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Towards Decision Making under Interval Uncertainty

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Abstract

In many practical situations, we know the exact form of the objective function, and we know the optimal decision corresponding to each values of the corresponding parameters x_i. What should we do if we do not know the exact values of x_i, and instead, we only know each x_i with uncertainty – e.g., with interval uncertainty? In this case, one of the most widely used approaches is to select, for each i, one value from the corresponding interval – usually, a midpoint – and to use the exact-case optimal decision corresponding to the selected values. Does this approach lead to the optimal solution to the interval-uncertainty problem? If yes, is selecting the midpoints the best idea? In this paper, we provide answers to these questions. It turns out that the selecting-a-valued-from-each-interval approach can indeed lead us to the optimal solution for the interval problem – but *not* if we select midpoints.

1 Formulation of the Practical Problem

Often, we know the ideal-case solution. One of the main objectives of science and engineering is to provide an optimal decision in different situations. In many practical situations, we have an algorithm that provides an optimal decision based under the condition that we know the exact values of the corresponding parameters $x_1, \ldots, x_n$.

In practice, we need to take uncertainty into account. In practice, we usually know x_i with some uncertainty. For example, often, for each i, we only know an interval $[\underline{x}_i, \overline{x}_i]$ that contains the actual (unknown) value x_i; see, e.g., [7].

A problem. In the presence of such interval uncertainty, how can we find the optimal solution?

One of the most widely used approaches uses the fact that under interval uncertainty, we can implement decisions corresponding to different combinations of values $x_i \in [\underline{x}_i, \overline{x}_i]$. If this indeed a way to the solution which is optimal under interval uncertainty? If yes, which values should we choose?

Often, practitioners select the midpoints, but is this selection the best choice? These are the questions that we answer in this paper.

2 Formulation of the Problem in Precise Terms

Decision making: a general description. In general, we need to make a decision $u = (u_1, \ldots, u_m)$ based on the state $x = (x_1, \ldots, x_n)$ of the system. According to decision theory, a rational person selects a decision that maximizes the value of an appropriate function known as utility; see, e.g., [1, 4, 5, 8].

We will consider situations when or each state x and for each decision u, we know the value of the utility $f(x, u)$ corresponding to us choosing u. Then, when we know the exact state x of the system, the optimal decision $u^{\rm opt}(x)$ is the decision for which this utility is the largest possible:

$$f(x, u^{\rm opt}(x)) = \max_u f(x, u). \tag{1}$$

Decision making under interval uncertainty. In practice, we rarely know the exact state of the system, we usually know this state with some uncertainty. Often, we do not know the probabilities of different possible states x, we only know the bounds on different parameters describing the state.

The bounds mean that for each i, instead of knowing the exact values of x_i, we only know the bounds $\underline{x}_i$ and $\overline{x}_i$ on this quantity, i.e., we only know that the actual (unknown) value x_i belongs to the interval $[\underline{x}_i, \overline{x}_i]$. The question is: what decision u should we make in this case?

We also assume that the uncertainty with which we know x is relatively small, so in the corresponding Taylor series, we can only keep the first few terms in terms of this uncertainty.

Decision making under interval uncertainty: towards a precise formulation of the problem. Because of the uncertainty with which we know the state x, for each possible decision u, we do not know the exact value of the utility, we only know that this utility is equal to $f(x, u)$ for some $x_i \in [\underline{x}_i, \overline{x}_i]$. Thus, all we know is that this utility value belongs to the interval

$$\left[\min_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n, u), \max_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n, u)\right]. \tag{2}$$

According to decision theory (see, e.g., [2, 3, 4]), if for every action a, we only know the interval $[f^-(a), f^+(a)]$ of possible values of utility, then we should select the action for which the following combination takes the largest possible value:

$$\alpha \cdot f^+(a) + (1 - \alpha) \cdot f^-(a), \tag{3}$$

where the parameter $\alpha \in [0,1]$ describes the decision maker's degree of optimism-pessimism:

- the value $\alpha = 1$ means that the decision maker is a complete optimist, only taking into account the best-case situations,
- the value $\alpha = 0$ means that the decision maker is a complete pessimist, only taking into account the worst-case situations, and
- intermediate value $\alpha \in (0,1)$ means that the decision maker takes into account both worst-case and best-case scenarios.

Resulting formulation of the problem. In these terms our goal is:

- given the function $f(x,u)$ and the bounds $\underline{x}$ and $\overline{x}$,
- to find the value u for which the following objective function takes the largest possible value:

$$\alpha \cdot \max_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1,\ldots,x_n,u) + (1-\alpha) \cdot \min_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1,\ldots,x_n,u) \to \max_u. \quad (4)$$

Comment. The simplest case when the state x is characterized by a single parameter and when a decision u is also described by a single number, was analyzed in [6].

3 Analysis of the Problem

We assumed that the uncertainty is small, and that in the corresponding Taylor expansions, we can keep only a few first terms corresponding to this uncertainty. Therefore, it is convenient to describe this uncertainty explicitly.

Let us denote the midpoint $\dfrac{\underline{x}_i + \overline{x}_i}{2}$ of the interval $[\underline{x}_i, \overline{x}_i]$ by $\widetilde{x}_i$. Then, each value x_i from this interval can be represented as $x_i = \widetilde{x}_i + \Delta x_i$, where we denoted $\Delta x_i \stackrel{\text{def}}{=} x_i - \widetilde{x}_i$. The range of possible values of Δx_i is $[\underline{x}_i - \widetilde{x}_i, \overline{x}_i - \widetilde{x}_i] = [-\Delta_i, \Delta_i]$, where we denoted $\Delta_i \stackrel{\text{def}}{=} \dfrac{\overline{x}_i - \underline{x}_i}{2}$.

The differences Δx_i are small, so we should be able to keep only the few first terms in Δx_i.

When all the values x_i are known exactly, the exact-case optimal decision is $u^{\text{opt}}(x)$. Since uncertainty is assumed to be small, the optimal decision $u = (u_1,\ldots,u_m)$ under interval uncertainty should be close to the exact-case optimal decision $\widetilde{u} = (\widetilde{u}_1,\ldots,\widetilde{u}_m) \stackrel{\text{def}}{=} u^{\text{opt}}(\widetilde{x})$ corresponding to the midpoints of all the intervals. So, the difference $\Delta u_j \stackrel{\text{def}}{=} u_j - \widetilde{u}_j$ should also be small. In terms of Δu_j, the interval-case optimal value u_j has the form $u_j = \widetilde{u}_j + \Delta u_j$. Substituting $x_i = \widetilde{x}_i + \Delta x_i$ and $u_j = \widetilde{u}_j + \Delta u_j$ into the expression $f(x,u)$ for the utility, and keeping only linear and quadratic terms in this expansion, we conclude that

$$f(x,u) = f(\widetilde{x} + \Delta x, \widetilde{u} + \Delta u) =$$

$$f(\widetilde{x},\widetilde{u})+\sum_{i=1}^{n} f_{x_i}\cdot \Delta x_i+\sum_{j=1}^{m} f_{u_j}\cdot \Delta u_j+$$
$$\frac{1}{2}\cdot \sum_{i=1}^{n}\sum_{i'=1}^{n} f_{x_i x_{i'}}\cdot \Delta x_i\cdot \Delta x_{i'}+\sum_{i=1}^{n}\sum_{j=1}^{m} f_{x_i u_j}\cdot \Delta x_i\cdot \Delta u_j+$$
$$\frac{1}{2}\cdot \sum_{j=1}^{m}\sum_{j'=1}^{m} f_{u_j u_{j'}}\cdot \Delta u_j\cdot \Delta u_{j'}, \tag{5}$$

where we denoted

$$f_{x_i}\stackrel{\rm def}{=}\frac{\partial f}{\partial x_i}(\widetilde{x},\widetilde{u}_j),\quad f_{u_j}\stackrel{\rm def}{=}\frac{\partial f}{\partial u_j}(\widetilde{x},\widetilde{u}),$$
$$f_{x_i x_{i'}}\stackrel{\rm def}{=}\frac{\partial^2 f}{\partial x_i\partial x_{i'}}(\widetilde{x},\widetilde{u}),\quad f_{x_i u_j}\stackrel{\rm def}{=}\frac{\partial^2 f}{\partial x_i\partial u_j}(\widetilde{x},\widetilde{u}),$$
$$f_{u_j u_{j'}}\stackrel{\rm def}{=}\frac{\partial^2 f}{\partial u_j\partial u_{j'}}(\widetilde{x},\widetilde{u}).$$

To find an explicit expression for the objective function (4), we need to find the maximum and the minimum of this objective function when u is fixed and $x_i\in[\underline{x}_i,\overline{x}_i]$, i.e., when $\Delta x_i\in[-\Delta_i,\Delta_i]$. To find the maximum and the minimum of a function of an interval, it is useful to compute its derivative. For the objective function (5), we have

$$\frac{\partial f}{\partial x_i}=f_{x_i}+\sum_{i'=1}^{n} f_{x_i x_{i'}}\cdot \Delta x_{i'}+\sum_{j=1}^{m} f_{x_i u_j}\cdot \Delta u_j. \tag{6}$$

In general, the value f_{x_i} is different from 0; possible degenerate cases when $f_{x_i}=0$ seem to be rare. On a simple example – when the state is described by a single quantity $x=x_1$ and the decision u is also described by a single quantity $u=u_1$ – let us explain why we believe that this degenerate case can be ignored.

4 Explaining Why, In General, We Have $f_{x_i}\neq 0$

Simple case. Let us assume that the state x is the difference $x=T-T_{\rm ideal}$ between the actual temperature T and the ideal temperature $T_{\rm ideal}$. In this case, $T=T_{\rm ideal}+x$.

Let u be the amount of degree by which we cool down the room. Then, the resulting temperature in the room is $T'=T-u=T_{\rm ideal}+x-u$. The difference $T'-T_{\rm ideal}$ between the resulting temperature T' and the ideal temperature $T_{\rm ideal}$ is thus equal to $x-u$.

It is reasonable to assume that the discomfort D depends on this difference d: $D=D(d)$. The discomfort is 0 when the difference is 0, and is positive when the difference is non-zero. Thus, if we expand the dependence $D(d)$ in Taylor series and keep only quadratic terms in this expansion $D(d) = d_0 + d_1 \cdot d + d_2 \cdot d^2$, we conclude that $d_0 = 0$, that $d_1 = 0$ (since the function $D(d)$ has a minimum at $d = 0$), and thus, that $D(d) = d_2 \cdot d^2 = d_2 \cdot (x - u)^2$, for some $d_2 > 0$.

So the utility – which is negative this discomfort – is equal to $f(x, u) = -d_2 \cdot (x - u)^2$.

In this case, for each state x, the exact-case optimal decision is $u^{\text{opt}}(x) = x$. Thus, at the point where $x = x_0$ and $u = u_0 = u^{\text{opt}}(x) = x_0$, we have

$$f_x = \frac{\partial f}{\partial x} = -2d_2 \cdot (x_0 - u_0) = 0.$$

So, we have exactly the degenerate case that we were trying to avoid.

Let us make the description of this case slightly more realistic. Let us show that if we make the description more realistic, the derivative f_x is no longer equal to 0.

Indeed, in the above simplified description, we only took into account the discomfort of the user when the temperature in the room is different from the ideal. To be realistic, we need to also take into account that there is a cost $C(u)$ associated with cooling (or heating).

This cost is 0 when $u = 0$ and is non-negative when $u \neq 0$, so in the first approximation, similarly to how we described $D(d)$, we conclude that $C(u) = k \cdot u^2$, for some $k > 0$. The need to pay this cost decreases the utility function which now takes the form

$$f(x, u) = -d_2 \cdot (x - u)^2 - k \cdot u^2.$$

For this more realistic utility function, the value $u^{\text{opt}}(x)$ that maximizes the utility for a given x can be found if we differentiate the utility function with respect to u and equate the derivative to 0. Thus, we get

$$2d_2 \cdot (u - x) + 2k \cdot u = 0,$$

hence $(d_2 + k) \cdot u - d_2 \cdot x = 0$, and

$$u^{\text{opt}}(x) = \frac{d_2}{d_2 + k} \cdot x.$$

For $x = x_0$ and $u = u_0 = u^{\text{opt}}(x_0) = \frac{d_2}{d_2 + k} \cdot x_0$, we thus get

$$f_x = \frac{\partial f}{\partial x} = 2(x_0 - u_0) = 2\left(x_0 - \frac{d_2}{d_2 + k} \cdot x_0\right) = \frac{2k}{d_2 + k} \cdot x_0 \neq 0.$$

So, if we make the model more realistic, we indeed get a non-degenerate case $f_x \neq 0$ that we consider in our paper.

5 Analysis of the Problem (continued)

We consider the non-degenerate case, when $f_{x_i} \neq 0$. Since we assumed that all the differences Δx_i and Δu_j are small, a linear combination of these differences is smaller than $|f_{x_i}|$. Thus, for all values Δx_i from the corresponding intervals $\Delta x_i \in [-\Delta_i, \Delta_i]$, the sign of the derivative $\dfrac{\partial f}{\partial x_i}$ is the same as the sign $s_{x_i} \stackrel{\text{def}}{=} \operatorname{sign}(f_{x_i})$ of the midpoint value f_{x_i}.

Hence:

- when $f_{x_i} > 0$ and $s_{x_i} = +1$, the function $f(x, u)$ is an increasing function of x_i; its maximum is attained when x_i is attained its largest possible values $\overline{x}_i$, i.e., when $\Delta x_i = \Delta_i$, and its minimum is attained when $\Delta x_i = -\Delta_i$;
- when $f_{x_i} < 0$ and $s_{x_i} = -1$, the function $f(x, u)$ is an decreasing function of x_i; its maximum is attained when x_i is attained its smallest possible values $\underline{x}_i$, i.e., when $\Delta x_i = -\Delta_i$, and its minimum is attained when $\Delta x_i = \Delta_i$.

In both cases, the maximum of the utility function $f(x, u)$ is attained when $\Delta x_i = s_{x_i} \cdot \Delta_i$ and its minimum is attained when $\Delta x_i = -s_{x_i} \cdot \Delta_i$. Thus,

$$\max_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n, u) = f(\widetilde{x}_1 + s_{x_1} \cdot \Delta_1, \ldots, \widetilde{x}_n + s_{x_n} \cdot \Delta_n, \widetilde{u} + \Delta u) =$$

$$f(\widetilde{x}, \widetilde{u}) + \sum_{i=1}^{n} f_{x_i} \cdot s_{x_i} \cdot \Delta_i + \sum_{j=1}^{m} f_{u_j} \cdot \Delta u_j +$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{i'=1}^{n} f_{x_i x_{i'}} \cdot s_{x_i} \cdot s_{x_{i'}} \cdot \Delta_i \cdot \Delta_{i'} + \sum_{i=1}^{n} \sum_{j=1}^{m} f_{x_i u_j} \cdot s_{x_i} \cdot \Delta_i \cdot \Delta u_j +$$

$$\frac{1}{2} \cdot \sum_{j=1}^{m} \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta u_j \cdot \Delta u_{j'}, \tag{7}$$

and

$$\min_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n, u) = f(\widetilde{x}_1 - s_{x_1} \cdot \Delta_1, \ldots, \widetilde{x}_n - s_{x_n} \cdot \Delta_n, \widetilde{u} + \Delta u) =$$

$$f(\widetilde{x}, \widetilde{u}) - \sum_{i=1}^{n} f_{x_i} \cdot s_{x_i} \cdot \Delta_i + \sum_{j=1}^{m} f_{u_j} \cdot \Delta u_j +$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{i'=1}^{n} f_{x_i x_{i'}} \cdot s_{x_i} \cdot s_{x_{i'}} \cdot \Delta_i \cdot \Delta_{i'} - \sum_{i=1}^{n} \sum_{j=1}^{m} f_{x_i u_j} \cdot s_{x_i} \cdot \Delta_i \cdot \Delta u_j +$$

$$\frac{1}{2} \cdot \sum_{j=1}^{m} \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta u_j \cdot \Delta u_{j'}. \tag{8}$$

Therefore, our objective function (4) takes the form

$$\alpha \cdot \max_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n, u) + (1-\alpha) \cdot \min_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n, u) =$$

$$f(\widetilde{x}, \widetilde{u}) + (2\alpha - 1) \cdot \sum_{i=1}^{n} f_{x_i} \cdot s_{x_i} \cdot \Delta_i + \sum_{j=1}^{m} f_{u_j} \cdot \Delta u_j +$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{i'=1}^{n} f_{x_i x_{i'}} \cdot s_{x_i} \cdot s_{x_{i'}} \cdot \Delta_i \cdot \Delta_{i'} + (2\alpha - 1) \cdot \sum_{i=1}^{n} \sum_{j=1}^{m} f_{x_i u_j} \cdot s_{x_i} \cdot \Delta_i \cdot \Delta u_j +$$

$$\frac{1}{2} \cdot \sum_{j=1}^{m} \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta u_j \cdot \Delta u_{j'}. \quad (9)$$

To find the interval-case optimal value $\Delta u_j^{\max} = u_j - \widetilde{u}_j$ for which the objective function (4) attains its largest possible value, we differentiate the expression (9) for the objective function (4) with respect to Δu_j and equate the derivative to 0. As a result, we get:

$$f_{u_j} + (2\alpha - 1) \cdot \sum_{i=1}^{n} f_{x_i u_j} \cdot s_{x_i} \cdot \Delta_i + \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta u_{j'}^{\max} = 0. \quad (10)$$

To simplify this expression, let us now take into account that for each $x = (x_1, \ldots, x_n)$, the function $f(x, u)$ attains its maximum at the known value $u^{\rm opt}(x)$. Differentiating expression (5) with respect to Δu_j and equating the derivative to 0, we get:

$$f_{u_j} + \sum_{i=1}^{n} f_{x_i u_j} \cdot \Delta x_i + \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta^{\rm opt} u_{j'} = 0, \quad (11)$$

where we denoted $\Delta^{\rm opt} u_j \stackrel{\rm def}{=} u_j^{\rm opt} - u_j$.

For $x = \widetilde{x}$, i.e., when $\Delta x_i = 0$ for all i, this maximum is attained when $u = \widetilde{u}$, i.e., when $\Delta u_j = 0$ for all j. Substituting $\Delta x_i = 0$ and $\Delta u_j = 0$ into the formula (11), we conclude that $f_{u_j} = 0$ for all j. Thus, the formula (10) takes a simplified form

$$(2\alpha - 1) \cdot \sum_{i=1}^{n} f_{x_i u_j} \cdot s_{x_i} \cdot \Delta_i + \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta u_{j'}^{\max} = 0. \quad (12)$$

In general, we can similarly expand $u_j^{\rm opt}(x)$ in Taylor series and keep only a few first terms in this expansion:

$$u_j^{\rm opt}(x_1, \ldots, x_n) = u_j^{\rm opt}(\widetilde{x}_1 + \Delta x_1, \ldots, \widetilde{x}_n + \Delta x_n) = \widetilde{u}_j + \sum_{i=1}^{n} u_{j,x_i} \cdot \Delta x_i, \quad (13)$$

where we denoted $u_{j,x_i} \stackrel{\text{def}}{=} \dfrac{\partial u_j^{\text{opt}}}{\partial x_i}$. Thus, for the exact-case optimal decision,

$$\Delta u_j^{\text{opt}} = u_j^{\text{opt}}(x) - \widetilde{u}_j = \sum_{i=1}^{n} u_{j,x_i} \cdot \Delta x_i. \tag{14}$$

Substituting this expression for Δu_j^{opt} into the formula (11), we conclude that

$$\sum_{i=1}^{n} f_{x_i u_j} \cdot \Delta x_i + \sum_{i=1}^{n} \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot u_{j',x_i} \cdot \Delta x_i = 0,$$

i.e., that

$$\sum_{i=1}^{n} \Delta x_i \cdot \left(f_{x_i u_j} + \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot u_{j',x_i} \right) = 0$$

for all possible combinations of Δx_i. Thus, for each i, the coefficient at Δx_i is equal to 0, i.e.,

$$f_{x_i u_j} + \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot u_{j',x_i} = 0$$

for all i and j, so

$$f_{x_i u_j} = - \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot u_{j',x_i} = 0. \tag{15}$$

Substituting the expression (15) into the formula (12), we conclude that

$$-(2\alpha - 1) \cdot \sum_{i=1}^{n} \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot u_{j',x_i} \cdot (s_{x_i} \cdot \Delta_i) + \sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \Delta_{j'}^{\max} = 0,$$

i.e., that

$$\sum_{j'=1}^{m} f_{u_j u_{j'}} \cdot \left(\Delta u_{j'}^{\max} - (2\alpha - 1) \cdot \sum_{i=1}^{n} u_{j',x_i} \cdot (s_{x_i} \cdot \Delta_i) \right) = 0.$$

This equality is achieved when

$$\Delta u_{j'}^{\max} = (2\alpha - 1) \cdot \sum_{i=1}^{n} u_{j',x_i} \cdot (s_{x_i} \cdot \Delta_i) \tag{16}$$

for all j'. So, the interval-case optimal values $u_j^{\max} = \widetilde{u}_j + \Delta u_j^{\max}$ can be described as

$$u_j^{\max} = \widetilde{u}_j + (2\alpha - 1) \cdot \sum_{i=1}^{n} u_{j,x_i} \cdot (s_{x_i} \cdot \Delta_i). \tag{17}$$

In general, as we have mentioned earlier (formula (13)), we have

$$u_j^{\rm opt}(\widetilde{x}_1+\Delta x_1,\ldots,\widetilde{x}_n+\Delta x_n)=\widetilde{u}_j+\sum_{i=1}^n u_{j,x_i}\cdot\Delta x_i. \qquad (18)$$

By comparing the formulas (17) and (18), we can see that $u_j^{\rm max}$ is equal to $u_j^{\rm opt}(s)$ when we take $s_j=\widetilde{x}_j+(2\alpha-1)\cdot s_{x_i}\cdot\Delta_i$, i.e., that

$$u_j^{\rm max}=u_j^{\rm opt}(\widetilde{x}_1+(2\alpha-1)\cdot s_{x_1}\cdot\Delta_1,\ldots,\widetilde{x}_n+(2\alpha-1)\cdot s_{x_n}\cdot\Delta_n). \qquad (19)$$

Here, s_{x_i} is the sign of the derivative f_{x_i}. We have two options:

- If $f_{x_i}>0$, i.e., if the objective function increases with x_i, then $s_{x_i}=1$, and the expression $s_i\stackrel{\rm def}{=}\widetilde{x}_i+(2\alpha-1)\cdot s_{x_i}\cdot\Delta_i$ in the formula (19) takes the form

$$s_i=\frac{\underline{x}_i+\overline{x}_i}{2}+(2\alpha-1)\cdot\frac{\overline{x}_i-\underline{x}_i}{2}=\alpha\cdot\overline{x}_i+(1-\alpha)\cdot\underline{x}_i. \qquad (20)$$

- If $f_{x_i}<0$, i.e., if the objective function decreases with x_i, then $s_{x_i}=-1$, and the expression $s_i=\widetilde{x}_i+(2\alpha-1)\cdot s_{x_i}\cdot\Delta_i$ in the formula (19) takes the form

$$s_i=\frac{\underline{x}_i+\overline{x}_i}{2}-(2\alpha-1)\cdot\frac{\overline{x}_i-\underline{x}_i}{2}=\alpha\cdot\underline{x}_i+(1-\alpha)\cdot\overline{x}_i. \qquad (21)$$

So, we arrive at the following recommendation.

6 Solution to the Problem

Formulation of the problem: reminder. We assume that we know the objective function $f(x,u)$ that characterizes our gain in a situation when the actual values of the parameters are $x=(x_1,\ldots,x_n)$ and we select an alternative $u=(u_1,\ldots,u_m)$.

We also assume that for every state x, we know the exact-case optimal decision $u^{\rm opt}(x)$ for which the objective function attains its largest possible value.

In a practical situation in which we only know that each value x_i is contained in an interval $[\underline{x}_i,\overline{x}_i]$, we need to find the alternative $u^{\rm max}=(u_1^{\rm max},\ldots,u_m^{\rm max})$ that maximizes the Hurwicz combination of the best-case and worst-case values of the objective function.

Description of the solution. The solution to our problem is to use the exact-case optimal solution $u^{\rm opt}(s)$ corresponding to an appropriate state $s=(s_1,\ldots,s_n)$.

Here, for the variables x_i for which the objective function is an increasing function of x_i, we should select

$$s_i=\alpha\cdot\overline{x}_i+(1-\alpha)\cdot\underline{x}_i, \qquad (22)$$

where α is the optimism-pessimism parameter that characterizes the decision maker.

For the variables x_i for which the objective function is a decreasing function of x_i, we should select

$$s_i = \alpha \cdot \underline{x}_i + (1 - \alpha) \cdot \overline{x}_i. \quad (23)$$

Comment. Thus, the usual selection of the midpoint s is only interval-case optimal for decision makers for which $\alpha = 0.5$; in all other cases, this selection is *not* interval-case optimal.

Discussion. Intuitively, the above solution is in good accordance with the Hurwicz criterion:

- when the objective function increases with x_i, the best possible situation corresponds to $\overline{x}_i$, and the worst possible situation corresponds to $\underline{x}_i$; thus, the Hurwicz combination corresponds to the formula (22);
- when the objective function decreases with x, the best possible situation corresponds to $\underline{x}_i$, and the worst possible situation corresponds to $\overline{x}_i$; thus, the Hurwicz combination corresponds to the formula (23).

This intuitive understanding is, however, not a proof – Hurwicz formula combines utilities, not parameter values.

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