if two conditions are satisfied, i.e., if the first condition A is satisfied *and* the second condition B is satisfied. It is therefore desirable to find out how confident is the expert that the corresponding "and"-statement $A \& B$ holds, or, more generally, that the "and"-combination $A_1 \& \ldots \& A_n$ holds. Ideally, we should extract these degrees from the experts. However, for n statements, we have $2^n - 1$ possible "and"-combinations. Already for $n = 100$, we thus get an astronomical number of combinations, there is no way to ask the expert about each of these combinations.

In situations when we cannot explicitly ask an expert about his/her degree of certainty in an "and"-combination $A \& B$, we need to estimate this degree based on the known degrees of certainty a and b in statements A and B. Let us denote this estimate by $f_\&(a, b)$. This function is known as an *"and"-operation*, or, alternatively, a *t-norm*.

The "and"-operation must satisfy many reasonable properties. For example, since $A \& B$ means the same as $B \& A$, it is reasonable to require that the estimates of confidence for these two statements are the same, i.e., that $f_\&(a, b) = f_\&(b, a)$ for all a and b. Similarly, since $A \& (B \& C)$ and $(A \& B) \& C$ are equivalent, it is reasonable to require that $f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$, i.e., that the "and"-operation be associative.

Product is one of the most efficient "and"-operations. In principle, there are many different "and"-operations [1, 2, 3]. Empirically, one of the most efficient "and"-operations is the algebraic product $f_\&(a, b) = a \cdot b$.

But why? In this paper, we provide a possible explanation of why the product is one of the most efficient "and"-operations.

2 Main Idea

Membership functions: reminder. Fuzzy logic was originally designed to describe imprecise ("fuzzy") words from natural language like "small". To describe the meaning of each such word, we assign, to each possible value of the corresponding quantity x, the degree $\mu(x) \in [0, 1]$ to which this value satisfies the property described by this word – e.g., the degree to which the value x is small.

The corresponding function $\mu(x)$ – known as the *membership function* – is usually selected in such a way that $\max_x \mu(x) = 1$. Membership functions that satisfy this property are known as *normalized*.

What if we take into account expert's reliability. For commonsense properties like "small" or "young", everyone is an expert. However, similar properties like "small" occur in expertise-related situations as well. For example, a complex medical rule may use, as a condition, that the size of a tumor is small. In such situations, an expert may be not 100% confident that his or her opinion correctly reflects the corresponding expertise.

In fuzzy logic, it is reasonable to describe the degree to which the expert is confident in his or her expertise by a number d_0 from the interval $[0, 1]$. In such situations, we should not fully trust the expert's opinion about each value x. Instead, we believe that x is actually small if the expert considers this value small *and* this expert is reliable.

In fuzzy logic, in general, the degree of confidence is a composite statement $A \& B$ is estimated by applying the corresponding "and"-operation (t-norm) $f_\&(a, b)$ to the degrees of confidence a and b in the original statements A and B. In particular, in our situation, we have $a = \mu(x)$ and $b = d_0$, thus the degree of belief that x is actually small can be computed as $\mu'(x) = f_\&(\mu(x), d_0)$.

The problem with this new membership function $\mu'(x)$ is that it is not normalized. Indeed, by the properties of an "and"-operation, we always have $f_\&(a, b) \leq b$. In particular, in our case, we have $\mu'(x) = f_\&(\mu(x), d_0) \leq d_0$, thus, $\max_x \mu'(x) \leq d_0$. The original function $\mu(x)$ was normalized, meaning

that there exists a value x_0 for which $\mu(x_0) = 1$. For this value, we have $\mu'(x_0) = f_\&(\mu(x_0), d_0) = f_\&(1, d_0) = d_0$, so $\max_x \mu'(x) = d_0$.

To get a normalized membership function from the non-normalized membership function $\mu'(x)$, we can use the following normalization procedure typically used in fuzzy logic:

$$\mu_N(x) = \frac{\mu'(x)}{\max_y \mu'(y)}.$$

Reasonable requirement. The only information that we have about the expert knowledge is contained in the original membership function $\mu(x)$. Thus, it makes sense to require that the new normalized membership function coincides with the original one, i.e., that $\mu_N(x) = \mu(x)$ for all x.

Let us analyze which "and"-operations satisfy this requirement.

3 Main Result

Definition. *We say that an "and"-operation $f_\&(a, b)$ is* reasonable *if for every membership function $\mu(x)$ and for every number $\mu_0 \in (0, 1)$, the following equality holds for every x: $\mu_N(x) = \mu(x)$, where*

$$\mu_N(x) = \frac{\mu'(x)}{\max_y \mu'(y)}$$

and $\mu'(x) = f_\&(\mu(x), d_0)$.

Proposition. *The "and"-operation is reasonable in the sense to Definition if and only if the "and"-operation is a product: $f_\&(a, b) = a \cdot b$.*

Proof. If $f_\&(a, b) = a \cdot b$, then $\mu'(x) = f_\&(\mu(x), d_0) = d_0 \cdot \mu(x)$. Thus,

$$\mu_N(x) = \frac{\mu'(x)}{\max_y \mu'(y)} = \frac{\mu'(x)}{d_o} = \mu(x),$$

i.e., this "and"-operation is indeed reasonable.

Vice verse, let us assume that the "and"-operation is reasonable. Then, for every membership function $\mu(x)$, for every value x, and for every value d_0, we have

$$\mu(x) = \frac{f_\&(\mu(x), d_0)}{d_0},$$

and thus, $f_\&(\mu(x), d_0) = \mu(x) \cdot d_0$. Therefore, $f_\&(a, b) = a \cdot b$ for every a and b, i.e., the "and"-operation is indeed the product.

Acknowledgments

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