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Need for a Large-N Array (and Wavelets and Differences) to Determine the Assumption-free 3-D Earth Model

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Abstract

One of the main objectives of geophysical seismic analysis is to determine the Earth's structure. Usually, to determine this structure, geophysicists supplement the measurement results with additional geophysical assumptions. An important question is: when is it possible to reconstruct the Earth's structure uniquely based on the measurement results only, without the need to use any additional assumptions? In this paper, we show that for this, one needs to use large-N arrays – 2-D arrays of seismic sensors. To actually perform this reconstruction, we need to use differences between measurements by neighboring sensor and we need to apply wavelet analysis to the corresponding seismic signals.

1 Formulation of the Problem

The main objective of geophysical seismic analysis is to determine the mechanical properties of the Earth based on the observed seismic waves – waves generated both by actual earthquakes and by test explosions; see, e.g., [1].

Usually, in addition to data, we use a large number of additional assumptions [1]. A fundamental question is: when can we reconstruct the mechanical structure directly from data, without the need to involve additional geophysical assumptions?

2 Wave Propagation is Isotropic Solids: Reminder

Seismic waves propagate in the solid matter; most of the Earth materials can be safely assumed to be isotropic. Thus, the propagation of seismic waves can be described by the equations of wave propagation in isotropic solids.

In general, a wave in a solid body is described by the displacement u_i of each particle in comparison to this particle's original position. The change of displacement with time is described by the Newton's second law, which, in this case, takes the form

$$\rho \cdot \ddot{u}_i = \sigma_{ji,j}, \tag{1}$$

where:

- $\dot{a}$ means time derivative $\dot{a} \stackrel{\text{def}}{=} \dfrac{\partial a}{\partial t}$,
- $a_{,j}$ means derivative over the j-th spatial coordinate $a_{,j} \stackrel{\text{def}}{=} \dfrac{\partial a}{\partial x_j}$,
- σ_{ij} is the Cauchy's stress tensor (describing force per unit area), and
- repeating index means summation, so that, e.g., $\sigma_{ji,j} \stackrel{\text{def}}{=} \sum\limits_{j=1}^{3} \sigma_{ji,j}$.

When we simply shift or rotate the solid body as a whole, no stress appears. Stress is caused by changes that go beyond shifts and rotations. Such changes are described by the *strain tensor*

$$\varepsilon_{ij} \stackrel{\text{def}}{=} \frac{1}{2} \cdot (u_{i,j} + u_{j,i}). \tag{2}$$

In general – with the exception large displacements – we can safely assume that the dependence of the stress on strain is linear. This assumption constitutes a well-known Hooke's law:

$$\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{kl}. \tag{3}$$

The set of the corresponding coefficients C_{ijkl} are known as the *stiffness tensor*.

In the isotopic case, the stiffness tensor takes the form

$$C_{ijkl} = K \cdot \delta_{ij} \cdot \delta_{kl} + \mu \cdot \left(\delta_{ik} \cdot \delta_{jl} + \delta_{il} \cdot \delta_{jk} - \frac{2}{3} \cdot \delta_{ij} \cdot \delta_{kl} \right), \tag{4}$$

where $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

The coefficient K is known as the *bulk modulus*, and μ is known as the *shear modulus*. Substituting the formula (4) into the expression (3), we conclude that

$$\sigma_{ij} = \lambda \cdot \delta_{ij} \cdot \varepsilon_{kk} + 2\mu \cdot \varepsilon_{ij}, \tag{5}$$

where $\lambda \stackrel{\text{def}}{=} K - \dfrac{2}{3} \cdot \mu$ is known as *Lamé's first parameter*. Due to (2), we have

$$\sigma_{ij} = \lambda \cdot \delta_{ij} \cdot u_{k,k} + \mu \cdot (u_{i,j} + u_{j,i}). \tag{6}$$

Substituting the expression (6) for ε_{ij} into the formula (1), we get the final equation:

$$\rho \cdot \ddot{u}_i = \lambda_{,i} \cdot u_{k,k} + \lambda \cdot u_{k,ki} + \mu_{,j} \cdot (u_{i,j} + u_{j,i}) + \mu \cdot (\mu_{i,jj} + u_{j,ij}). \qquad (7)$$

Comment. Instead of λ and μ, practitioners often use *Yound's modulus*

$$E \stackrel{\text{def}}{=} \frac{\mu \cdot (2\lambda + 2\mu)}{\lambda + \mu} \qquad (8)$$

and *Poisson ratio*

$$\nu \stackrel{\text{def}}{=} \frac{\lambda}{2(\lambda + \mu)}. \qquad (9)$$

Vice versa, if we know the Young's modulus K and the Poisson's ratio ν, we can determine λ and μ by using the following formulas:

$$\lambda = \frac{E \cdot \nu}{(1 + \nu) \cdot (1 - 2\nu)}, \quad \mu = \frac{E}{2 + 2\nu}. \qquad (10)$$

3 Analysis of the Problem and the Resulting Recommendation

What are the unknowns. The equation (7) involves three quantities: the density ρ and the Lamé's parameters λ and μ. In general, the values of each of these three parameters ρ, λ, and μ changes from one spatial location to another. Thus, to fully determine the mechanical properties of the Earth, we need to find three functions of three spatial variables: $\rho(x, y, z)$, $\lambda(x, y, z)$, and $\mu(x, y, z)$. These three functions are the unknowns.

Comment. In geophysics, it is often assumed that the density ρ can be described in terms of the P-wave velocity V_P as $\rho = \alpha \cdot V_p^{\beta}$, where $\beta = 0.25$. The velocity V_P can be described in terms of λ and μ as

$$V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \qquad (11)$$

This relation was first discovered by G. H. F. Gardner and L. W. Gardner in [5] and is thus known as the *Gardner's relation.*

Due to this relation, we conclude that

$$\rho = \alpha \cdot \left(\frac{\lambda + 2\mu}{\rho} \right)^{1/8}, \qquad (12)$$

hence $\rho^{9/8} = \alpha \cdot (\lambda + 2\mu)^{1/8}$ and

$$\rho = \alpha^{8/9} \cdot (\lambda + 2\mu)^{1/9}. \qquad (13)$$

Thus, if we assume Gardner's relation, the density ρ can be described in terms of λ and μ, so we get only two unknown functions of three variables.

In general, when can we uniquely reconstruct all the unknowns from measurements? Each measurement result is a relation between the unknown quantities. Thus, after n measurements, we have n equations to reconstruct all the unknown quantities.

It is is known that to be able to uniquely determine all the unknowns, we need to have at least as many equations as there are unknowns – otherwise, the corresponding system of equations will be under-determined.

Let us apply this general idea to our case.

How many measurements do we need to uniquely determine all the desired mechanical characteristics. If we assume that each spatial variable has N possible values, then we have N^3 possible combinations of three spatial variables. Thus, to uniquely reconstruct a function of three variables, we need to determine N^3 values. To uniquely reconstruct two or three functions of three variables, we thus need to have $k \cdot N^3$ values, where k is either 2 or 3.

So, to uniquely reconstruct two or three functions of three variables, we thus need to have at least $k \cdot N^3$ measurements.

For an area of 100 km size, we would like to have information with the accuracy of at least 1 km – and ideally, even better. Thus, we have $N \approx 100$.

What information do we have to determine these unknowns and why this is usually not enough. If we have a single sensor, then we measure the displacement u_i (or strain ε_{ij}) at the location of this sensor at different moments of time. As a result, we get a function of one variable – time. By knowing the value of this function at all N moments of time, we thus get N measurement results.

After we observe several (s) seismic events, we get several functions of one variable, so we get $s \cdot N$ measurement results. It is easy to see that even when we have observed dozens of earthquakes, we still have $s \cdot N \ll k \cdot N^3$. Thus, having a single sensor is not enough to uniquely reconstruct the mechanical structure of the Earth – we need additional geophysical assumptions.

In seismic experiments, when we set an artificial explosion and measure the seismic waves generated by this explosion, we usually set up a large number of sensors along a line. These sensors come closely after one another, so, in effect, what we have after the measurements is a function of *two* variables:

- one variable is the time, and
- another variable is the distance along this line.

As a result, after each measurement, we get N^2 values. This is much larger than $s \cdot N$, but still much smaller than the desired number $k \cdot N^3$. Thus, with the traditional seismic experiments, we are still unable to uniquely determine the Earth's structure.

Need to large-N arrays. To uniquely determine the Earth's structure, we need the measurement results to form a function of three variables. Since each

sensor provided a function of two variables, we thus need to have sensor forming a 2-D structure. In other words, we need sensors forming not a 1-D array but a 2-D array.

Such an arrangement has indeed been recently actively used under the name of the *large-N array*; see, e.g., [4, 6] and references therein. Thus, large-N arrays are indeed needed to uniquely reconstruct the Earth's structure.

Are large-N arrays sufficient? We have shown that without the large-N arrays, we cannot uniquely determine the Earth's structure, so large-N arrays are necessary. But are they sufficient?

To find three unknowns, we need at least 3 equations, but, of course, not all systems of 3 equations have a unique solution. Similarly here, the fact that we have at least as many equations as unknowns does not necessarily mean that we can uniquely determine all the unknowns.

We are optimistic, however, since, as many papers –including our own – have shown, a lot of things *can* be determined based on the large-N array measurements [2, 4, 6].

How can we actually reconstruct the Earth's structure? In the corresponding partial differential equation (7), we use derivatives – which assume that we known the values of the corresponding functions for all possible values of t, x, y, and z. In practice, we only have a discrete set of sensors, so we know the values of the displacements u_i only at finitely many spatial locations. Similarly, we only know the values u_i corresponding to several discrete moments of time.

So, to solve the corresponding partial differential equation, we need to approximate the corresponding derivatives based on the available discrete data.

With respect to spatial derivatives – we have few points, so the best we can do is to use numerical differentiation $u' \approx \dfrac{u(x+\Delta x)-u(x)}{\Delta x}$ as a reasonable approximation for the corresponding spatial derivative u'. In other words, we need to consider differences $u(x+\Delta x)-u(x)$ between the signals measured by the neighboring sensors. This is exactly what we did in our paper [2] to detect the location of the fault based on the corresponding large-N seismic measurements.

With respect to time derivatives, we have much more points, and these points are much closer related. As a result, the difference $u(t+\Delta t)-u(t)$ between the displacements at two neighboring moments of time t and $t+\Delta t$ is so small that it is usually much smaller than the noise. As a result, if we use numerical differentiation and simply take the differences between the values measured at neighboring moments of time, we get noise instead of meaningful data. To get meaningful data, we need to filter a signal. For this filtering, we need to represent the signal in the vicinity of given point. The functions used for such local representation are known as *wavelets*, so what we need to get a reasonable approximation to time derivatives is to use wavelet analysis. It is worth mentioning that wavelet techniques has indeed been used successfully in seismic analysis; see, e.g., [3].

Conclsion. To uniquely recoinstruct Earth's structure from seismic information, we need to use large-N arrays, and we need to use differences between signals measured by neighboring sensors and to use wavelet techniques to process the signals measured by different sensors.

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