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Abstract Many equations in economics and finance are very complex. As a result, existing methods of solving these equations are very complicated and time-consuming. In many practical situations, more efficient algorithms for solving new complex equations appear when it turns out that these equations can be reduced to equations from other application areas – equations for which more efficient algorithms are already known. It turns out that some equations in economics and finance can be reduced to equations from physics – namely, from quantum physics. The resulting approach for solving economic equations is known as *quantum econometrics*. In quantum physics, the main objects are described by complex numbers; so, to have a reduction, we need to come up with an economic interpretation of these complex numbers. It turns out that in many cases, the most efficient interpretation comes when we separately interpret the absolute value (modulus) and the phase of each corresponding quantum number; the resulting techniques are known as *Bohmian econometrics*. In this paper, we use an algebraic approach – namely, the idea of invariance and symmetries – to explain why such an interpretation is empirically the best.

1 Formulation of the Problem

Quantum econometrics: a brief reminder. Economic and financial phenomena are very complex. As a result, many corresponding equations are very difficult to solve.

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Good news is that, as theoretical computer science has shown, all classes of sufficiently complex problems (they are known as NP-hard) can be reduced to each other; see, e.g., [18, 20, 21]. Thus, once we have come up with an efficient method of solving complex problems from one application area, we can reduce problems from other areas to these problems and thus, get a good algorithm for solving problems from other areas as well.

So, a natural way to solve complex problems in economics and finance is to reduce them to complex problems in other application areas – problems for which solutions are mostly known. One such area is physics – analysis of the physical world. Its equations are often very complex, and still, during several centuries of physics, researchers have found reasonably efficient algorithms for solving many of these equations. Thus, to solve complex problems in economics and finance, it makes sense to reduce them to solvable complex problems from physics.

It is known that to get an adequate description of a physical phenomenon, it is necessary to take quantum effects into account, i.e., to get into the domain of quantum physics; see, e.g., [12]. Because of this, the most complex physical equations are equations of quantum physics. Thus, a natural idea is to reduce complex equations arising in economics and finance to complex quantum equations – namely, to complex quantum equations for solving which we have efficient algorithms. Then, by solving the corresponding quantum equation and translating the solutions back into the language of economics and finance, we can get efficient algorithms for solving complex economic and financial problems. This idea – known as *quantum econometrics* – has indeed been implemented, and it has led to many efficient econometric algorithms; see, e.g., [1, 2, 3, 4, 5, 14, 15, 16, 17, 22, 23] (see also [20, 24, 25]).

Mathematics behind quantum physics: a brief reminder. Quantum physics is probabilistic by nature. In contrast to pre-quantum physics, where each particle at each moment of time has the exact location and the exact velocity – both of which can be, in principle, measured with any given accuracy – quantum physics takes into account that the only way to find the location of an elementary particle is to send another elementary particle to interact with it.

When we send a bunch of photons (a laser beam) to measure the distance to the Moon, this bunch affects the Moon a little bit, but the overall energy of this bunch is so small in comparison with the Moon's mass that we can safely ignore this effect. However, when a photo interacts with, e.g., a proton, their masses are more or less comparable and thus, the effects of the interaction on the location of the proton can no longer be ignored. As a result of these effects, we will never get the exact original location of the particle, i.e., this exact location cannot be measured. As a result of the measurement, we get different locations with different probability. This is how the state of a particle is described in quantum physics: by describing the probability of finding its location at different spatial places x, i.e., by a probability density function (pdf) $\rho(x)$ describing the particle's spatial distribution.

In addition to this pdf, we need to know the probability distribution corresponding to measuring velocity, and distributions corresponding to measuring other physical characteristics. These probability distributions must be consistent with each

other. It turns out that each tuple of consistent pdfs can be described by a complex-valued function $\psi(x)$ for which $\rho(x) = |\psi(x)|^2$.

Why complex numbers in quantum physics: an explanation based on common sense and relatively simple physics. Why complex numbers? It is known that they naturally appear when we try to simplify physical equations by combining location x and velocity v into a single quantity. Let us explain this phenomenon on the simple example of a 1-D oscillator: e.g., a particle attached to some location by a coil or by a rubber band. Let us assume that this particle is originally at location 0. When it moves to location $x \neq 0$, a force $f(x)$ (depending on x) moves it back to 0. So, Newton's equation for this particle takes the form $m \cdot \ddot{x} = f(x)$, where m is the particle's mass, $\dot{x}$, as usual, denotes derivative of the location over time (i.e., velocity), and $\ddot{x}$ denotes the second derivative (i.e., acceleration).

Because of the force pushing the particle back to 0, deviations from x do not have a chance to get large. Thus, the value x is small. So, terms quadratic and higher order in x are so small that they can be safely ignored. Thus, we can expand the function $f(x)$ in Taylor series in terms of x and keep only linear terms in this expansion: $f(x) \approx f_0 + f_1 \cdot x$. When the particle is at $x = 0$, there is no force, so $f_0 = 0$. When $x > 0$, the force moves the particle in the opposite direction, so $f_1 < 0$ and thus, $f_1 = -|f_1|$. Thus, the Newton's equations take the form $m \cdot \ddot{x} = -|f_1| \cdot x$, i.e., the form

$$\ddot{x} = -k \cdot x, \tag{1}$$

where we denoted $k \stackrel{\text{def}}{=} |f_1|/m$.

This equation is second-order: it uses the second derivative over time. It is known that the higher the order, the more difficult it is to solve this equation. The easiest to solve are first-order equations $\dot{x} = q \cdot x$, for which a general solution is the exponential function $x(t) = A \cdot \exp(q \cdot t)$. Thus, to simplify the solution of equation (1), it is reasonable to reduce this equation to first order. A known way to do it is:

- to replace the original variable x with a linear combination $v = x + c \cdot \dot{x}$, and
- to select c for which (1) becomes the equation of the first order.

From the definition of the combination v and from the equation (1), we conclude that

$$\dot{v} = \dot{x} + c \cdot \ddot{x} = \dot{x} - c \cdot k \cdot x. \tag{2}$$

This will be the first order equation when the coefficient c is selected in such a way that the right-hand side of this new equation (2) is proportional to v. To get the right coefficient at x, we need to multiply v by $-c \cdot k$, resulting in $-c \cdot k \cdot v = -(c \cdot k) \cdot x - c^2 \cdot k \cdot \dot{x}$. For this expression to coincide with the right-hand side of the equation (2), we must have $-c^2 \cdot k = 1$, i.e., we must have $c^2 = -1/k$ and thus, $c = \sqrt{1/k} \cdot \mathrm{i}$, where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$. Thus, the desired linear combination becomes a complex number $v = x + c \cdot \sqrt{1/k} \cdot \mathrm{i} \cdot \dot{x}$.

This complex representation is indeed very useful in physics: it represents the oscillator solution $x(t) = A \cdot \cos(\omega \cdot t + \varphi)$ as a real part of an easy-to-derive complex

expression $A \cdot \exp(\mathrm{i} \cdot (\omega \cdot t + \varphi))$ – an expression that uses the exponential function, the solution of the first-order equations.

Similar reduction is actively used in electrodynamics, and, in general, it is used to solve a general system of linear differential equations

$$\dot{x}_i = a_{i1} \cdot x_1 + \ldots + a_{in} \cdot x_m, \;\; i = 1, \ldots, n,$$

that can be reduced to simple equations if, instead of the original variables x_i, we consider their linear combinations corresponding to eigenvectors of the matrix a_{ij} – and these eigenvectors are, in general, complex.

Bohmian econometrics: a reminder. To reduce an economic or financial problem to a problem in quantum physics, we need to find an economic meaning for the complex-valued function $\psi(x)$.

To describe a complex number $z = a + \mathrm{i} \cdot b$, we need to describe two real numbers. Thus, to describe each complex value $z = \psi(x)$ in economic terms, we need to find two real-valued functions of this number z that will be interpreted economically.

There are many different ways to describe a complex number $z = a + \mathrm{i} \cdot b$ by two real numbers. From the mathematical viewpoint, the most natural idea is to describe the number z by its real part a and its imaginary part b. Another option – motivated by the geometric representation of a complex number as a 2-D point (a,b) – is to instead describe its absolute value $\rho = |z| = \sqrt{a^2 + b^2}$ and the angle φ between the real axis and the direction from from 0 to the point (a,b); then $z = \rho \cdot \exp(\mathrm{i} \cdot \varphi)$. These are the two simplest representations, motivated by high-school mathematics; we can use more complex ones: we can take any two functions $f_1(a,b)$ and $f_2(a,b)$ (of course, as long as these functions are not dependent on each other).

Many such interpretations have indeed been tried, and it turned out that the most efficient is the representation of each complex values $z = \psi(x)$ as a pair (ρ, φ) for which $z = \rho \cdot \exp(\mathrm{i} \cdot \varphi)$; see, e.g., [8, 9, 10, 11]. This representation is called *Bohmian*, because of the physicist and philosopher David Bohm who used ρ and φ to provide a physical interpretation of the wave function; see, e.g., [6, 7]; thus, the corresponding applications to econometrics are knows as *Bohmian econometrics.*

Challenge: why Bohmian representation works better than others? The above empirical fact prompts a natural question: why does Bohmian representation work better than all other possible representations of complex numbers?

What we do in this paper. In this paper, we provide an explanation for this empirically observed efficiency. This explanation is based on what, from the mathematical viewpoint, are algebraic ideas – namely, on the ideas of invariance (symmetry) with respect to some reasonable transformations.

For that purpose, we first explain why it makes sense to consider symmetries in general, what are the reasonable transformations, and finally, which numerical characteristics of complex numbers are invariant under such transformations.

2 Why Symmetries: A General Explanation

Simple symmetry: an example. Why are we sure that if we leave a pen in the air, it will fall down? Because we tried it yesterday, we tried it the day before yesterday, we tried it today – and we noticed that the result is the same every day, it does not change if we go one day ahead or one day back.

So, based on this invariance ("not changing"), and the fact that the pen fell yesterday, we conclude that it will follow tomorrow as well, as well as the day after tomorrow, etc.

Simple symmetries: general case. How do we gain any knowledge at all?

In general, once we have observed some phenomenon in different situations, different in terms of location, orientation, time, etc., we conclude that this phenomenon does not change (= is invariant) if we change location, orientation, etc. – and thus, that under similar circumstances, the same phenomenon will repeat.

Complex symmetries. The same idea explains how do we learn that a certain physical equation holds.

How do we learn that, e.g., Ohm's law is true? We observed the corresponding phenomenon under different circumstances, and we noticed that this law is always true. This law does not change if we change location, orientation, starting time, etc. Thus, we conclude that it will always be true.

Geometric symmetries. If we rotate a system, or shift it to a new location, or shift it in time – by starting the same operations after some time – the results are usually the same. Thus, most physical systems are invariant under such geometric transformations.

Scaling: another typical physical symmetry. Equations relate numerical values of physical quantities. These numerical values, however, depend not only on the quantity itself, but also on the measuring unit that we have selected.

For example, if to measure distances, we use meters instead of kilometers, then the actual distances will not change, but all the numerical values will multiply by 1000. In general, if we replace the original measuring unit by a new unit which is λ times smaller, then all the numerical values of this quantities will multiply by λ: instead of an original value q, we will have a new value $v' = \lambda \cdot q$. The corresponding transformation is known as *scaling*.

Sometimes, there is a physically motivated unit. However, in most situations, the choice of the unit is rather arbitrary – and different countries may use different units. In such situations, it is reasonable to require that the physical laws do not change if we simply change the measuring unit.

This *scale-invariance* is actually very important in physics: e.g., it is known that, based on scale-invariance (plus a few other reasonable requirements), we can derive the equations of gravitation (Einstein's General Relativity Theory), electromagnetism (Maxwell's equations), quantum mechanics (Schroedinger's equations), and many many others; see, e.g., [13, 19].

3 What Are the Reasonable Transformations of the Complex Value $\psi(x)$?

What are reasonable symmetries here? We want to reduce complex problems from economics and finance to problems from quantum physics. The main object of quantum physics is the complex-valued wave function $\psi(x)$.

So, to come up with the desired reduction, we need to interpret each complex value $\psi(x)$ in econometric terms. To do that, it is helpful to know what are the physically reasonable transformations of this value.

Let us first consider the general typical physical transformations that we listed in the previous section.

Geometric transformations do not change this complex value. As we have mentioned earlier, the value of the wave function $\psi(x)$ is, in effect, a square root of the probability density function (pdf) $\rho(x)$ that describes the particle's spatial distribution.

When we perform a rotation or a shift, the corresponding pdf does not change. Thus, it makes sense to assume that the value $\psi(x)$ also does not change under such transformations.

Effect of scaling. The probability density function (pdf) of a 3-D probability distribution is probability per unit volume. When we replace the original measuring unit by a new one which is λ times smaller, probabilities do not change, but numerical values of the volumes are multiplied by λ^3. Thus, the pdf is multiplied by λ^{-3}. It is therefore reasonable to assume that the wave function – which is, in effect, a square root of the pdf – is multiplied by the square root of this number, i.e., by $\lambda^{-3/2}$:

$$z \to \lambda^{-3/2} \cdot z.$$

It is reasonable to require that the physical laws do not change under such transformations $\psi \to \lambda^{-3/2} \cdot \psi$.

Since λ can be any positive real number, the values $k \stackrel{\text{def}}{=} \lambda^{-3/2}$ can also be arbitrary real numbers. Thus, we can simply say that the physical laws be invariant with respect to an arbitrary transformation $\psi \to k \cdot \psi$, for any $k > 0$.

Additional physical symmetry specific for quantum physics. In quantum physics, in addition to geometric transformations and scaling, there is another natural transformation that does not change physics: $\psi \to \exp(\mathrm{i} \cdot \alpha) \cdot \psi$ for any real number α; see, e.g., [12]. This transformation changes only the phase φ of the complex number $z = \rho \cdot \exp(\mathrm{i} \cdot \varphi)$; it is therefore called a *phase transformation*.

Summarizing. We need to consider two reasonable transformations: scaling $\psi \to k \cdot \psi$ and the phase transformation $\psi \to \exp(\mathrm{i} \cdot \alpha) \cdot \psi$.

Let us now consider which functions of ψ are invariant under each of these transformations. We will show that this leads us exactly to the functions used in Bohmian econometrics.

4 Invariant Functions of ψ Are Exactly the Functions Corresponding to Bohmian Econometrics

Definition 1. *We say that a function f from complex numbers to real numbers is* scale-invariant *if $f(k \cdot z) = f(z)$ for all complex numbers z and for all $k > 0$.*

Definition 2. *We say that a function f from complex numbers to real numbers is* phase-invariant *if $f(\exp(\mathrm{i} \cdot \alpha) \cdot z) = f(z)$ for all complex numbers z and for all real numbers α.*

Proposition 1. *A function $f(z)$ is scale-invariant if and only if it depends only on the phase of the complex number, i.e., if $f(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) = F(\varphi)$ for some function $F(x)$.*

Proposition 2. *A function $f(z)$ is phase-invariant if and only if it depends only on the modulus of the complex number, i.e., if $f(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) = F(\rho)$ for some function $F(x)$.*

Proof of Propositions 1 and 2. For Proposition 1, due to scale-invariance with $k = \rho$, we have $f(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) = f(\exp(\mathrm{i} \cdot \varphi))$. Thus, the proposition is true for $F(x) \stackrel{\text{def}}{=} f(\exp(\mathrm{i} \cdot x))$. Vice versa, one can easily show that for any function $F(x)$ from real numbers to real numbers, the function $f(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) \stackrel{\text{def}}{=} F(\varphi)$ is scale-invariant.

For Proposition 2, due to phase-invariance with $\alpha = \varphi$, we have

$$f(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) = f(\rho).$$

Thus, the proposition is true for $F(x) \stackrel{\text{def}}{=} f(x)$. Vice versa, one can easily show that for any function $F(x)$ from real numbers to real numbers, the function

$$f(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) \stackrel{\text{def}}{=} F(\rho)$$

is phase-invariant.

The propositions are proven.

What we did. We have just shown why Bohmian characteristics – absolute value and phase – are reasonable: because they are invariant with respect to natural transformations.

What we have not yet done. We have not yet proven that the Bohmian characteristics are, in some reasonable sense, optimal – and this is what we need to prove to fulfil our main promise: to explain the empirical phenomenon. We will do this in the next section.

5 Functions Corresponding to Bohmian Econometrics Are Optimal: A Proof

We want our results to be as general as possible. Of course, we could always come up with *some* optimality criterion and show that the Bohmian characteristics are optimal with respect to this particular criterion. But then, one could come up with a different optimality criterion for which the optimal characteristics would be different. To make a really convincing explanation of why Bohmian characteristics are empirically optimal, we would like to prove a stronger statement: that the Bohmian characteristics are optimal under *all* reasonably optimality criteria.

To prove such a statement, we need to start with a most general definition of an optimality criterion.

What is an optimality criterion. Usually, an optimality criterion is described by an objective function $J(x)$: an alternative b is better than the alternative a if $J(a) < J(b)$.

However, this is not the only way to describe optimality: e.g., if for a company making a decision, $J(x)$ is the profit, and we have two alternatives a and b with exactly the same expected profit, it is reasonable to use another criterion to select between a and b: e.g., to minimize the risk $R(x)$. In this case, the resulting optimality criterion is not a numerical one, it is more complex: b is better than a if:

- either $J(a) < J(b)$,
- or $J(a) = J(b)$ and $R(a) > R(b)$.

What is in common between all possible optimality criteria – numerical or more complex ones – is that they allow us to compare alternatives:

- Sometimes, we conclude that the alternative b is better than the alternative a: we will denote it by $a < b$.
- Sometimes, we conclude that these alternatives are of the same quality; we will denote this by $a \sim b$.
- Sometimes, the selected criterion does not allow us to decide which alternative is better; this also happens.

Of course, if b is better than a and c is better than b, then c should be better than a. Similar arguments lead to the following natural definition:

Definition 3. *Let A be a set. By an* optimality criterion *on the set A, we mean a pair of relations $(<, \sim)$ on this set that satisfy the following properties:*

- *if $a \sim b$, then $b \sim a$;*
- *if $a \sim b$ and $b \sim c$, then $a \sim c$;*
- *if $a < b$, then $a \not\sim b$ and $b \not< a$;*
- *if $a < b$ and $b < c$, then $a < c$;*
- *if $a < b$ and $b \sim c$, then $a < c$;*
- *if $a \sim b$ and $b < c$, then $a < c$.*

We are interested in final optimality criteria. The main purpose of selecting an optimality criterion is to find the best alternative, i.e., an alternative a_{opt} for which, for every other alternative a, we have $a < a_{\text{opt}}$ or $a \sim a_{\text{opt}}$.

Definition 4. *We say that an alternative a_{opt} is* optimal *with respect to an optimality criterion $(<, \sim)$ if for every alternative a, we have $a < a_{\text{opt}}$ or $a \sim a_{\text{opt}}$.*

From this viewpoint, if an optimality criterion does not select any alternative as optimal, then this criterion is useless.

Similarly, if the criterion selects several different alternatives as optimal, then we can use some other criterion to select between them (as we showed on the example of profit and risk). This means that a criterion that has several different optimal alternatives is not final – we need to supplement it with another criterion for selecting between the "optimal" ones. If after this supplementation, we still have several different optimal alternatives, we can use this non-uniqueness to optimize something else – and this process continues until we get a final optimality criterion, for which exactly one alternative is optimal.

Definition 5. *We say that the optimality criterion is* final *if there exists exactly one alternative which is optimal with respect to this criterion.*

Optimality criterion must be invariant. If we have natural symmetries T on the set of all alternatives, then it is reasonable to require that the criterion be invariant with respect to these symmetries.

For example, in financial situations, a reasonable symmetry is changing a monetary unit: e.g., replacing US dollars with Thai Bahts. If an alternative is better when we measure profits and risks in dollars, it should still be better if instead we use Bahts.

In general, we arrive at the following definition.

Definition 6. *We say that the optimality criterion is invariant with respect to a transformation T if:*

- *$a < b$ implies $T(a) < T(b)$, and*
- *$a \sim b$ implies that $T(a) \sim T(b)$.*

Now, we can show that functions corresponding to Bohmian econometrics are indeed optimal.

Proposition 3. *Let $(<, \sim)$ be a scale-invariant final optimality criterion on the set of all functions f from complex numbers to real numbers. Then, the optimal function $f_{\text{opt}}(z)$ depends only on the phase of the complex number: $f_{\text{opt}}(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) = F(\varphi)$ for some function $F(x)$.*

Proposition 4. *Let $(<, \sim)$ be a phase-invariant final optimality criterion on the set of all functions f from complex numbers to real numbers. Then, the optimal function $f_{\text{opt}}(z)$ depends only on the modulus of the complex number: $f_{\text{opt}}(\rho \cdot \exp(\mathrm{i} \cdot \varphi)) = F(\rho)$ for some function $F(x)$.*

Proof of Propositions 3 and 4. Let us first prove that if a final optimality criterion is invariant with respect to some transformation T, then the optimal alternative a_{opt} is invariant relative to T, i.e., that $T(a_{\text{opt}}) = a_{\text{opt}}$.

Indeed, optimality implies, in particular, that for every alternative a and for the inverse transformation T^{-1}, we have either $T^{-1}(a) < a_{\text{opt}}$ or $T^{-1}(a) \sim a_{\text{opt}}$. Since the criterion is invariant with respect to T, we thus conclude that $a < T(a_{\text{opt}})$ or $a \sim T(a_{\text{opt}})$. This is true for all alternatives a; so, the alternative $T(a_{\text{opt}})$ is also optimal. But since the optimality criterion is final, there can be only one optimal alternative, so we must have $T(a_{\text{opt}}) = a_{\text{opt}}$.

Thus, in the case of scale-invariance (Proposition 3), we conclude that the optimal function must also be scale-invariant – and hence, the desired result follows from Proposition 1. Similarly, in the case of phase-invariance (Proposition 4), we conclude that the optimal function must also be phase-invariant – and hence, the desired result follows from Proposition 2.

The statements are proven.

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