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Andrzej Pownuk
University of Texas at El Paso, ampownuk@utep.edu

Olga Kosheleva
University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu

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TOWARDS DECISION MAKING UNDER GENERAL UNCERTAINTY

Andrzej Pownuk
Ph.D. (Phys.-Math.), Instructor, e-mail: amponwuk@utep.edu
Olga Kosheleva
Ph.D. (Phys.-Math.), Associate Professor, e-mail: olgak@utep.edu
Vladik Kreinovich
Ph.D. (Phys.-Math.), Professor, e-mail: vladik@utep.edu

University of Texas at El Paso, El Paso, Texas 79968, USA

Abstract. There exist techniques for decision making under specific types of uncertainty, such as probabilistic, fuzzy, etc. Each of the corresponding ways of describing uncertainty has its advantages and limitations. As a result, new techniques for describing uncertainty appear all the time. Instead of trying to extend the existing decision making idea to each of these new techniques one by one, we attempt to develop a general approach that would cover all possible uncertainty techniques.

Keywords: decision making, general uncertainty, fuzzy uncertainty, probabilistic uncertainty.

1. Formulation of the Problem

Need for decision making under uncertainty. The ultimate goal of science and engineering is to make decisions, i.e., to select the most appropriate action.

Situations when we have full information about possible consequences of each action are rare. Usually, there is some uncertainty. It is therefore important to make decisions under uncertainty.

There are many different techniques for describing uncertainty. There are many different techniques for describing uncertainty: probabilistic, fuzzy (see, e.g., [4, 10, 12]), possibilistic, interval-valued or, more generally, type-2 fuzzy (see, e.g., [6, 7]), complex-valued fuzzy [2], etc. For many of these techniques, there are known methods for decision making under the corresponding uncertainty.

All the current techniques for describing uncertainty have their advantages and their limitations. Because of the known limitations, new – more adequate – techniques for describing uncertainty appear all the time. For each of these techniques, we need to understand how to make decisions under the corresponding uncertainty.

A problem that we try to solve in this paper. At present, this understanding mostly comes technique-by-technique. A natural question is: can we develop a general framework that would allow us to make decision under general uncertainty?

The main objective of this paper is to develop such a general formalism.

Towards a precise formulation of the problem. Let us start with a monetary problem. Suppose that we need to make a financial decision, such as investing a given amount of money in a certain financial instrument (such as shares or bonds).

If we knew the exact consequences of this action, then we would know exactly how much money we will have after a certain period of time. This happens, e.g., if we simply place the given amount in a saving account with a known interest rate.

In most situations, however, we are uncertainty of the possible financial consequences of this action. In other words, for each investment scheme, there are several possible consequences, with monetary amounts $x_1, \ldots, x_n$. By using an appropriate uncertainty technique, we can describe our degree of certainty that the i-th alternative is possible by the corresponding value μ_i. Depending on the formalism for describing uncertainty,

- a value μ_i can be a number – e.g., when we use probabilistic or fuzzy uncertainty,
- it can be an interval – when we use interval-valued fuzzy,
- it can be a complex number – if we use complex-valued fuzzy,
- it can be a fuzzy set – if we use type-2 fuzzy techniques, etc.

For another investment scheme, we can have n' different possible consequences, with monetary values $x'_1, \ldots, x'_{n'}$ and degrees of certainty $\mu'_1, \ldots, \mu'_{n'}$.

To make a decision, we need to compare this investment, in particular, with situations like placing money in a saving account, in which we simply get a fixed amount of money after the same period of time.

- If this fixed amount of money is too small, then investing in an uncertain financial instrument is clearly better.
- If this fixed amount of money is sufficiently large, then getting this fixed amount of money is clearly better than investing in an uncertain financial instrument.

There should be a threshold value of the fixed amount at which we go from the instrument being preferable to a fixed amount being preferable. This threshold fixed amount of money is thus equivalent, to the user, to the investment in an uncertain instrument.

So, for each uncertain investment, in which we get:

- the amount x_1 with degree of possibility μ_1,
- the amount x_2 with degree of possibility μ_2,
- $\ldots$,
- amount x_n with degree of possibility μ_n,

we have an equivalent amount of money. We will denote this equivalent amount of money by $f(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n)$.

Our goal is to find out how this equivalent amount of money depends on the values x_i and μ_i. Once we know the equivalent amount of money corresponding to each uncertain investment, we will be able to select the best of the possible investments: namely, it is natural to select the investment for which the corresponding equivalent amount of money is the largest possible.

What about non-financial decision making situations? It is known (see, e.g., [3, 5, 8, 11]) that decisions of a rational person can be described as optimizing a certain quantity called *utility*.

Thus, in general, we have the following problem: for each uncertain situation, in which we get:

- utility x_1 with degree of possibility μ_1,
- utility x_2 with degree of possibility μ_2,
- $\ldots$,
- utility x_n with degree of possibility μ_n,

we have an equivalent utility value. We will denote this equivalent utility value by $f(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n)$.

Our goal is thus to find out how this equivalent utility value depends on the values x_i and μ_i. Once we know the equivalent utility value corresponding to each possible decision, we will be able to select the best of the possible decisions: namely, it is natural to select the decision for which the corresponding equivalent utility value is the largest possible.

Comment. In the following text, to make our thinking as understandable as possible, we will most talk about financial situations – since it is easier to think about money than about abstract utilities. However, our reasoning is applicable to utilities as well.

2. Analysis of the Problem

First reasonable assumption: additivity. We are interested in finding a function $f(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n)$ of $2n$ variables.

Suppose that the money that we get from the investment comes in two consequent payments. In the i-th alternative, we first get the amount x_i, and then – almost immediately – we also get the amount y_i.

We can consider the resulting investment in two different ways. First, we can simply ignore the fact that the money comes in two installments, and just take into account that in each alternative i, we get the amount $x_i + y_i$. This way, the equivalent amount of money is equal to

$$f(x_1 + y_1, \ldots, x_n + y_n, \mu_1, \ldots, \mu_n).$$

Alternatively, we can treat both installments separately:

- in the first installment, we get x_i with uncertainty μ_i,
- in the second installment, we get y_i with uncertainty μ_i.

Thus:

- the first installment is worth the amount $f(x_1,\ldots,x_n,\mu_1,\ldots,\mu_n)$, and
- the second installment is worth the amount $f(y_1,\ldots,y_n,\mu_1,\ldots,\mu_n)$.

The overall benefit is the sum of the amounts corresponding to both installments. So, in this way of description, the overall money value of the original investment is equal to the sum of the money values of the two installments:

$$f(x_1,\ldots,x_n,\mu_1,\ldots,\mu_n)+f(y_1,\ldots,y_n,\mu_1,\ldots,\mu_n).$$

The equivalent benefit of the investment should not depend on the way we compute it, so the two estimates should be equal:

$$f(x_1+y_1,\ldots,x_n+y_n,\mu_1,\ldots,\mu_n)=$$

$$f(x_1,\ldots,x_n,\mu_1,\ldots,\mu_n)+f(y_1,\ldots,y_n,\mu_1,\ldots,\mu_n).$$

Functions satisfying this property are known as *additive*. Thus, we can say that for each combination of values $\mu_1,\ldots,\mu_n$, the dependence on $x_1,\ldots,x_n$ is additive.

Second reasonable assumption: bounds. No matter what happens, we get at least $\min\limits_i x_i$ and at most $\max\limits_i x_i$. Thus, the equivalent benefit of an investment cannot be smaller than $\min\limits_i x_i$ and cannot be larger than $\max\limits_i x_i$:

$$\min_i x_i\le f(x_1,\ldots,x_n,\mu_1,\ldots,\mu_n)\le\max_i x_i.$$

What we can conclude form the first two assumptions. It is known (see, e.g., [1]) that every bounded additive function is linear, i.e., that we have

$$f(x_1,\ldots,x_n,\mu_1,\ldots,\mu_n)=\sum_{i=1}^n c_i(\mu_1,\ldots,\mu_n)\cdot x_i.$$

So, instead of a function of $2n$ variables, we now have a simpler task for finding n functions $c_i(\mu_1,\ldots,\mu_n)$ of n variables.

Nothing should depend on the ordering of the alternatives. The ordering of the alternatives is arbitrary, so nothing should change if we change this ordering. For example, if we swap the first and the second alternatives, then instead of

$$c_1(\mu_1,\mu_2,\ldots)\cdot x_1+c_2(\mu_1,\mu_2,\ldots)\cdot x_2+\ldots$$

we should have

$$c_2(\mu_2,\mu_1,\ldots)\cdot x_1+c_1(\mu_2,\mu_1,\ldots)\cdot x_2+\ldots$$

These two expression must coincide, so the coefficients at x_1 must coincide, and we must have

$$c_2(\mu_2, \mu_1, \ldots) = c_1(\mu_1, \mu_2, \ldots).$$

In general, we should thus have

$$c_i(\mu_1, \ldots, \mu_n) = \mu_1(\mu_i, \mu_1, \ldots \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n).$$

Thus, the above expression should have the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_1(\mu_i, \mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n) \cdot x_i.$$

Now, the problem is to find a single function $c_1(\mu_1, \ldots, \mu_n)$ of n variables.

Combining alternatives with the same outcomes. Based on the above formula, the value $c_1(\mu_1, \mu_2, \ldots, \mu_n)$ corresponds to $f(1, 0, \ldots, 0)$, i.e., to a situation when we have:

- the value 1 with degree of possibility μ_1,
- the value 0 with degree of possibility μ_2,
- $\ldots$,
- the value 0 with degree of possibility μ_n.

In alternatives 2 through n, we have the same outcome 0, so it makes sense to consider them as a single alternative. To find the degree of possibility of this combined alternatives, we need to apply some "or"-operation $\oplus$ to the degrees of possibility $\mu_2, \ldots, \mu_n$ of individual alternatives.

For probabilities, this combination operation is simply the sum $a \oplus b = a + b$, for fuzzy, it is a t-conorm, etc. In general, the degree of certainty of the combined alternative is equal to $\mu_2 \oplus \ldots \oplus \mu_n$. Thus, the equivalent value of this situation is equal to $c_1(\mu_1, \mu_2 \oplus \ldots \oplus \mu_n)$. So, we have

$$c_1(\mu_1, \mu_2, \ldots, \mu_n) = c_1(\mu_1, \mu_2 \oplus \ldots \oplus \mu_n),$$

and the above expression for the equivalent benefit takes the following form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_1(\mu_i, \mu_1 \oplus \mu_{i-1} \oplus \mu_{i+1} \oplus \ldots \oplus \mu_n) \cdot x_i.$$

Now, the problem is to find a single function $c_1(\mu_1, \mu_2)$ of two variables.

Let us simplify this problem even further.

Yet another reasonable requirement. Let us consider a situation in which we have three alternatives, i.e., in which, we get:

- the amount x_1 with degree of possibility μ_1,

- the amount x_2 with degree of possibility μ_2, and
- the amount x_3 with degree of possibility μ_3.

According to the above formula, for this situation, the equivalent benefit is equal to

$$c_1(\mu_1, \mu_2 \oplus \mu_3) \cdot x_1 + c_1(\mu_2, \mu_1 \oplus \mu_3) \cdot x_2 + c_1(\mu_3, \mu_1 \oplus \mu_2) \cdot x_2.$$

On the other hand, we can consider an auxiliary situation A in which we get:

- the amount x_1 with degree of possibility μ_1 and
- the amount x_2 with the degree of possibility μ_2.

This situation is equivalent to the amount

$$x_A = c_1(\mu_1, \mu_2) \cdot x_1 + c_1(\mu_2, \mu_1) \cdot x_2,$$

and the degree of possibility of this auxiliary situation can be obtained by applying the corresponding "or"-operation to the degrees μ_1 and μ_2 and is thus, equal to $\mu_A = \mu_1 \oplus \mu_2$.

By replacing the first two alternatives in the original 3-alternative situation with the equivalent alternative, we get the equivalent situation, in which we get:

- the value x_A with degree of possibility μ_A and
- the value x_3 with degree of possibility μ_3.

For this equivalent situation, the equivalent amount is equal to

$$c_1(\mu_A, \mu_3) \cdot x_A + c_1(\mu_3, \mu_A) \cdot x_3.$$

Substituting the expressions for x_A and μ_A into this formula, we conclude that the equivalent amount is equal to

$$c_1(\mu_1 \oplus \mu_2, \mu_3) \cdot (c_1(\mu_1, \mu_2) \cdot x_1 + c_1(\mu_2, \mu_2) \cdot x_2) + c_1(\mu_3, \mu_1 \oplus \mu_2) \cdot x_3 =$$

$$c_1(\mu_1 \oplus \mu_2, \mu_3) \cdot c_1(\mu_1, \mu_2) \cdot x_1 + c_1(\mu_1 \oplus \mu_2, \mu_3) \cdot c_1(\mu_2, \mu_2) \cdot x_2 +$$
$$c_1(\mu_3, \mu_1 \oplus \mu_2) \cdot x_3.$$

We get two expressions for the same equivalent amount. These expressions must coincide. This means, in particular, that the coefficients at x_1 at both expressions must coincide, i.e., that we should have

$$c_1(\mu_1, \mu_2 \oplus \mu_3) = c_1(\mu_1 \oplus \mu_2, \mu_3) \cdot c_1(\mu_1, \mu_2).$$

What can we extract from this requirement. Let us consider an auxiliary function $c(a, b) \stackrel{\text{def}}{=} c_1(a, b \ominus a)$, where $b \ominus a$ is an inverse to $\oplus$, i.e., the value for which $a \oplus (b \ominus a) = b$.

By definition of the new operation $\ominus$, we have

$$b = (a \oplus b) \ominus b.$$

Thus, we have

$$c(a, a \oplus b) = c_1((a \oplus b) \ominus a) = c_1(a, b).$$

In other words, for every a and b, we have

$$c_1(a, b) = c(a, a \oplus b).$$

Substituting this expression for $c_1(a, b)$ into the above formula, we conclude that

$$c(\mu_1, \mu_1 \oplus \mu_2 \oplus \mu_3) = c(\mu_1 \oplus \mu_2, \mu_1 \oplus \mu_2 \oplus \mu_3) \cdot c_1(\mu_1, \mu_1 \oplus \mu_2).$$

The left-hand side depends only on two values $x \stackrel{\text{def}}{=} \mu_1$ and $z \stackrel{\text{def}}{=} \mu_1 \oplus u_2 \oplus \mu_3$, and does not depend on the value $y \stackrel{\text{def}}{=} \mu_1 \oplus \mu_3$:

$$c(x, z) = c(y, z) \cdot c(x, y).$$

Thus, if we fix some value y_0, we conclude that

$$c(x, z) = g(z) \cdot h(x),$$

where we denoted $g(z) \stackrel{\text{def}}{=} c(y_0, z)$ and $h(x) \stackrel{\text{def}}{=} c(x, y_0)$.

Describing $c_1(a, b)$ in terms of the auxiliary function $c(a, b)$, we can transform the expression for the equivalent monetary value to

$$\sum_{i=1}^{n} c(\mu_i, \mu_1 \oplus \ldots \oplus \mu_n) \cdot x_i.$$

Substituting the expression $c(x, z) = g(z) \cdot h(x)$ into this formula, we conclude that the equivalent monetary value takes the form

$$\sum_{i=1}^{n} h(\mu_i) \cdot g \cdot x_i,$$

where we denoted $g \stackrel{\text{def}}{=} g(\mu_1 \oplus \ldots \oplus \mu_n)$.

For the case when $x_1 = x_2 = \ldots = x_n$, the boundedness requirement implies that the equivalent value is equal to x_1. Thus, we have

$$x_1 = \sum_{i=1}^{n} h(\mu_i) \cdot g \cdot x_1.$$

Dividing both sides by x_1, we conclude that

$$1 = g \cdot \sum_{i=1}^{n} h(\mu_i)$$

and hence, that

$$g = \frac{1}{\sum\limits_{i=1}^{n} h(\mu_i)}.$$

So, the equivalent monetary value is equal to the following expression:

$$\frac{\sum\limits_{i=1}^{n} h(\mu_i) \cdot x_i}{\sum\limits_{i=1}^{n} h(\mu_i)}.$$

So, now we are down to a single unknown function $h(\mu)$.

3. Conclusions

General conclusion. We need to decide between several actions. For each action, we know the possible outcomes $x_1, \ldots, x_n$, and for each of these possible outcomes i, we know the degree of possibility μ_i of this outcome. The above analysis shows that the benefit of each action can then be described by the following formula

$$\frac{\sum\limits_{i=1}^{n} h(\mu_i) \cdot x_i}{\sum\limits_{i=1}^{n} h(\mu_i)},$$

for an appropriate function $h(\mu)$.

How can we find the function $h(\mu)$? If we have two alternatives with the same outcome $x_1 = x_2$, then we can:

- either treat them separately, leading to the terms

$$h(\mu_1) \cdot g(\mu_1 \oplus \mu_2 \oplus \ldots) \cdot x_1 + h(\mu_2) \cdot g(\mu_1 \oplus \mu_2 \oplus \ldots) \cdot x_1 + \ldots$$

- or treat them as a single alternative x_1, with degree of possibility $\mu_1 \oplus \mu_2$, thus leading to the term

$$h(\mu_1 \oplus \mu_2) \cdot g(\mu_1 \oplus \mu_2 \oplus \ldots) \cdot x_1.$$

These two expressions must coincide, so we must have

$$h(\mu_1 \oplus \mu_2) = h(\mu_1) + h(\mu_2).$$

Let us show, on two specific cases, what this leads to.

Probabilistic case. In this case, the values μ_i are probabilities, and as we have mentioned, we have $\mu_1 \oplus \mu_2 = \mu_1 + \mu_2$. So, the above condition takes the form

$$h(\mu_1 + \mu_2) = h(\mu_1) + h(\mu_2).$$

Thus, in the probabilistic case, the function $h(\mu)$ must be additive.

The higher probability, the more importance should be given to the corresponding alternative, so the function $h(\mu)$ should be monotonic. It is known (see, e.g., [1]) that every monotonic additive function is linear, so we must have $h(\mu) = c \cdot \mu$ for some constant μ. Thus, the above formula for the equivalent amount takes the form

$$\frac{\sum\limits_{i=1}^{n} c \cdot \mu_i \cdot x_i}{\sum\limits_{i=1}^{n} c \cdot \mu_i}.$$

For probabilities, $\sum\limits_{i=1}^{n} \mu_i = 1$. So, dividing both the numerator and the denominator by c, we conclude that the equivalent benefit has the form

$$\sum_{i=1}^{n} \mu_i \cdot x_i.$$

This is exactly the formula for the expected utility that appears when we consider the decision of rational agents under probabilistic uncertainty [3, 5, 8, 11].

Fuzzy case. In the fuzzy case, $a \oplus b$ is a t-conorm. It is known (see, e.g., [9]) that every t-comrm can be approximated, with arbitrary accuracy, by an Archimedean t-conorm, i.e., by a function of the type $G^{-1}(G(a) + G(b))$, where $G(a)$ is a strictly increasing continuous function and G^{-1} denotes the inverse function. Thus, from the practical viewpoint, we can safely assume that the actual t-conorm operation $a \oplus b$ is Archimedean:

$$a \oplus b = G^{-1}(G(a) + G(b)).$$

In this case, the condition $a \oplus b = c$ is equivalent to

$$G(a) + G(b) = G(c).$$

The requirement that

$$h(\mu_1 \oplus \mu_2) = h(\mu_1) + h(\mu_2)$$

means that if $a \oplus b = c$, then

$$h(a) + h(b) = h(c).$$

In other words, if $G(a) + G(b) = G(c)$, then

$$h(a) + h(b) = h(c).$$

If we denote $A \stackrel{\text{def}}{=} G(a)$, $B \stackrel{\text{def}}{=} G(b)$, and $C \stackrel{\text{def}}{=} h(c)$, then $a = G^{-1}(A)$, $b = G^{-1}(B)$, $c = G^{-1}(C)$, and the above requirement takes the following form: if $A+B = C$, then

$$h(G^{-1}(A)) + h(G^{-1}(B)) = h(G^{-1}(C)).$$

So, for the auxiliary function $H(A) \stackrel{\text{def}}{=} h(G^{-1}(A))$, we have $A+B=c$ implying that $H(C)=H(A)+H(B)$, i.e., that $H(A+B)=H(A)+H(B)$. The function $H(A)$ is monotonic and additive, hence $H(A)=k\cdot A$ for some constant k.

So, $H(A)=h(G^{-1}(A))=k\cdot A$. Substituting $A=G(a)$ into this formula, we conclude that

$$h(G^{-1}(G(a))=h(a)=k\cdot G(a).$$

Thus, in the fuzzy case, the equivalent monetary value of each action is equal to

$$\frac{\sum\limits_{i=1}^{n} k\cdot G(\mu_i)\cdot x_i}{\sum\limits_{i=1}^{n} k\cdot G(\mu_i)}.$$

Dividing both the numerator and the denominator by the constant k, we get the final formula

$$\frac{\sum\limits_{i=1}^{n} G(\mu_i)\cdot x_i}{\sum\limits_{i=1}^{n} G(\mu_i)},$$

where $G(a)$ is a "generating" function of the t-conorm, i.e., a function for which the t-conorm has the form

$$G^{-1}(G(a)+G(b)).$$

Fuzzy case: example. For example, for the algebraic sum t-conorm

$$a\oplus b=a+b-a\cdot b,$$

we have

$$1-a\oplus b=(1-a)\cdot(1-b)$$

and thus,

$$-\ln(1-a\oplus b)=(-\ln(1-a))+(-\ln(1-b)),$$

so we have $G(a)=-\ln(1-a)$.

Thus, the formula for the equivalent amount takes the form

$$\frac{\sum\limits_{i=1}^{n} \ln(1-\mu_i)\cdot x_i}{\sum\limits_{i=1}^{n} \ln(1-\mu_i)}.$$

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