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Preferences (Partial Pre-Orders) on Complex Numbers – in View of Possible Use in Quantum Econometrics

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Abstract

In economic application, it is desirable to find an optimal solution – i.e., a solution which is preferable to any other possible solution. Traditionally, the state of an economic system has been described by real-valued quantities such as profit, unemployment level, etc. For such quantities, preferences correspond to natural order between real numbers: all things being equal, the more profit the better, and the smaller unemployment, the better. Lately, it turned out that to adequately describe economic phenomena, it is often convenient to use complex numbers. From this viewpoint, a natural question is: what are possible orders on complex numbers? In this paper, we show that the only possible orders are orders on real numbers.

1 Formulation of the Problem

What is econometrics: a brief reminder. Econometrics describes, in quantitative terms, human economics-related behavior: what people prefer, how they actually behave, and – if their current behavior is not optimal in relation to their own preferences – what behavior should be optimal.

Order in traditional econometrics. From the mathematical viewpoint, preference $a \le b$ or, equivalently, $b \ge a$ (meaning that b is at least as good as a) is an *pre-order* relation, i.e., a relation which is:

- reflexive, i.e., $a \le a$, and
- transitive, i.e., if $a \le b$ and $b \le c$, then $a \le c$.

In some cases, preferences form an *order*, i.e., $a \le b$ and $b \le a$ imply $a = b$. However, this is not always the case: a person can consider two different alternatives a and b to be equally good.

Traditional econometric models use real-valued quantities such as profit, income, productivity, etc. Such real-life quantities are easy to compare: e.g., the larger the profit, the better for the company; the smaller the unemployment rate, then – everything else being equal – the better for the economy, etc.

For real numbers, the natural pre-order is *linear* in the sense that for every a and b, we can have either $a \le b$ or $b \le a$. However, in general, pre-orders describing human preferences do not have to be linear: sometimes a person cannot decide whether the first of the two alternatives a and b is better or the second is better – and this person is not sure that these two alternatives are equivalent. In this case, we have $a \not\le b$ and $b \not\le a$.

Quantum econometrics. It turns out that in many practical situations, economic phenomena can be described by quantum-type formalisms (see, e.g., [11]), with quantities described by complex numbers; see, e.g., [1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 18, 20] (see also [17, 21, 22]).

Natural question. As we have mentioned, one of the main topics of econometrics is describing and using preferences – i.e., in mathematical terms, order relations. From this viewpoint, since we now allow complex-values quantities, a natural question is: what are possible extensions of the natural real-numbers order to pre-orders on the set of complex numbers?

This is the question that we will analyze in this paper. Specifically, we show that the only possible orders are orders coinciding with the original order on the real numbers – i.e., no non-trivial extension to complex numbers is possible.

2 Definitions and the Main Result

Motivations. In mathematical terms, complex numbers form a *field*. This means:

- that both addition and multiplication are commutative ($a+b=b+a$ and $a\cdot b=b\cdot a$) and associative ($a+(b+c)=(a+b)+c$ and $a\cdot(b\cdot c)=(a\cdot b)\cdot c$),
- that these two operations are distributive: $a\cdot(b+c)=a\cdot b+a\cdot c$),
- that there exists elements 0 and 1 for which $a+0=a$ and $a\cdot 1=a$ for every a,
- that every element a has an additive inverse $-a$ for which $a+(-a)=0$, and
- that every non-zero element $a \ne 0$ has a multiplicative inverse a^{-1} for which $a\cdot a^{-1}=1$.

For a field, the sum $a+(-b)$ is usually denoted by $a-b$.

It is usually required that orders $\le$ on a field are consistent with the algebraic operations, in the following sense (see, e.g., [6, 12, 19]):

- if $a \le b$ then $a+c \le b+c$;

- if $a \geq 0$ and $b \geq 0$, then $a \cdot b \geq 0$.

Sometimes, fields have natural transformations T with respect to which nothing changes: addition turns into addition, multiplication into multiplication, and physical meaning remains the same. In this case, it is reasonable to require that the order is also not changed under this operation, i.e., that $a \leq b$ implies $T(a) \leq T(b)$.

For complex numbers $z = a + b \cdot \mathrm{i}$, where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$, such an operation is *complex conjugation* that transforms z into $z^* \stackrel{\text{def}}{=} a = b \cdot \mathrm{i}$. Thus, we arrive at the following definition.

Definition 1. *Let A be a set.*

- *By a* pre-order $\leq$ *on the set A, we mean a binary relation which is reflexive ($a \leq a$) and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$).*
- *By an* order*, we mean a pre-order for which if $a \leq b$ and $b \leq a$, then $a = b$.*
- *We say that a pre-order is* linear *if for every a and b, we have either $a \leq b$ or $b \leq a$.*

Definition 2. *By a* consistent pre-order *on the set of all complex numbers, we mean a pre-order that satisfies the following properties:*

- *for real numbers, this pre-order coincides with a natural order;*
- *if $a \leq b$, then, for every c, we have $a + c \leq b + c$;*
- *if $a \geq 0$ and $b \geq 0$, then $a \cdot b \geq 0$;*
- *if $a \leq b$, then $a^* \leq b^*$.*

Known result. The following result is known.

Proposition 1. *A consistent pre-order cannot be linear.*

Proof. Indeed, suppose that a consistent pre-order is linear. Then either $\mathrm{i} \geq 0$ or $\mathrm{i} \leq 0$.

In the first, case, from $a = \mathrm{i} \geq 0$ and $b = \mathrm{i} \geq 0$, we conclude that $a \cdot b = -1 \geq 0$, while we know that $-1 < 0$.

In the second case, from $a = \mathrm{i} \leq b = 0$, we conclude, for $c = -\mathrm{i}$, that $a + c \leq b + c$, i.e., that $0 \leq -\mathrm{i}$ or, equivalently, that $-\mathrm{i} \geq 0$. Now, from $a = -\mathrm{i} \geq 0$ and $b = -\mathrm{i} \geq 0$, we conclude that $a \cdot b = -1 \geq 0$, while we know that $-1 < 0$.

In both cases, we have a contradiction, which proves that a consistent pre-order cannot be linear.

Main result. Let us now formulate our general result about consistent pre-orders that may be partial.

Proposition 2. *The only consistent pre-order is the original real-numbers pre-order in which $a \leq b$ if and only if the difference $b - a$ is a non-negative real number.*

3 Proof of the Main Result

1°. Let us prove that if $a \ge 0$ and $b \ge 0$, then $a + b \ge 0$.

Indeed, from $0 \le a$, by using $c = b$, we conclude $b \le a + b$, so by transitivity, we get $0 \le a + b$. i.e., $a + b \ge 0$.

2°. Let $\le$ be a consistent pre-order on the set of all complex numbers.

By definition, $a \le b$ implies that $a + c \le b + c$. In particular, for $c = -a$, we conclude that $a \le b$ implies that $0 \le b - a$.

Vice versa, if $0 \le b - a$, then, by taking $c = a$, we conclude that $a \le b$.

Thus, $a \le b$ if and only if $b - a \ge 0$. So, to describe a consistent pre-order, it is sufficient to describe the set of all the elements a for which $a \ge 0$.

In our case, we will thus prove that $a \ge 0$ if and only if a is a non-negative real number.

3°. First, let us prove that if $z = a + b \cdot \mathrm{i} \ge 0$ and $z \ne 0$, then $a > 0$.

Indeed, from $z \ge 0$, we can conclude that $z^* = a - b \cdot \mathrm{i} \ge 0$. Thus, due to Part 1 of this proof, we get $z + z^* = 2a \ge 0$, hence $a \ge 0$.

Let us show that we cannot have $a = 0$. Indeed, if $a = 0$, then we would have $z = b \cdot \mathrm{i}$. Since $z \ne 0$, this means that $b \ne 0$. Then, we would have $z \cdot z = -b^2 \ge 0$, but we know that $-b^2 < 0$ – a contradiction.

4°. We have shown that $a > 0$. Multiplying the complex number $a + b \cdot \mathrm{i} \ge 0$ by a positive number a^{-1}, we conclude that the product $v = 1 + t \cdot \mathrm{i} \ge 0$, where we denoted $t \stackrel{\text{def}}{=} b \cdot a^{-1}$.

5°. Let us now prove that if $1 + t \cdot \mathrm{i} \ge 0$, then $|t| < 1$ and $1 + s \cdot \mathrm{i} \ge 0$, where $s \stackrel{\text{def}}{=} \dfrac{2t}{1 - t^2}$.

Indeed, if $1 + t \cdot \mathrm{i} \ge 0$, then the product of two such numbers should also be greater than or equal to 0, i.e., we should have $(1 - t^2) + 2t \cdot \mathrm{i} \ge 0$. From Part 3 of this proof, we can now conclude that $1 - t^2 > 0$, hence $|t| \le 1$.

Since $1 - t^2 > 0$, we can multiply the new complex number $(1 - t^2) + 2t \cdot \mathrm{i} \ge 0$ by $1/(1 - t^2) > 0$ and get the desired number $1 + s \cdot \mathrm{i}$.

Here, $1 - t^2 \le 1$ hence $|s| \ge 2|t|$.

6°. If we have $a + b \cdot \mathrm{i} \ge 0$ for some $b \ne 0$, then, from Part 4 of this proof, we get a new complex number $1 + t \cdot \mathrm{i} \ge 0$ for some $t \ne 0$.

By using Part 5 of this proof, we can get a new complex number $1 + s \cdot \mathrm{i}$ with $|s| \ge 2|t|$. Then, we can again apply the same procedure to the number $1 + s \cdot \mathrm{i}$ and a new non-negative complex number for which the absolute value of the imaginary part is at least as large as $|s|$ – and thus, at least 4 times as large as $|t|$. By repeating the same procedure k times, we get a complex number $1 + d \cdot \mathrm{i}$ with $|d| \ge 2^k \cdot |t|$. If $t > 0$, then for sufficiently large k, we will have $2^k \cdot |t| > 1$ hence $|d| > 1$ – and we know, from Part 5 of this proof, that we must have $|d| < 1$, a contradiction.

This contradiction shows that the case $b \ne 0$ is impossible, so the only complex numbers $z \ge 0$ are numbers with 0 imaginary part – i.e., real numbers.

The proposition is proven.

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