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# Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets? Fuzzy Sets And/Or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?

Christian Servin, Gerardo Muela, and Vladik Kreinovich

**Abstract** Fuzzy sets are naturally ordered by the subsethood relation  $A \subseteq B$ . If we only know which set which fuzzy set is a subset of which – and have no access to the actual values of the corresponding membership functions – can we detect which fuzzy sets are crisp? In this paper, we show that this is indeed possible. We also show that if we start with interval-valued fuzzy sets, then we can similarly detect type-1 fuzzy sets and crisp sets.

## 1 Formulation of the Problem

**Fuzzy sets: a brief reminder.** A *fuzzy set* is usually defined as a function  $\mu : U \rightarrow [0, 1]$  from some set  $U$  (called *Universe of discourse*) to the interval  $[0, 1]$ ; see, e.g., [1, 2, 3]. This function is also known as a *membership function*.

A fuzzy set  $A$  with a membership function  $\mu_A(x)$  is called a *subset* of a fuzzy set  $B$  with a membership function  $\mu_B(x)$  if  $\mu_A(x) \leq \mu_B(x)$  for all  $x$ . The subsethood relation is an *order* in the sense that it is reflexive ( $A \subseteq A$ ), asymmetric ( $A \subseteq B$  and  $B \subseteq A$  imply  $A = B$ ), and transitive ( $A \subseteq B$  and  $B \subseteq C$  imply  $A \subseteq C$ ).

Traditional (*crisp*) sets  $S$  can be viewed as particular cases of fuzzy sets, with their characteristic functions playing the role of membership functions:  $\mu_S(x) = 1$  if  $x \in S$  and  $\mu_S(x) = 0$  if  $x \notin S$ .

**A natural question: can we detect crisp sets based only on the subsethood ordering of fuzzy sets?** If we have a class  $F$  of all fuzzy sets, and for each fuzzy

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set  $A$  and for each element  $x \in U$ , we know the value  $\mu_A(x)$  of the corresponding membership function, then we can easily detect which of the fuzzy sets are crisp: a fuzzy set is crisp if for every  $x \in U$ , we have either  $\mu_A(x) = 0$  or  $\mu_A(x) = 1$ .

Suppose now that we have a class  $F$  of all fuzzy sets with the subsethood ordering  $A \subseteq B$  – but we have no access to the actual values of the corresponding membership functions. Based only on this ordering relation  $A \subseteq B$ , can we then detect crisp sets?

**What if we only consider interval-valued fuzzy sets.** A similar question can be asked if we consider interval-valued fuzzy sets, for which the value of the membership function is a subinterval of the interval  $[0, 1]$ :  $\mu(x) = [\underline{\mu}(x), \overline{\mu}(x)] \subseteq [0, 1]$ , and  $A \subseteq B$  means that  $\underline{\mu}_A(x) \leq \underline{\mu}_B(x)$  and  $\overline{\mu}_A(x) \leq \overline{\mu}_B(x)$  for all  $x$ .

**What we do in this paper.** In this paper, we prove that in both cases – when we consider fuzzy sets and when we consider interval-valued fuzzy sets – we can indeed detect crisp sets and type-1 fuzzy sets based only on the subsethood relation  $A \subseteq B$ .

## 2 What If We Consider $[0, 1]$ -Based Fuzzy Sets

**Our plan.** To describe crisp sets in terms of the subsethood relation  $A \subseteq B$ , we will follow the following four steps:

- first, we will prove that the empty set  $\emptyset$  can be uniquely determined based on the subsethood relation;
- second, we will show that 1-element crisp sets, i.e., sets of the type  $\{x_0\}$ , can be thus determined,
- third, we will prove that 1-element fuzzy sets, i.e., fuzzy sets  $A$  for which for some  $x_0 \in U$ , we have  $\mu_A(x_0) > 0$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ , can be determined based on the subsethood relation, and
- finally, we prove that crisp sets can be uniquely determined based on the subsethood relation.

**First step: how to detect an empty set?** An empty set  $\emptyset$  is a fuzzy set for which  $\mu_\emptyset(x) = 0$  for all  $x \in U$ . The detection of an empty set can be made based on the following simple result:

**Proposition 1.** *A fuzzy set  $A$  is an empty set if and only if  $A \subseteq B$  for all fuzzy sets  $B$ .*

**Proof.**

1°. Let us first prove that when  $A = \emptyset$ , then  $A \subseteq B$  for all fuzzy sets  $B$ .

Indeed, for every fuzzy set  $B$ , we have  $0 \leq \mu_B(x)$  for all  $x$  and thus,  $\mu_\emptyset(x) = 0 \leq \mu_B(x)$  for all  $x$ , i.e., we indeed have  $\emptyset \subseteq B$ .

2°. Let us now prove that, vice versa, if for some fuzzy set  $A$ , we have  $A \subseteq B$  for every possible fuzzy set  $B$ , then  $A = \emptyset$ .

Indeed, in particular, the property  $A \subseteq B$  is true for the case when  $B$  is the empty set. In this case, from the fact that  $\mu_A(x) \leq \mu_B(x) = \mu_\emptyset(x) = 0$ , we conclude that  $\mu_A(x) = 0$  for all  $x$ , i.e., that  $A$  is indeed the empty set.

The proposition is proven.

**Second step: how to detect 1-element crisp sets based on the subsethood relation.** Let us prove the following auxiliary result.

**Proposition 2.** *A non-empty fuzzy set  $A$  is a one-element crisp set if and only if the following two conditions are satisfied:*

- *the class  $\{B : B \subseteq A\}$  is linearly ordered and*
- *for no proper superset  $A'$  of  $A$ , the class  $\{B : B \subseteq A'\}$  is linearly ordered.*

**Proof.**

1°. Let us first prove that every 1-element crisp set, i.e., every set of the type  $A = \{x_0\}$ , satisfies the above two properties.

1.1°. Let us prove the first property: that the class  $\{B : B \subseteq A\}$  is linearly ordered.

Indeed, for the given set  $A$ , we have  $\mu_A(x_0) = 1$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ . So, if  $B \subseteq A$ , i.e., if  $\mu_B(x) \leq \mu_A(x)$  for all  $x$ , this means that  $\mu_B(x) = 0$  for all  $x \neq x_0$ . Thus, for such sets  $B$ , the only non-zero value of the membership function may be attained when  $x = x_0$ .

So, if we have two sets  $B \subseteq A$  and  $B' \subseteq A$ , then for these two sets,  $\mu_B(x) = \mu_{B'}(x) = 0$  for all  $x \neq x_0$ . Thus:

- if  $\mu_B(x_0) \leq \mu_{B'}(x_0)$ , then, as one can easily check, we have  $\mu_B(x) \leq \mu_{B'}(x)$  for all  $x$ , i.e. we have  $B \subseteq B'$ , and
- if  $\mu_{B'}(x_0) \leq \mu_B(x_0)$ , then, as one can easily check, we have  $\mu_{B'}(x) \leq \mu_B(x)$  for all  $x$ , we have  $B' \subseteq B$ .

Thus, for every two fuzzy sets  $B$  and  $B'$  from the class  $\{B : B \subseteq A\}$ , we have either  $B \subseteq B'$  or  $B' \subseteq B$ . So, this class is indeed linearly ordered.

1.2°. Let us now prove that no proper superset  $A'$  of the 1-element set  $A = \{x_0\}$  has the property that the class  $\{B : B \subseteq A'\}$  is linearly ordered.

For the set  $A = \{x_0\}$ , we have  $\mu_A(x_0) = 1$  and  $\mu_A(x) = 0$  for all other  $x$ . If  $A'$  is a superset of  $A$ , this means that  $\mu_{A'}(x) = 1$ . The fact that  $A'$  is a proper superset means that  $A' \neq A$ , thus we have  $\mu_{A'}(x') > 0$  for some  $x' \neq x_0$ . In this case, we can define the following fuzzy set  $B$ :  $\mu_B(x') = \mu_{A'}(x')$  and  $\mu_B(x) = 0$  for all  $x \neq x_0$ . Then, we have  $B \subseteq A'$ ,  $A \subseteq A'$ , but  $B \not\subseteq A$  (since  $\mu_B(x') > 0$  and thus,  $\mu_B(x') \not\leq \mu_A(x') = 0$ ) and  $A \not\subseteq B$  (since  $1 = \mu_A(x_0) \not\leq \mu_B(x_0) = 0$ ). Thus, the class  $\{B : B \subseteq A'\}$  is indeed not linearly ordered.

2°. Let us prove that, vice versa, if a fuzzy set  $A$  has the above two properties, then it is a one-element crisp set.

2.1°. Let us first prove, by contradiction, that we can only have one element  $x$  for which  $\mu_A(x) > 0$ . Indeed, if  $\mu_A(x_1) > 0$  and  $\mu_A(x_2) > 0$  for some  $x_1 \neq x_2$ , then we can take the following fuzzy sets  $B_1$  and  $B_2$ :

- $\mu_{B_1}(x_1) = \mu_A(x_1)$  and  $\mu_{B_1}(x) = 0$  for all other  $x$ , and
- $\mu_{B_2}(x_2) = \mu_A(x_2)$  and  $\mu_{B_2}(x) = 0$  for all other  $x$ .

Here,  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_2 \not\subseteq B_1$  and  $B_1 \not\subseteq B_2$  – which contradicts to our assumption that the class  $\{B : B \subseteq A\}$  is linearly ordered.

2.2°. Due to Part 2.1, we have  $\mu_A(x_0) > 0$  for at most one element  $x_0$ ; for all  $x \neq x_0$ , we have  $\mu_A(x) = 0$ . Let us prove, by contradiction, that  $\mu_A(x_0) = 1$ , i.e., that  $A$  is indeed a one-element crisp set.

Indeed, if  $\mu_A(x_0) < 1$ , then we can consider the following proper superset  $A' \supseteq A$ :  $\mu_{A'}(x_0) = (1 + \mu_A(x_0))/2 < 1$  and  $\mu_{A'}(x) = 0$  for all other  $x$ . Similarly to Part 1.1 of this proof, we can prove that for this superset  $A'$ , the class  $\{B : B \subseteq A'\}$  is linearly ordered – which contradicts to our assumption that such a proper superset does not exist.

The proposition is proven.

**Third step: how to detect 1-element fuzzy sets based on the subethood relation.**

We say that a fuzzy set is a *1-element set* if for some  $x_0 \in X$ , we have  $\mu_A(x_0) > 0$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ . Let us prove the following auxiliary result.

**Proposition 3.** *A non-empty fuzzy set  $A$  is a one-element fuzzy set if and only if the class  $\{B : B \subseteq A\}$  is linearly ordered.*

**Proof.**

1°. Arguments similar to Part 1.1 of the proof of Proposition 2 show that if  $A$  is a one-element fuzzy set, then the class  $\{B : B \subseteq A\}$  is linearly ordered.

2°. Vice versa, if  $A$  is not an empty set and not a one-element fuzzy set, this means that there exist at least two values  $x_1 \neq x_2$  for which  $\mu_A(x_1) > 0$  and  $\mu_A(x_2) > 0$ . We can then take the following fuzzy sets  $B_1$  and  $B_2$ :

- $\mu_{B_1}(x_1) = \mu_A(x_1)$  and  $\mu_{B_1}(x) = 0$  for all  $x \neq x_1$ , and
- $\mu_{B_2}(x_2) = \mu_A(x_2)$  and  $\mu_{B_2}(x) = 0$  for all  $x \neq x_2$ .

Then  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ . Thus, the class  $\{B : B \subseteq A\}$  is not linearly ordered.

The proposition is proven.

**Final result: how to detect crisp sets based on the subethood relation.** Let us prove the following auxiliary result.

**Theorem 1.** *A fuzzy set  $A$  is crisp if and only if every one-element fuzzy subset  $B \subseteq A$  can be embedded in a one-element crisp subset of  $A$ .*

*Comment.* In other words,

$$A \text{ is crisp} \Leftrightarrow \forall A (B \text{ is a one-element fuzzy subset of } A \Rightarrow \exists C ((B \subseteq C \subseteq A) \& (C \text{ is a 1-element crisp set}))).$$

**Proof.**

1°. Let  $A$  be a crisp set, and let  $B \subseteq A$  be a 1-element fuzzy set. By definition, this means that for some  $x_0$ , we have  $\mu_B(x_0) > 0$  and  $\mu_B(x) = 0$  for all other  $x$ .

Since the set  $A$  is crisp, the only possible values of  $\mu_A(x_0)$  are 0 and 1. From  $\mu_B(x_0) \leq \mu_A(x_0)$ , we conclude that  $\mu_A(x_0) > 0$  and thus, that  $\mu_A(x_0) = 1$ . So,  $x_0 \in A$  and hence  $B \subseteq \{x_0\} \subseteq A$ .

2°. Vice versa, if  $A$  is not a crisp set, this means that for some element  $x_0$ , we have  $0 < \mu_A(x_0) < 1$ . In this case, we can take the following 1-element fuzzy set  $B \subseteq A$ :  $\mu_B(x_0) = \mu_A(x_0)$  and  $\mu_B(x) = 0$  for all  $x \neq x_0$ . Here,  $B \subseteq A$ , but the only 1-element crisp set  $C$  containing  $B$  is the set  $C = \{x_0\}$ , and this 1-element crisp set is *not* a subset of the original set  $A$ :  $C \not\subseteq A$ .

The theorem is proven.

### 3 What If We Consider Interval-Valued Fuzzy Sets

**First step: how to detect an empty set.** An empty set  $\emptyset$  is an interval-valued fuzzy set for which  $\mu_\emptyset(x) = [0, 0]$  for all  $x \in U$ . The detection of an empty set can be made based on the following result:

**Proposition 4.** *An interval-valued fuzzy set  $A$  is an empty set if and only if  $A \subseteq B$  for all interval-valued fuzzy sets  $B$ .*

**Proof** is similar to proof of Proposition 1.

**Second step: how to detect special 1-element interval-valued fuzzy sets based on the subsethood relation.** Let's introduce an auxiliary notion. We say that an interval-valued fuzzy set  $A$  is *special* if for some element  $x_0$ , we have  $\mu_A(x_0) = [0, a]$  for some number  $a > 0$  and  $\mu_A(x) = [0, 0]$  for all  $x \neq x_0$ .

**Proposition 5.** *A non-empty interval-valued fuzzy set  $A$  is special if and only if the class  $\{B : B \subseteq A\}$  is linearly ordered.*

**Proof.**

1°. For special sets (in the sense of the above definition), the fact that the class  $\{B : B \subseteq A\}$  is linearly ordered can be proven similarly to Part 1.1 of the proof of Proposition 2.

2°. Let us now prove that, vice versa, if for some non-empty interval-valued fuzzy set  $A$ , the class  $\{B : B \subseteq A\}$  is linearly ordered, then the set  $A$  is special.

2.1°. Since  $A$  is non-empty, there exists an element  $x_0$  for which  $\mu_A(x_0) \neq [0, 0]$ . Let us prove, by contradiction, that for every other element  $x \neq x_0$ , we have  $\mu_A(x) = [0, 0]$ .

Indeed, if we had  $\mu_A(x_1) \neq [0, 0]$  for some  $x_1 \neq x_0$ , then we would be able to take the following two sets  $B_0$  and  $B_1$ :

- $\mu_{B_0}(x_0) = \mu_A(x_0)$  and  $\mu_{B_0}(x) = [0, 0]$  for all  $x \neq x_0$ , and

- $\mu_{B_1}(x_1) = \mu_A(x_1)$  and  $\mu_{B_1}(x) = [0, 0]$  for all  $x \neq x_1$ .

In this case,  $B_0 \subseteq A$  and  $B_1 \subseteq A$ , but  $B_0 \not\subseteq B_1$  and  $B_1 \not\subseteq B_0$ . This contradicts our assumption that the class  $\{B : B \subseteq A\}$  is linearly ordered.

2.2°. To complete the proof of the proposition, we need to prove that the value  $\mu_A(x_0) = [\underline{\mu}_A(x_0), \overline{\mu}_A(x_0)]$  has the form  $[0, a]$  for some  $a > 0$ , i.e., that  $\underline{\mu}_A(x_0) = 0$ .

We will prove it by contradiction. Suppose that, vice versa,  $\underline{\mu}_A(x_0) > 0$ . In this case, we can take the following sets  $B_1$  and  $B_2$ :

- $\mu_{B_1}(x_0) = [0.5 \cdot \underline{\mu}_A(x_0), 0.5 \cdot \underline{\mu}_A(x_0)]$  and  $\mu_{B_1}(x) = 0$  for all  $x \neq x_0$ , and
- $\mu_{B_2}(x_0) = [0, \underline{\mu}_A(x_0)]$  and  $\mu_{B_2}(x) = 0$  for all  $x \neq x_0$ .

Then,  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ . This contradicts our assumption that the class  $\{B : B \subseteq A\}$  is linearly ordered.

The proposition is proven.

**Third step: how to detect 1-element type-1 fuzzy sets based on the subsethood relation.** We say that an interval-valued fuzzy set is a *1-element type-1* fuzzy set if there exists an element  $x_0$  for which  $\mu_A(x_0) = [a, a]$  for some  $a > 0$  and  $\mu_A(x) = [0, 0]$  for all  $x \neq x_0$ .

**Proposition 6.** *A non-empty interval-valued fuzzy set  $A$  is a 1-element type-1 set if and only if it satisfies the following three properties:*

- *the set  $A$  is not special (in the sense of the above definition),*
- *there exists a special set  $B \subseteq A$  for which the class  $\{C : B \subseteq C \subseteq A\}$  is linearly ordered, and*
- *for no proper superset  $A'$  of  $A$ , the class  $\{C : B \subseteq C \subseteq A'\}$  is linearly ordered.*

**Proof** is similar to the proof of Proposition 2.

**Final result.** Since we have subsethood, we also have union: the union of  $A_\alpha$  is the  $\subseteq$ -smallest set that contains all  $A - \alpha$ . We can thus define type-1 fuzzy sets as unions of 1-element type-1 fuzzy sets. Once we can detect type-1 fuzzy sets, we can use techniques from the previous section to detect crisp sets. Thus, *we can indeed detect type-1 fuzzy sets and crisp sets based only on subsethood relation between interval-valued fuzzy sets.*

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