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Towards Analytical Techniques For Optimizing Knowledge Acquisition, Processing, Propagation, And Use In Cyberinfrastructure

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TOWARDS ANALYTICAL TECHNIQUES FOR OPTIMIZING KNOWLEDGE ACQUISITION, PROCESSING, PROPAGATION, AND USE IN CYBERINFRASTRUCTURE

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I dedicate this work to my family,

for their love, patience and understanding

during the many hours

when I was away in my path of learning:

to my sons Sam and Joseph,

to my parents Hortensia and Antonio,

to my sister Martha, and

to my brothers Antonio, Victor, and Javier.

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by

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Abstract

For many decades, there has been a continuous progress in science and engineering applications. A large part of this progress comes from the new knowledge that researchers acquire, propagate, and use. This new knowledge has revolutionized many aspects of our life, from driving to communications to shopping.

Somewhat surprisingly, there is one area of human activity which is the least impacted by the modern technological progress: the very processes of acquiring, processing, and propagating information. When we decide where to place sensors, which algorithm to use for processing the data – we rely mostly on our own intuition and on the opinion of the experts. As a result, knowledge-related methods that we select are often far from optimal. To make effective recommendations, it is necessary to build realistic models of the corresponding processes, and then use these models to find optimal ways of controlling these processes.

The need for such models is well understood. There are many numerical models of knowledge acquisition, processing, and propagations. Some of these models have been successfully used to enhance the corresponding processes. However, these applications are limited by the fact that most of these models are based on detailed numerical simulation of the corresponding processes, which make the resulting models very time-consuming to use. It is therefore necessary to develop *analytical* models for the corresponding knowledge-related processes, models that would allow easier optimization and application.

The main purpose of this thesis is to develop analytical models for all the knowledge-related processes, from knowledge acquisition to knowledge processing and knowledge propagation.

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Chapter 1

Introduction

Knowledge-related processes are important. For many decades, there has been a continuous progress in science and engineering applications. A large part of this progress comes from the new knowledge that researchers acquire, propagate, and use. This new knowledge has revolutionized many aspects of our life. We rely on automated computer-based systems when we drive our cars, when we fly planes, when we communicate with people over the web.

Knowledge-related processes need to be enhanced. Somewhat surprisingly, there is one area of human activity which is the least impacted by the modern technological progress: the very processes of acquiring, processing, and propagating information. When we decide where to place sensors, which algorithm to use for processing the data acquired via these sensors, what is the best way to propagate information (e.g., which teaching techniques is the best for each task) – we rely mostly on our own intuition and on the opinion of the experts, and not – as in many other areas – on computer-based systems.

As a result, the methods of data acquisition, processing, and propagation that we select are often far from being optimal. This non-optimality is not surprising: when a driver navigates in a new town, clearly a GPS-based navigator will select a much better route than a driver would come up based on his or her intuition.

State-of-the-art in modeling and enhancing knowledge-related processes: successes and limitations. To make effective recommendation on how to acquire, process, and propagate knowledge, it is necessary to build realistic models of the corresponding

processes, and then use these models to find optimal ways of controlling these processes.

This need is well understood. There are many numerical models of knowledge acquisition, processing, and propagations; see, e.g., [2, 9, 10, 28, 29, 41, 69, 80, 89, 101, 106, 109]. Some of these models have been successfully used to enhance the corresponding processes.

However, these applications are limited by the fact that most of these models are based on detailed numerical simulation of the corresponding processes. Knowledge-related processes are complex, and therefore, the existing simulations are very complex – and optimizations based on these models are also very complex and time-consuming. These models have been successfully used in large-scale applications: e.g., to optimize how a large research laboratory works – but they are not yet fully ready for everyday applications to day-to-day practical decisions on where to place sensors, which algorithms to select, how to best propagate knowledge – decisions which researchers face all the time.

Need for analytical models. To help researchers make these decisions, we need to develop not only *numerical* simulation models, we also need to develop *analytical* models for the corresponding knowledge-related processes, models that would allow easier optimization and application.

It is mostly analytical models which have led to successful applications of science and engineering. For example, in chemistry, we can write down Schroedinger's equations which simulate how electrons, atoms, and molecules interact – but for complex molecules, this requires large computations on high performance computers. In most practical applications, chemists use simplified analytical models to predict and control the results of chemical reactions. Similarly, in statistical physics, we can write Newton's equations for the motion of all the molecules – and sometimes it is necessary – but most applications use analytical macromodels operating with temperature, pressure, and other macro-characteristics instead of an atom-by-atom descriptions.

What we do in this thesis: main objective and chapter-by-chapter structure. The main purpose of this thesis is to develop analytical models for all the knowledge-related processes, from knowledge acquisition to knowledge processing and knowledge propagation. Of course, this is vast area of research, and we are not aiming at covering all possible aspects which can be described by analytical models. Our goal is more practical: to develop analytical models for at least some of the knowledge-related processes related to the interdisciplinary cyberinfrastructure research performed at UTEP's Cyber-ShARE Center.

In this thesis, we present the preliminary results of our work, and we present plans for the future works which will hopefully lead to a PhD dissertation.

In Chapter 2, we start with data acquisition. The main source of knowledge is processing data. Data comes from sensors. Within a limited budget, it is extremely important to make sure that the use of the sensors is optimized so that we get the largest possible amount of useful data from these sensors.

Traditionally, most data comes from stationary sensors, i.e., sensors which we place at fixed locations. For such sensors, it is important to come up with the optimal placement, the placement which would lead to the largest amount of useful data. We analyze this problem in Section 2.1, on the example of placing bio-weapon detectors, and in Section 2.2, on the example of placing environmental sensors.

The problem of optimal use becomes more technically challenging if we take into account the possibility of using mobile sensors, i.e., sensors which we can move along different trajectories. In this case, it is important to come up with optimal trajectories, i.e., the trajectories which would lead to the largest amount of useful data. We analyze this problem in Section 2.3, on the example of Unmanned Aerial Vehicles (UAVs) patrolling the border.

In all these cases, it is important to make sure that not only we have an algorithm producing the optimal placement or optimal trajectory: we also need to make sure that the corresponding algorithms are computationally efficient, i.e., that the corresponding optimization algorithms can produce the resulting optimal setting in reasonable time. The more sensors we need to place, the more computations we need and therefore, the more

important it is for the computation time to be reasonable. We analyze this problem in Section 2.4, again on the example of security problems.

Once the data is collected, we need to process this data. Problems related to data processing are analyzed in Chapter 3. For processing, we need to use computers – and the more data we collect, the more computer power we need. It is therefore important to optimally distribute this computing power. This is the problem that we will analyze in Section 3.1.

In many cases, data processing is a creative process, it goes beyond simple application of known algorithms. To come up with the best ways of processing data, of extracting knowledge from the data, we need creative teams – teams combining domain expertise and computer expertise. Just like with computers, simply bringing people together does not always improve their efficiency. It is therefore important to make sure that people collaborate in the most efficient way. This aspect of data and knowledge processing is analyzed in Section 3.2.

Once we have transformed data into knowledge, we need to propagate this knowledge – so that other researchers can use and enhance this knowledge. Problems related to data propagation are analyzed in Chapter 4.

To propagate data, first, we need to motivate people to learn the new knowledge, we need to make sure that the idea is propagated to more and more people. To ensure that, we need to analyze the process of idea propagation; this is done in Section 4.1.

Once a person is willing to learn the corresponding techniques and ideas, we can start the actual learning. For this learning to be successful, we need to get a good understanding of where the person stands now, what is his/her level of knowledge in the corresponding areas. This assessment problem is analyzed in Section 4.2.

Once this information is known, we need to actually present this information to the interested folks – and use appropriate feedback to modify (if needed) the speed with which this knowledge is presented. Issues related to the material's presentation are analyzed in Sections 4.3 and 4.4. Specifically, in Section 4.3, we consider the problem from the global

viewpoint: e.g., in what order we should present different parts of the material. In Section 4.4, we consider this problem from the local viewpoint: what is the best way to present different items. Finally, in Section 4.5, we analyze the problems related to feedback.

In Chapter 5, we analyze the problems related to using data. How can we use the acquired knowledge? In many practical situations, we have a well-defined problem, with a clear well-formulated objective. Such problems are typical in engineering: we want a bridge which can withstand a given load, we want a car with a given fuel efficiency, etc. There exist many techniques for solving such well-defined optimization problems.

However, in many practical situations, it is important to also take into account subjective user preferences. This subjective aspect of decision making is known as *Kansei engineering*. This aspect is what we analyze in Chapter 5.

Finally, Chapter 6 overviews our future plans, which include both further theoretical developments and practical applications of our analytical models and resulting recommendations.

Chapter 2

Data Acquisition: Towards Optimal Use of Sensors

The main source of knowledge is processing data. Data comes from sensors. Within a limited budget, it is extremely important to make sure that the use of the sensors is optimized so that we get the largest possible amount of useful data from these sensors.

Traditionally, most data comes from stationary sensors, i.e., sensors which we place at fixed locations. For such sensors, it is important to come up with the optimal placement, the placement which would lead to the largest amount of useful data. We analyze this problem in Section 2.1, on the example of placing bio-weapon detectors, and in Section 2.2, on the example of placing environmental sensors.

The problem of optimal use becomes more technically challenging if we take into account the possibility of using mobile sensors, i.e., sensors which we can move along different trajectories. In this case, it is important to come up with optimal trajectories, i.e., the trajectories which would lead to the largest amount of useful data. We analyze this problem in Section 2.3, on the example of Unmanned Aerial Vehicles (UAVs) patrolling the border.

In all these cases, it is important to make sure that not only we have an algorithm producing the optimal placement or optimal trajectory: we also need to make sure that the corresponding algorithms are computationally efficient, i.e., that the corresponding optimization algorithms can produce the resulting optimal setting in reasonable time. The more sensors we need to place, the more computations we need and therefore, the more important it is for the computation time to be reasonable. We analyze this problem in Section 2.4, again on the example of security problems.

2.1 Optimal Use of Stationary Sensors: Case Study of Optimal Placement of Bio-Weapon Detectors

In this section, we analyze the problem of the optimal use of stationary sensors, on the example of optimal placement of bio-weapon detectors. Biological weapons are difficult and expensive to detect. Within a limited budget, we can afford a limited number of bio-weapon detector stations. It is therefore important to find the optimal locations for such stations. A natural idea is to place more detectors in the areas with more population – and fewer in desert areas, with fewer people. However, such a commonsense analysis does not tell us how many detectors to place where. To decide on the exact placement of bio-weapon detectors, we formulate the placement problem in precise terms, and come up with an (almost) explicit solution to the resulting optimization problem.

The results from this section were first published in [49].

Formulation of the practical problem. Biological weapons are difficult and expensive to detect. Within a limited budget, we can afford a limited number of bio-weapon detector stations. It is therefore important to find the optimal locations for such stations.

Commonsense analysis of the problem. A natural idea is to place more detectors in the areas with more population – and fewer in areas with fewer people, e.g., in the desert areas. However, such a commonsense analysis does not tell us how many detectors to place where. To decide on the exact placement of bio-weapon detectors, we must formulate the placement problem in precise terms.

Objective function. The above commonsense idea is based on a (reasonable) assumption that the adversary's objective is to kill as many people as possible. Vice versa, our objective is to minimize the potential effect of a bio-weapon attack.

Comment. In this chapter, we mainly concentrate on the above objective function. This objective function may not always fully describe the adversary's objectives. For example, one of the objectives of political terrorism may be extra publicity for the cause. From this

viewpoint, an adversary may prefer a scenario with a smaller number of victims if several of these victims are well-known. It is therefore desirable to formulate the objective functions that describe this (and similar) approaches, and extend our optimization analysis to the case of such more complex objective functions.

Towards precise formulation of the problem: what is known. Since the objective is to target as many people as possible, to analyze this situation, we need to know how many people live at different locations. In precise terms, we assume that we know, for every possible location x, the population density $\rho(x)$ in the vicinity of this location.

We assume that we know the number N of detectors that we can afford to place in the given territory.

We also assume that we know the efficiency of a bio-weapons detector station. We will estimate this efficiency by the distance d_0 at which this station can detect an outbreak of a disease.

For many diseases, $d_0 = 0$ – we can only detect a disease when the sources of this disease reach the detecting station.

However, it is quite possible that for some diseases, we have a super-sensitive equipment that is able to detect the concentration of the bio-weapons agent at a level below the threshold that makes this agent dangerous to the population. In this case, we can detect the coming disease before it starts affecting people in the direct vicinity of the station – i.e., in effect, we have $d_0 > 0$.

For simplicity, we assume that the disease spreads equally fast in all directions.

Comment. This is also a somewhat simplifying assumption, since in reality, a disease spreads

- either with human movements – in which case in the vicinity of an interstate it spreads faster in the direction of the interstate,
- or with wind – in which case it spreads faster in the direction of the prevailing winds.

How we can describe the detector placement. On a large-scale basis, we need to decide how many detectors to place in different areas. In other words, we need to find the *density* $\rho_d(x)$ of detector placement – the number of detectors per unit of area (e.g., a square mile).

Under this description, the number of detectors in an area of size Δx is approximately equal to $\rho_d(x) \cdot \Delta x$, so the overall number of detectors can be obtained by adding these amounts, as $\int \rho_d(x)\, dx$. Thus, the constraint that we have exactly N detecting stations can be described as

$$\int \rho_d(x)\, dx = N. \tag{2.1.1}$$

Optimal placement of sensors: at the vertices of a hexagonal grid. We want to place the sensors in such a way that the largest distance D to a sensor is as small as possible. Alternatively, if D is fixed, we want to minimize the number of sensors for which every point is at a distance $\le D$ from one of the sensors. In geometric terms, this means that every point on a plane belongs to a circle of radius D centered on one the sensors – and thus, the whole plane is covered by such circles. Out of all such coverings, we want to find the covering with the smallest possible number of sensors.

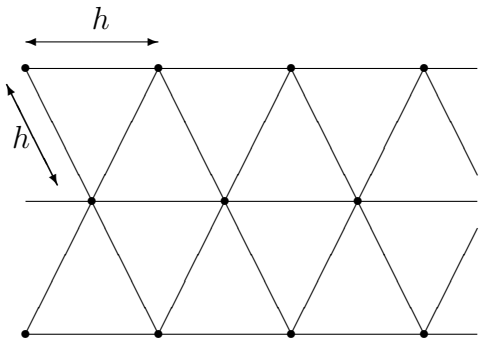
It is known that the smallest such number is provided by an equilateral triangle grid, i.e., a grid formed by equilateral triangles; see, e.g., [46]. Hence, in this section, we will select such a grid.

Locations of detector stations are assumed to be known to the adversary. Bio-weapon detector stations are not easily concealable. Thus, we assume that the adversary knows the locations of different stations.

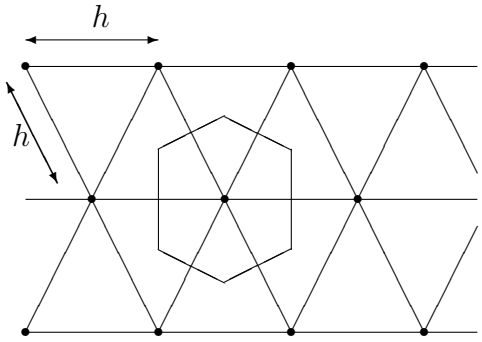
How to estimate the effect of placing bio-weapons at a location x. Let us assume that we have already decided how many detectors to place in different regions, i.e., that we have already selected the density function $\rho_d(x)$.

Within a small region of area A, we have $A \cdot \rho_d(x)$ detectors. Thus, if we, e.g., place these detectors on a grid with distance h between the two neighboring ones in each direction, we

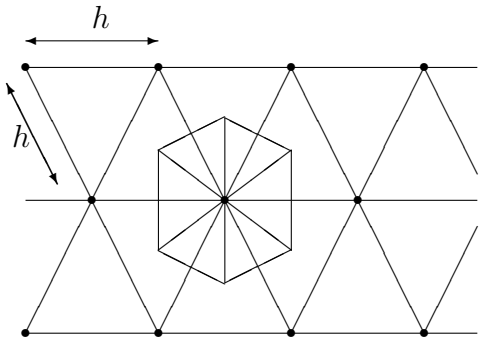
have:



For this placement, the set of all the points which are closest to a given detector forms a hexagonal area:



This hexagonal area consists of 6 equilateral triangles with height $h/2$:



In each triangle, the height $h/2$ is related to the size s by the formula

$$\frac{h}{2} = s \cdot \cos(60^\circ) = s \cdot \frac{\sqrt{3}}{2}, \tag{2.1.2}$$

hence

$$s = \frac{h}{\sqrt{3}} = h \cdot \frac{\sqrt{3}}{3}. \tag{2.1.3}$$

Thus, the area A_t of each triangle is equal to

$$A_t = \frac{1}{2} \cdot s \cdot \frac{h}{2} = \frac{1}{2} \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{2} \cdot h^2 = \frac{\sqrt{3}}{12} \cdot h^2. \tag{2.1.4}$$

So, the area A_s of the whole set is equal to 6 times the triangle area:

$$A_s = 6 \cdot A_t = \frac{\sqrt{3}}{2} \cdot h^2. \tag{2.1.5}$$

Each point from the region is the closest to one of the points from the detector grid, so the region of area A is thus divided into $A \cdot \rho_d(x)$ (practically) disjoint sets of area $\frac{\sqrt{3}}{2} \cdot h^2$. So, the area of the region is equal to the sum of the areas of these sets:

$$A = (A \cdot \rho_d(x)) \cdot \frac{\sqrt{3}}{2} \cdot h^2. \tag{2.1.6}$$

Dividing both sides of this equality by A, we conclude that

$$1 = \rho_d(x) \cdot \frac{\sqrt{3}}{2} \cdot h^2, \tag{2.1.7}$$

and hence, that

$$h = \frac{c_0}{\sqrt{\rho_d(x)}}, \tag{2.1.8}$$

where we denote

$$c_0 \stackrel{\text{def}}{=} \sqrt{\frac{2}{\sqrt{3}}}. \tag{2.1.9}$$

From the viewpoint of the adversary, it is desirable to place the bio-weapon at a location which is the farthest away from the detectors – so that it will take the longest time to be detected. For the grid placement, this location is at one of the vertices of the hexagonal

zone – at which the distance from each neighboring detector is equal to $s = h \cdot \dfrac{\sqrt{3}}{3}$. By using formula (2.1.8), we can determine s in terms of $\rho_d(x)$, as

$$s = \frac{c_1}{\sqrt{\rho_d(x)}}, \tag{2.1.10}$$

where we denote

$$c_1 = \frac{\sqrt{3}}{3} \cdot c_0 = \frac{\sqrt[4]{3} \cdot \sqrt{2}}{3}. \tag{2.1.11}$$

Once the bio-weapon is placed at this location, it starts spreading until its spread area reaches the threshold distance d_0 from the detector. In other words, it spreads for the distance $s - d_0$. During this spread, the disease covers the circle of radius $s - d_0$ and area $\pi \cdot (s - d_0)^2$.

By using the known population density $\rho(x)$, we can conclude that the number of affected people $n(x)$ is equal to

$$n(x) = \pi \cdot (s - d_0)^2 \cdot \rho(x). \tag{2.1.12}$$

Substituting the expression (2.1.10) into this formula, we conclude that

$$n(x) = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x). \tag{2.1.13}$$

Adversary's choice of the location. According to our assumption about the adversary's objective function, the adversary wants to maximize the number of affected people. Thus, the adversary will select a location x for which this number $n(x)$ (as described by the expression (2.1.13)) is the largest possible. The resulting damage n is thus equal to the largest of the values $n(x)$:

$$n = \max_x \left(\pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x) \right). \tag{2.1.14}$$

Our objective. Our objective is to minimize this overall damage, i.e., to select the detector placement $\rho_d(x)$ so as to minimize this value n.

In other words, we want to *minimize* the worst-possible (maximal) damage. This *minimax* formulation is typical for *zero-sum games*, in which the interests of the two sides are exactly opposite; see, e.g., [70].

Thus, we arrive at the following problem:

Resulting formulation of the problem in precise terms. We are given the population density $\rho(x)$, the value d_0, and the total number of detectors N. We want to find a function $\rho_d(x)$ that minimizes the expression (2.1.14) under the constraint $\int \rho_d(x)\,dx = N$.

Analysis of the resulting optimization problem. The damage is determined by the maximum n of the function $n(x)$. Let us assume that we have already selected the optimal detector density function, i.e., the function $\rho_d(x)$ that minimizes the desired objective function n.

Let us show that the damage function $n(x)$ corresponding to this selection is constant. We will prove this by contradiction. If the function $n(x)$ is not constant, this means that at some locations x, the values $n(x)$ are smaller than the maximum n. In this case, we can slightly increase the detector density at the locations where $n(x) = n$, at the expense of slightly decreasing the location density at locations where $n(x) < x$.

The value of the expected damage $n(x)$ monotonically decreases with the detector density $\rho_d(x)$. This mathematical observation is in perfect accordance with common sense: the more detectors we place at some location, the earlier we will be able to detect bio-weapons and thus, the smaller will be the resulting damage.

Thus, the above re-arrangement of detectors will decrease the value of $n(x)$ at all locations where $n(x) = n$ – and slightly increase at all other locations. As a result, after this detector relocation, the overall maximum $n = \max\limits_x n(x)$ will decrease. This possibility contradicts to our initial assumption that the value n is the smallest possible. Thus, the function $n(x)$ is indeed constant.

Let us denote this constant by n_0. Then, from the formula (2.1.13) for $n(x)$, we conclude that

$$n_0 = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x). \tag{2.1.15}$$

Thus, we conclude that

$$\left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 = \frac{n_0}{\pi \cdot \rho(x)}, \tag{2.1.16}$$

$$\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 = \frac{c_2}{\sqrt{\rho(x)}}, \tag{2.1.17}$$

where we denote

$$c_2 \stackrel{\text{def}}{=} \frac{\sqrt{n_0}}{\sqrt{\pi}}. \tag{2.1.18}$$

Thus, we get

$$\frac{c_1}{\sqrt{\rho_d(x)}} = d_0 + \frac{c_2}{\sqrt{\rho(x)}}, \tag{2.1.19}$$

$$\sqrt{\rho_d(x)} = \frac{c_1}{d_0 + \frac{c_2}{\sqrt{\rho(x)}}}, \tag{2.1.20}$$

and

$$\rho_d(x) = \frac{c_1^2}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}} \right)^2}, \tag{2.1.21}$$

From (2.1.11), we conclude that

$$c_1^2 = \frac{2 \cdot \sqrt{3}}{9}, \tag{2.1.22}$$

hence

$$\rho_d(x) = \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}} \right)^2}. \tag{2.1.23}$$

The value c_2 must be determined from the equation (2.1.1).

Thus, we arrive at the following solution:

Solution: the optimal detector location is characterized by the detector density

$$\rho_d(x) = \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}}\right)^2},$$

where the parameter c_2 must be determined from the equation

$$\int \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}}\right)^2} \, dx = N. \tag{2.1.24}$$

Case of $d_0 = 0$. As we have mentioned earlier, in some cases, we have $d_0 = 0$. In this case, the formula (2.1.23) takes a simplified form

$$\rho_d(x) = C \cdot \rho(x) \tag{2.1.25}$$

for some constant C. In this case, the detector density is exactly proportional to the population density.

Substituting the expression (2.1.25) into the constraint (2.1.1), we conclude that

$$N = C \cdot N_p, \tag{2.1.26}$$

where $N_p = \int \rho(x) \, dx$ is the total population. Thus, $C = \frac{N}{N_p}$ and the optimal detector placement (2.1.25) takes the form

$$\rho_d(x) = \frac{N}{N_p} \cdot \rho(x). \tag{2.1.27}$$

Towards more relevant objective functions. In our computations, we assumed that the main objective of the adversary is to maximize the number of people affected by the bio-weapon, i.e., to maximize the value $\int_A \rho(x) \, dx$, where A is the region were people become affected before the bio-weapon is detected.

As we have mentioned, the actual adversary's objective function may differ from this simplified objective function. For example, the adversary may take into account that different locations have different publicity potential. In this case, instead of maximizing the total number of affected people, the adversary may want to maximize the weighted value $\int_A \widetilde{\rho}(x)\,dx$, where $\widetilde{\rho}(x) \stackrel{\text{def}}{=} w(x) \cdot \rho(x)$, and the weight $w(x)$ describes the publicity-related importance of the location x.

From the purely mathematical viewpoint, once we have fixed the weight functions $w(x)$, we get the exact same problem as before – with the only difference that we now have "effective population density" $\widetilde{\rho}(x)$ instead of the original density $\rho(x)$. Thus, if we know the exact weight function $w(x)$, then we find the optimal detector density $\rho_d(x)$ by substituting the effective population density $\widetilde{\rho}(x)$ instead of $\rho(x)$ into the above formulas.

2.2 Optimal Use of Stationary Sensors: Case Study of Optimal Placement of Environmental Sensors

In this section, we analyze the problem of the optimal use of stationary sensors, on another example: of optimal placement of environmental sensors. Specifically, we show that under reasonable assumption, the spatial variability of a field $f(x)$, i.e., the expected value

$$F(z) \stackrel{\text{def}}{=} E[(f(x+z) - f(x))^2],$$

has the form $F(z) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} \cdot z_i \cdot z_j\right|^{\alpha}$. We explain how to find g_{ij} and α from the observations, and how to optimally place sensors in view of this spatial variability.

The results of this section were first published in [52].

Need to describe spatial variability. To understand climate trends, we need to describe not only the values of temperature, humidity, wind speed and direction at a single location, we also need to know how these characteristics change from one location to the other. In other words, we need to describe spatial variability of the corresponding characteristics.

There is a similar need in other application areas. For example, to understand the brain activity within a region, in addition to describing brain activity at certain locations, we also need to describe how this brain activity changes from one location to the other, i.e., we need to describe spatial variability of the corresponding characteristics.

How to describe spatial variability: use of random variables. In general, we have a characteristic $f(x)$ that takes different values at different locations x. Since we cannot exactly predict the exact future values $f(x)$, it is reasonable to consider them random variables. Random variables $f(x)$ corresponding to different locations x form a *random field.*

How to describe spatial variability: use of normal distributions. The values $f(x)$ are determined by a large number of different factors. In statistics, the joint effect of many small independent factors is – due to the Central Limit Theorem – well described by a normal distribution; see, e.g., [94]. Thus, it is reasonable to assume that the variables $f(x)$ are normally distributed.

A normal distribution is uniquely determined by its first two moments, i.e., by the expected values $E[f(x)]$ and $E[f(x) \cdot f(y)]$. The values $E[f(x)]$ and $E[(f(x))^2]$ describe the behavior at a single location. Thus, to describe spatial variability, it is sufficient to describe the values $E[f(x) \cdot f(y)]$ for $x \neq y$. Since we know the values $E[(f(x))^2]$ and $E[(f(y))^2]$, describing $E[f(x) \cdot f(y)]$ is equivalent to describing the following expected value:

$$C(x, y) \stackrel{\text{def}}{=} E[(f(y) - f(x))^2] = E[(f(y))^2] + E[(f(x))^2] - 2E[f(x) \cdot f(y)].$$

Homogeneity. Locally, the distribution is usually homogenous, i.e., does not change after a shift. Thus, if we change x to $x + z$ and y to $y + z$, we should get the same value $C(x, y)$: $C(x + z, y + z) = C(x, y)$. For $z = -z$, this leads to $C(x, y) = C(0, y - x)$. So, to describe spatial variability, it is sufficient to describe the function

$$F(z) \stackrel{\text{def}}{=} C(0, z) = E[(f(x + z) - f(x))^2].$$

Comment. For $z = 0$, the above definition leads to $F(0) = 0$.

Other natural requirements. It is reasonable to assume that $F(z)$ continuously depends on z.

It is also reasonable to assume that there is spatial variability, i.e., that $F(z) > 0$ for $z > 0$.

Another requirement is that $f(x)$ is very close to $f(y)$ only for close x and y. Formally, we will require that for some value $F_0 > 0$, the set $\{z : F(z) \le F_0\}$ is bounded.

Comment. It should be mentioned that the spatial distribution is often *anisotropic*, i.e., depends on the direction. For example, a North-South oriented mountain range goes through the city of El Paso. The closeness to the mountain affects temperature, rainfall, wind, etc. As a result, the meteorological characteristics change much more when we move in the East-West direction than when we move in the North-South one.

We need to select a few-parametric family of functions $F(z)$. In different practical situations, we have different functions $F(z) \ge 0$. To describe all such situations, it is desirable to have a parametric family $\mathcal{F}$ of possible functions $F(z)$.

Often, we only have a limited amount of data, so we can only statistically significantly determine a small number of parameters of the function $F(z)$. For example, in environmental sciences, we have a limited number of observations in remote areas such as most areas of Arctic and Antarctica. In brain research, we also often only have limited data. To cover such situations, it is desirable to have simple, few-parametric families $\mathcal{F}$.

Desired properties of few-parametric families. The numerical value of a physical characteristic depends on the choice of a measuring unit. For example, for length, if we change from inches to cm, the numerical values increase by 2.54. In general, if we use a new unit which is λ times smaller than the previous one, then numerical values $f(x)$ increase by λ, and the resulting values of $F(z)$ increase by λ^2. In principle, we can have an arbitrary positive value $C = \lambda^2$, so it is reasonable to require that the family $\mathcal{F}$ contains, with every function $F(z)$, also all functions $C \cdot F(z)$ for every $C > 0$.

Another possible change is a change in spatial coordinates. In some applications, the

usual coordinates work best, in other applications, polar, cylindrical, or other coordinates are more appropriate. Locally, each smooth coordinate transformation $x_i \to f_i(x_1, \ldots, x_n)$ can be well approximated by a linear function $x_i \to \sum_{j=1}^{n} a_{ij} \cdot x_j + a_i$, i.e., in matrix terms, $x \to Ax + a$. Under this transformation, the difference $z = y - x$ is replaced with Az. It is therefore reasonable to require that the the family $\mathcal{F}$ contains, with every function $F(z)$, also all functions $F(Az)$ for all non-degenerate matrices A.

It turns out that these two requirements are sufficient to determine few-parametric families $\mathcal{F}$ with the smallest possible number of parameters.

Proposition. *Let $\mathcal{F}$ be a $\frac{n \cdot (n+1)}{2}$-parametric family of continuous functions $F(z)$ from $\mathbb{R}^n$ to $\mathbb{R}$ for which $F(z) = 0$, $F(z) > 0$ for $z \neq 0$, and for some $F_0 > 0$, the set $\{z : F(z) \leq F_0\}$ is bounded. Let us also assume that the family $\mathcal{F}$ contains, with every function $F(z)$, also all functions $C \cdot F(z)$ for all $C > 0$ and all functions $F(Az)$ for all non-degenerate matrices A. Then, every function $F \in \mathcal{F}$ has the form*

$$F(z) = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} \cdot z_i \cdot z_j \right|^{\alpha}$$

for some real values α and g_{ij}.

Proof. In this proof, similarly to [62], we will use ellipsoids centered at 0, i.e., ellipsoids $E = \{z : \sum g_{ij} \cdot z_i \cdot z_j \leq 1\}$. We will call them *c-ellipsoids* (c for *centered*). To describe all such c-ellipsoids, we need to describe all symmetric matrices g_{ij}, so the family of c-ellipsoids is $\frac{n \cdot (n+1)}{2}$-dimensional. The border $\{z : \sum g_{ij} \cdot z_i \cdot z_j = 1\}$ of an ellipsoid E will be denoted by ∂E.

1°. Let $F \in \mathcal{F}$. Let us first prove that there is a c-ellipsoid E_0 on whose border ∂E_0 we have $F(z) = F_0$ for all $z \in \partial E_0$.

1.1°. By definition of the class $\mathcal{F}$, the set $S \stackrel{\text{def}}{=} \{z : F(z) \leq F_0\}$ is bounded, and each function $F \in \mathcal{F}$ is continuous. Since $F(z)$ is continuous, the set S is closed.

Every bounded set can be enclosed into a c-ellipsoid. It is known (see, e.g., [13]) that, among all ellipsoids containing a given closed bounded set, there is exactly one ellipsoid

with the smallest volume.

Let E_0 denote the c-ellipsoid with the smallest volume that contains the set S. We will say that this ellipsoid *corresponds* to the function $F(z)$.

Comment. The existence of the smallest-volume ellipsoid follows from the fact that every continuous function on a compact set attains its minimum. Uniqueness follows from the fact that if we have two c-ellipsoids E and E' of the same volume containing the same set, then we can select coordinates in which both matrices are diagonal, i.e., have the form $\sum g_i \cdot z_i^2 \le 1$ and $\sum g'_i \cdot z_i^2 \le 1$; then, for $g''_i = \dfrac{g_i + g'_i}{2}$, the ellipsoid $\sum g''_i \cdot z_i^2 \le 1$ also contains the bounded set and, as can be easily shown, has a strictly smaller volume than E and E'.

1.2°. It is known that every c-ellipsoid E in appropriate affine coordinates becomes a unit ball $\{z : \sum z_i^2 \le 1\}$. In other words, every ellipsoid can be obtained from a unit ball by an appropriate affine transformation. By combining the affine transformations corresponding to E and to E_0, we conclude that E can be obtained from the ellipsoid E_0 by an affine transformation $z \to Az$.

Under an affine transformation, the ratio of volumes is preserved. So, since E_0 the c-ellipsoid with the smallest volume contains the set $S = \{z : F(z) \le F_0\}$, E is the c-ellipsoid with the smallest volume containing the set $S' = \{z : F'(z) \le F_0\}$, where $F'(z) \stackrel{\text{def}}{=} F(Az) \in \mathcal{F}$.

Different ellipsoids correspond to different functions $F'(z)$, so we have as many such functions $F'(z)$ as there are ellipsoids – i.e., a $\dfrac{n \cdot (n+1)}{2}$-dimensional family.

1.3°. There are many affine transformations (rotations) that preserve the ball; in particular, for every two points on a unit sphere, there is a rotation that transforms one into another.

Thus, there are many affine transformations that preserve every ellipsoid E. In particular, for every two points $z, z' \in \partial E$ on this ellipsoid's border, there is an affine transformation that preserves ∂E and transforms z into z'.

For the ellipsoid E_0, let us denote, by G_0, the group of all affine transformations that

preserve ∂E_0.

1.4°. Let us show that the border ∂E_0 of the ellipsoid E_0 contains some points from the set $S \stackrel{\text{def}}{=} \{z : F(z) \le z_0\}$.

We will prove this by contradiction. Let us assume that the border ∂E_0 of the ellipsoid E_0 does not contain any points from the set S. Then, we can proportionally shrink E_0 and get a new c-ellipsoid with the smaller volume that still contains S. This contradicts to the fact that E_0 has the smallest volume. The statement is proven.

1.5°. Let us prove that for all $z \in \partial E_0 \cap S$, we have $F(z) = F_0$.

Indeed, since $z \in S$, by definition of the set S, we have $F(z) \le F_0$. On the other hand, since z belongs to the border ∂E_0, the point z is a limit of points z_n from outside E_0: $z_n \to z$. Outside E_0, there are no points from S, so for all $z_n \notin E_0$, we have $F(z_n) > F_0$. Since the function $F(z)$ is continuous, in the limit $z_n \to z$, we get $F(z) \ge F_0$. From $F(z) \le F_0$ and $F(z) \ge F_0$, we conclude that $F(z) = F_0$.

1.6°. Finally, let us prove that every point $z \in \partial E_0$ belongs to the set S; due to Part 1.5 of this proof, this will imply that $F(z) = F_0$ for all $z \in \partial E_0$.

We will prove this statement by contradiction. Let us assume that not every point $z \in \partial E_0$ belongs to the set S. Since transformations from G_0 transform every point $z \in \partial E_0$ into every other point $z' \in \partial E_0$, this means that not all transformations from G_0 preserve the intersection $\partial E_0 \cap S$. Thus, transformations that preserve the intersection form a subgroup $G_0' \subset G_0$. Subgroups of the group of rotations are well known, they have smaller dimension than G_0. Thus, we have a finite-parametric family of transformations (of dimension ≥ 1) that preserve ∂E_0 and turn the set $S = \{z : F(z) \le F_0\}$ into a different set S' – i.e., which turn $F(z)$ into a different function $F'(z)$ for which the ellipsoid E_0 is the same. Thus, we have an at least 1-dimensional family of functions $F'(z)$ corresponding to E_0.

By applying an affine transformation, we get a similar family of functions for every ellipsoid. The family of ellipsoids is already $\dfrac{n \cdot (n+1)}{2}$-dimensional, and for each of them,

there is an $\geq$ 1-dimensional family of functions – thus, we get a $\geq \left(\frac{n\cdot(n+1)}{2}+1\right)$-dimensional family of functions $F'(z)$ – which contradicts to our assumption that the whole family $\mathcal{F}$ is no more than $\frac{n\cdot(n+1)}{2}$-dimensional. This contradiction shows that indeed $\partial E \subseteq S$.

2°. The ellipsoid E_0 corresponding to the function $F(z)$ has the form $\{z : \|z\|^2 \leq 1\}$, where $\|z\|^2 \stackrel{\text{def}}{=} \sum\limits_{i,j} g_{ij}\cdot z_i \cdot z_j$. Let us prove that the function $F(z)$ has the form $F(z) = h(\|z\|)$ for some function $h(t)$ from real numbers to real numbers.

In other words, we need to prove that for every value v, the function $F(z)$ has a constant value on the border $\partial E_v \stackrel{\text{def}}{=} \{z : \|z\|^2 = v\}$ of the ellipsoid $E_v \stackrel{\text{def}}{=} \{z : \|z\|^2 \leq v\}$ which is obtained from E_0 by an appropriate dilation (homothety).

Indeed, if the function $F(z)$ had two different values on different points $z, z' \in \partial E_v$, then, similarly to Part 1.6 of this proof, we would be able to apply appropriate affine transformations and get a $\geq$ 1-parametric family of functions $F'(z)$ corresponding to the same ellipsoid E_0 and thus, a $\geq \left(\frac{n\cdot(n+1)}{2}+1\right)$-dimensional family of functions $F'(z)$ – which contradicts to our assumption that $\dim(\mathcal{F}) \leq \frac{n\cdot(n+1)}{2}$.

3°. To complete the proof, let us show that $h(t) = \text{const}\cdot t^\alpha$.

Let us consider the functions $F(z)$ corresponding to all
c-ellipsoids E which have the same volume $V(E)$ as E_0: $V(E) = V(E_0)$. The dimension of the family of all such ellipsoids is $\frac{n\cdot(n+1)}{2} - 1$.

For every function $F(z) = h(\|z\|) \in \mathcal{F}$, and for every two real numbers $C > 0$ and $k > 0$, the family $\mathcal{F}$ contains the function $C\cdot F(k\cdot z) = C\cdot h(k\cdot\|z\|)$. The corresponding transformations form a 2-dimensional multiplicative group.

The resulting family of functions cannot be fully 2-dimensional, since then, by considering such a family for every ellipsoid E with $V(E) = V(E_0)$, we would have a family of dimension

$$\geq \left(\frac{n\cdot(n+1)}{2} - 1\right) + 2 = \frac{n\cdot(n+1)}{2} + 1 > \frac{n\cdot(n+1)}{2}$$

inside the family $\mathcal{F}$. Thus, in the 2-dimensional transformation group, there is a ≥ 1-dimensional subgroup that keeps the function $h(t)$ invariant.

All subgroups of the 2-dimensional transformation group are well known, so we have $C(k) \cdot h(k \cdot t) = h(t)$ for some $C(k)$, and hence, $h(k \cdot t) = C^{-1}(k) \cdot h(t)$. It is known (see, e.g., [1]), that every continuous function that satisfies this functional equation has the form $h(t) = A \cdot t^{\alpha}$ for some A and α. The statement is proven, and so is our main result.

Mathematical comment: relation to Riemannian geometry. In general, the values g_{ij} describing spatial variability differ from one location to another. Thus, to describe spatial variability, we need to describe the values $g_{ij}(x)$ corresponding to different locations x. Mathematically, this is equivalent to describing a Riemannian metric.

How to determine g_{ij} and α from the empirical data? Based on the recorded values $f(x,t)$ at different locations x at different times $t = 1, \ldots, T$, we can estimate $C(z) = E[(f(x+z) - f(x))^2]$ as

$$C(z) = \frac{1}{T} \cdot \sum_{t=1}^{T} (f(x+z,t) - f(x,t))^2.$$

We can then use the following iterative procedure to find g_{ij} and α. Initially, we take $g_{ij}^{(0)} = \delta_{ij}$, i.e., $g_{ii}^{(0)} = 1$ and $g_{ij}^{(0)} = 0$ when $i \neq j$. At each iteration k, we start with the values $g_{ij}^{(k-1)}$, and do the following.

First, we estimate $\alpha^{(k)}$ from the condition $C(z) \approx \left| \sum g_{ij}^{(k-1)} \cdot z_i \cdot z_j \right|^{\alpha}$ We can find this α by taking the logarithms of both sides and applying the Least Squares Method to the resulting system of linear equations with unknown α:

$$\ln C(z) \approx \alpha \cdot \ln \left(\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}^{(k-1)} \cdot z_i \cdot z_j \right).$$

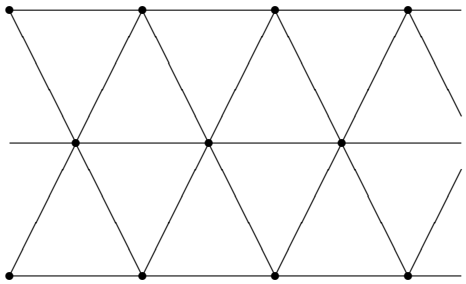
Once $\alpha^{(k)}$ is computed, we estimate $g_{ij}^{(k)}$ by applying the Least Squares Method to the following system of linear equations with unknown g_{ij}: $(C(z))^{1/\alpha^{(k)}} \approx \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} \cdot z_i \cdot z_j$.

Towards optimal sensor location. We want to place the sensors so as to reconstruct the value of $f(x)$ at all locations x with the desired accuracy ε. (Thus, in the spatial direction along which $f(x)$ changes faster we should place sensors more frequently.)

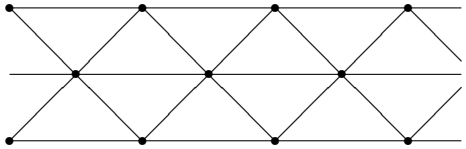
In precise terms, we want to place sensors in such a way that for each spatial location x, there is a sensor location s for which

$$E[(f(x)-f(s)]^2 = \left|\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\cdot(x_i-s_i)\cdot(x_j-s_j)\right|^{\alpha} \le \varepsilon^2.$$

For every symmetric matrix g_{ij}, there are affine coordinates – formed by its eigenvectors – in which this matrix become a unit matrix. In this case, the above condition simply means that every location must be ε-close to a sensor location. We have already mentioned, in Section 2.1, that under such condition, the asymptotically smallest number of sensors is provided by an equilateral triangle grid, i.e., a grid formed by equilateral triangles [46].



Hence, in general, the sensor grid can be obtained from the equilateral triangle one by an appropriate affine transformation.



In other words, we should place sensors along the grid parallel to eigenvectors of the matrix g_{ij}.

Mathematical comment: a similar problem of spatial distribution. Instead of spatial *variation*, we can consider a similar problem of spatial *distributions*, i.e., the problem of describing low-dimensional affine-invariant families of probability density functions – families that contain, with every function $\rho(x)$, the function $(\det(A))^{-1}\cdot\rho(Ax+a)$. Similar ellipsoid

arguments – but with general ellipsoids instead of c-ellipsoids – show that in this case, every distribution from the corresponding family has the form $\rho(x) = h(\|x - a\|)$ for some function $h(t)$ and some vector a, where $\|z\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} \cdot z_i \cdot z_j$ for some values g_{ij}.

2.3 Optimal Use of Mobile Sensors: Case Study of Unmanned Aerial Vehicles Patrolling the Border

In this section, we analyze the problem of the optimal use of mobile sensors, on the example of Unmanned Aerial Vehicles (UAVs) patrolling the border.

The results from this section were first published in [49]

Patrolling the border: a practical problem. Remote areas of international borders can be (and are) used by the adversaries: to smuggle drugs, to bring in weapons. It is therefore desirable to patrol the border, to minimize such actions.

Even with the current increase in the number of border patrol agents, it is not possible to effectively man every single segment of the border. It is therefore necessary to rely on other types of surveillance.

Unmanned Aerial Vehicles (UAVs) are an efficient way of patrolling the border:

- from every location along the border, they provide an overview of a large area, and
- if needed at a different location, they can move reasonably fast to the new location, without being slowed down by clogged roads or rough terrain.

However, while the area covered by the UAV is large, it is still limited. Due to resource limitations, we cannot have all the points on the border under a constant UAV surveillance. Thus, within a portion of the border that is covered by a UAV, it is necessary to keep the UAV moving.

How to describe UAV patrolling strategies. For simplicity, let us assume that the UAV can fly reasonably fast along the border, so that for each point, the interval between two consequent overflies does not exceed the time $2T$ needed to successfully cross the border area back-and-forth.

In the ideal case, this would means that the UAV is capable of detecting all adversaries – and thus, preventing all border violations. In reality, however, a fast flying UAV can miss the adversary. It is therefore desirable to select a trajectory that would minimize the effect of this miss.

The faster the UAV goes pass a certain location, the less time it spends in the vicinity of this location, the more probable it is that the UAV will miss the adversary. From this viewpoint, an important characteristic of the trajectory is the velocity $v(x)$ with which the UAV passes through the location x. So, by a patrolling strategy, we will mean a function $v(x)$ that describes how fast the UAV flies at different locations.

This strategy must be selected in such a way that a total time for a UAV to go from one end of the area to another one is equal to the given value T. The time during which a UAV passes from the location x to the location $x + \Delta x$ is equal to

$$\Delta t = \frac{\Delta x}{v(x)}. \tag{2.3.1}$$

Thus, the overall flight time is equal to the sum of these times, i.e., to

$$T = \int \frac{dx}{v(x)}, \tag{2.3.2}$$

where the integral is taken over the whole length of the border segment.

From the mathematical viewpoint, an arbitrary non-negative function $v(x)$ can describe the velocity at different locations. In practice, not every function $v(x)$ can be implemented, since the UAV has the largest possible velocity V, so we must have $v(x) \le V$ for all x.

From the computational viewpoint, it is convenient, instead of the velocity $v(x)$, to use its reciprocal

$$s(x) \stackrel{\text{def}}{=} \frac{1}{v(x)}. \tag{2.3.3}$$

In the geosciences, this reciprocal is called *slowness*; see, e.g., [5] and references therein; we will use this term in this section as well.

In terms of slowness, the requirement that the overall time be equal to T has a simpler form

$$T = \int s(x)\,dx. \tag{2.3.4}$$

In terms of slowness $s(x)$, the velocity limitation

$$v(x) \le V \tag{2.3.5}$$

takes the form $s(x) \ge S$, where $S \stackrel{\text{def}}{=} \dfrac{1}{V}$. Since $s(x) \ge S$, the value $s(x)$ can be represented as $S + \Delta s(x)$, where $\Delta s(x) \stackrel{\text{def}}{=} s(x) - S$ satisfy the simpler constraint $\Delta s(x) \ge 0$.

In terms of $\Delta s(x)$, the requirement that the overall time be equal to T has a simpler form

$$T = S \cdot L + \int \Delta s(x)\,dx, \tag{2.3.6}$$

where L is the total length of the piece of the border that we are defending, or, equivalently,

$$T_0 = \int \Delta s(x)\,dx, \tag{2.3.7}$$

where $T_0 \stackrel{\text{def}}{=} T - S \cdot L$.

Probability of detection. In order to select a reasonable patrolling strategy, we must find out, for each strategy, what is the probability that under this strategy, the adversary can still cross the border.

Let h denote a distance at which the UAV can still see. This means that when the adversary is trying to cross at location x, a UAV can, in principle, observe this adversary when it is located in the zone between $x - h$ and $x + h$. The width of this zone is equal to

$$(x + h) - (x - h) = 2h. \tag{2.3.8}$$

We have denoted the UAV's velocity at location x by $v(x)$. So, the time that it takes for a UAV to cross the zone of width $2h$ is equal to

$$t_{\text{obs}} = \frac{2h}{v(x)}. \tag{2.3.9}$$

In terms of slowness, this expression takes a simpler form

$$t_{\text{obs}} = 2h \cdot s(x). \tag{2.3.10}$$

Let Δt denote the time during which a UAV takes one snapshot of the underlying area. In these terms, during the crossing time t_{obs}, the UAV can take

$$n(x) = \frac{t_{\text{obs}}}{\Delta t} = \frac{2h}{\Delta t} \cdot s(x) \tag{2.3.11}$$

snapshots.

Let p_1 be the probability that an adversary can avoid detection based on a single snapshot. Then, to avoid detection during several snapshots means to avoid detection during the first snapshot, during the second snapshot, etc. It is reasonable to assume that the misses corresponding to different snapshots are statistically independent. Under this assumption, the probability $p(x)$ to be missed under $n(x)$ snapshots is equal to the product of $n(x)$ probabilities of a miss corresponding to different snapshots, i.e., equal to

$$p(x) = p_1^{n(x)}. \tag{2.3.12}$$

Substituting the above expression for $n(x)$ in terms of $s(x)$, we conclude that

$$p(x) = p_1^{(2h/\Delta t) \cdot s(x)}, \tag{2.3.13}$$

i.e., that

$$p(x) = \exp(-k \cdot s(x)), \tag{2.3.14}$$

where we denoted

$$k \stackrel{\text{def}}{=} \frac{2h}{\Delta t} \cdot |\ln(p_1)|. \tag{2.3.15}$$

Relative importance of different locations. We also need to take into account that different locations along the border have different importance.

For example, if smugglers succeed in bringing drugs to the vicinity of the city of El Paso, they can store in a safe place and distribute it without exposure. On the other hand,

if they bring the same shipment in the remote desert area, they still need to bring it close to a town or a city, and risk being detected while they are transporting this shipment.

In the case of smugglers, this importance can be described in monetary terms: a shipment available in city can be sold for a much larger amount than a shipment available at some remote location from which it still has to be transported to a city. The corresponding price $w(x)$ of the shipment successfully transported across the border at a point with coordinate x can be used as a measure of potential benefit, for the adversary, of penetrating the border at this particular location.

For other types of border penetration, we can also similarly estimate the potential benefit to the adversary.

We will start our analysis with a simplified case when we know the exact value of $w(x)$ for all x. After that, we will explain how to deal with a more realistic case, when we only know $w(x)$ with uncertainty.

Decision making: reminder. We assume that the adversary has observed the UAV, so the adversary knows the slowness function $s(x)$ and is, thus, capable of computing the probability $p(x)$ of avoiding detection. How does an adversary make decisions based on this knowledge?

A standard way to describe preferences of a decision maker is to use the notion of *utility*; see, e.g., [22, 23, 44, 65, 87]. To describe the utility of an outcome A, we need to select two extreme outcomes: a very unfavorable alternative A_- and a very favorable outcome A_+.

We assume that all outcomes A in which we are interested are better than A_- and worse than A_+. If we denote the relation "the decision maker prefers A' to A" by $A \le A'$, then we can describe this assumption as $A_- \le A \le A_+$.

Then, for each probability $p \in [0,1]$, we can consider a *lottery* $L(p)$ in which we have A_+ with probability p and A_- with the remaining probability $1-p$.

For $p = 1$, the lottery $L(p)$ coincides with A_+, so we have $A \le A(1)$. For $p = 0$, the lottery $L(p)$ coincides with A_-, so we have $A(0) \le A$. The larger p, i.e., the larger the

probability of a beneficial event A_+, the more beneficial is the lottery $L(p)$ for the decision maker. So, if $p < q$, then $L(p) < L(q)$.

Let p_0 be the infimum (greatest lower bound) of the set of all the values p for which $A \le L(p)$. Then:

- When $p < p_0$, then for $\widetilde{p} = (p + p_0)/2$, we have $\widetilde{p} < p_0$ and thus, by definition of the infimum, we cannot have $A \le L(\widetilde{p})$. Thus, we have $L(\widetilde{p}) \le A$. Since $p < \widetilde{p}$, we have $L(p) < L(\widetilde{p}) \le A$ and thus, $L(p) < A$.

- When $p > p_0$, then, since p_0 is the greatest lower bound, p is not a lower bound, i.e., there exists a value $\widetilde{p}$ for which $A \le L(\widetilde{p})$ and $\widetilde{p} < p$. Since $\widetilde{p} < p$, we have $L(\widetilde{p}) < L(p)$ hence $A < L(p)$.

Thus, we have the value p_0 that has the following property:

- when $p < p_0$, the corresponding lottery is worse than the event A:

$$L(p) < A; \tag{2.3.16}$$

- when $p > p_0$, the corresponding lottery is better than the event A:

$$L(p) > A. \tag{2.3.17}$$

This threshold value p_0 is called the *utility* of the event A. The utility is usually denoted by $u(A)$.

We can simplify the above somewhat complicated relation between A and p_0 by saying that the event $L(p_0)$ is *equivalent* to A. We will denote this equivalence by $A \sim L(p_0)$.

The notion of utility depends on the choice of the outcomes A_- (for which utility is 0) and A_+ (for which utility is 1). In principle, we select different outcomes A'_- and A'_+. One can show that the new value $u'(A)$ is linearly related to the old one: $u'(A) = a \cdot u(A) + b$, where:

- $b = u'(A_-)$ is the utility of A_- in the new scale, and

- $a + b = u'(A_+)$ is the utility of A_+ in the new scale, so we can determine a as $u'(A_+) - u'(A_-)$.

In other words, utility is defined modulo an arbitrary linear transformation

$$u(A) \to u'(A) = a \cdot u(A) + b. \tag{2.3.18}$$

In practice, we can rarely predict the exact consequences of each decision. The consequences depend on the circumstances. For example, if we decide whether to take an umbrella or not, the consequences of this decision depend on whether it will rain or not. In the ideal situation, we know the probabilities $p_1, \ldots, p_n$ of different possible consequences $E_1, \ldots, E_n$. In other words, the action leads to E_1 with probability p_1, to E_2 with probability p_2, ..., and to E_n with probability p_n.

By definition of the utility, the event E_1 is equivalent to a lottery $L(u(E_1))$ in which we get A_+ with probability $u(E_1)$, the event E_2 is equivalent to a lottery $L(u(E_2))$ in which we get A_+ with probability $u(E_2)$, etc. Thus, the original action is equivalent to the composite lottery, in which:

- with probability p_1, we get a lottery that results in A_+ with probability $u(E_1)$, and in A_- otherwise;
- with probability p_2, we get a lottery that results in A_+ with probability $u(E_2)$, and in A_- otherwise;
- ...

In this composite lottery, we get either A_+ or A_-, and the probability of getting A_+ can be easily computed as

$$u \stackrel{\text{def}}{=} p_1 \cdot u(E_1) + p_2 \cdot u(E_2) + \ldots + p_n \cdot u(E_n). \tag{2.3.19}$$

Thus, the original action is equivalent to the lottery $L(u)$. By definition of the utility, this means that the utility of the action is equal to u.

From the mathematical viewpoint, u is the expected value of the utility of different consequences, so we can conclude that the utility of an action is the expected value of utilities of its consequences.

Strategy selected by the adversary. We have already mentioned that utility is defined modulo an arbitrary linear transformation. For convenience, let us select the utility scale in such a way that for the adversary, the utility of not being able to cross the border is 0.

In this scale, let $w(x)$ denote the utility of the adversary succeeding in crossing the border at location x. We have assumed that we know the exact value of $w(x)$ for every location x.

According to decision theory, the adversary will select a location x at which the expected utility

$$u(x) = p(x) \cdot w(x) = \exp(-k \cdot s(x)) \cdot w(x) \tag{2.3.20}$$

is the largest possible.

Thus, for each slowness function $s(x)$, the adversary's gain $G(s)$ is equal to

$$G(s) = \max_x u(x) = \max_x \left[\exp(-k \cdot s(x)) \cdot w(x)\right]. \tag{2.3.21}$$

Towards an optimal strategy for patrolling the border. Our objective is to select a strategy $s(x)$ for which the gain $G(s)$ is the smallest possible.

Let x_m be the location at which the utility $u(x) = \exp(-k \cdot s(x)) \cdot w(x)$ attains its largest possible value. If close to x_m, we have a point x_0 for which $u(x_0) < u(x_m)$ and $s(x_0) > S$, then we can slightly decrease the slowness $s(x_0)$ at the vicinity of x_0 (i.e., go faster in this vicinity) and use the resulting time to slow down (i.e., to go slower) at all locations x at which $u(x) = u(x_m)$. As a result, we slightly decrease the value $u(x_m) = \exp(-k \cdot s(x_m)) \cdot w(x_m)$.

Yes, we also slightly increase the value

$$u(x_0) = \exp(-k \cdot s(x_0)) \cdot w(x_0), \tag{2.3.22}$$

but for small changes, this value is still smaller that $u(x_m)$ and thus, does not affect the maximum $\max_x u(x)$. As a result, the gain $G(s)$ decreases (this argument is similar to the one presented in [47]).

So, when the adversary's gain is minimized, we get

$$u(x) = u_0 = \text{const} \tag{2.3.23}$$

hence

$$\exp(-k \cdot s(x)) = \frac{u_0}{w(x)}, \tag{2.3.24}$$

thence

$$s(x) = \frac{1}{k} \cdot (\ln(w(x)) - \ln(u_0)) \tag{2.3.25}$$

and

$$\Delta s(x) = \frac{1}{k} \cdot \ln(w(x)) - \Delta_0, \tag{2.3.26}$$

where

$$\Delta_0 \stackrel{\text{def}}{=} -\frac{1}{k} \cdot \ln(u_0) - S. \tag{2.3.27}$$

When this value gets to $s(x) = S$ and $\Delta s(x) = 0$, we get $\Delta s(x) = S$. Thus, we conclude that

$$\Delta s(x) = \max\left(\frac{1}{k} \cdot \ln(w(x)) - \Delta_0, 0\right). \tag{2.3.28}$$

The value Δ_0 can be determined from the condition that

$$\int \Delta s(x)\, dx =$$

$$\int \max\left(\frac{1}{k} \cdot \ln(w(x)) - \Delta_0, 0\right)\, dx = T_0. \tag{2.3.29}$$

Since this integral monotonically decreases with Δ_0, we can use bisection to find the appropriate value Δ_0; see, e.g., [18].

Towards taking fuzzy uncertainty into account. The above algorithm is based on the assumption that we know the exact value of the adversary's gain $w(x)$ at different locations. In reality, as we have mentioned, we only have expert estimates for $w(x)$. To formalize these estimates, we can use fuzzy techniques; see, e.g., [50, 73].

Once we have the fuzzy values $w(x)$, we can apply Zadeh's extension principle to the above crisp formulas and thus, come up with fuzzy recommendations about the slowness, such as "go somewhat slow here", "go fast", etc. It is well known (see, e.g., [50, 73]) that Zadeh's extension principle is equivalent to processing α-cuts. Specifically, if we know a relation $y = f(x_1, \ldots, x_n)$ between the inputs $x_1, \ldots, x_n$ and the desired value y, and we know the fuzzy values $X_1, \ldots, X_n$ of the inputs, then the resulting fuzzy value Y of the output can be obtained as follows: for every $\alpha \in (0, 1]$, we have

$$Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)) = \{f(x_1, \ldots, x_n) : x_1 \in X_1(\alpha), \ldots, x_n \in X_n(\alpha)\}, \quad (2.3.30)$$

where for each fuzzy value Z with a membership function $\mu_Z(z)$, its α-cut $Z(\alpha)$ is defined as

$$Z(\alpha) \stackrel{\text{def}}{=} \{z : \mu_Z(z) \geq \alpha\}. \quad (2.3.31)$$

When a fuzzy value is a fuzzy number, each α-cut is an interval $Z(\alpha) = [\underline{Z}(\alpha), \overline{Z}(\alpha)]$. When all the inputs are fuzzy numbers, the above formula takes the simplified form

$$[\underline{Y}(\alpha), \overline{Y}(\alpha)] = \{f(x_1, \ldots, x_n) : x_i \in [\underline{X}_i(\alpha), \overline{X}_i(\alpha)]. \quad (2.3.32)$$

When the function $y = f(x_1, \ldots, x_n)$ is an increasing function of all its variables, then its largest value is attained when all its inputs attain their largest values, and its smallest value is attained when all its inputs attain their smallest values. In other words, the desired α-cut has the form $[\underline{Y}(\alpha), \overline{Y}(\alpha)]$, where

$$\underline{Y}(\alpha) = f(\underline{X}_1(\alpha), \ldots, \underline{X}_n(\alpha)); \quad (2.3.33)$$

$$\overline{Y}(\alpha) = f(\overline{X}_1(\alpha), \ldots, \overline{X}_n(\alpha)). \quad (2.3.34)$$

Similarly, when the function $y = f(x_1, \ldots, x_n)$ is an increasing function of the variables $x_1, \ldots, x_k$ and decreasing in $x_{k+1}, \ldots, x_n$), then the α-cut has the form $[\underline{Y}(\alpha), \overline{Y}(\alpha)]$, where

$$\underline{Y}(\alpha) = f(\underline{X}_1(\alpha), \ldots, \underline{X}_k(\alpha), \overline{X}_{k+1}(\alpha), \ldots, \overline{X}_n(\alpha)); \tag{2.3.35}$$

$$\overline{Y}(\alpha) = f(\overline{X}_1(\alpha), \ldots, \overline{X}_k(\alpha), \underline{X}_{k+1}(\alpha), \ldots, \overline{X}_n(\alpha)). \tag{2.3.36}$$

In our case, for each location x, we know the fuzzy value $W(x)$ of the corresponding gain. This means that for each degree α, we know the corresponding α-cut $W(x)(\alpha) = [\underline{W(x)}(\alpha), \overline{W(x)}(\alpha)]$.

In the crisp case, based on the gains $w(x)$, we first compute the value Δ_0 and then the corresponding changes $\Delta s(x)$ in the UAV's slowness. Thus, in the fuzzy case, we need to find the α-cuts for Δ_0 and then, α-cuts for $\Delta s(x)$.

According to the above formula for Δ_0, its value is an increasing function of all the inputs $w(x)$. Thus, we conclude that for every α, the α-cut for Δ_0 has the form $[\underline{\Delta}_0(\alpha), \overline{\Delta}_0(\alpha)]$, where $\underline{\Delta}_0(\alpha)$ can be determined from the condition that

$$\int \max\left(\frac{1}{k} \cdot \ln(\underline{W(x)}(\alpha)(x)) - \underline{\Delta}_0(\alpha), 0\right) dx = T_0, \tag{2.3.37}$$

and $\overline{\Delta}_0(\alpha)$ can be determined from the condition that

$$\int \max\left(\frac{1}{k} \cdot \ln(\overline{W(x)}(\alpha)(x)) - \overline{\Delta}_0(\alpha), 0\right) dx = T_0. \tag{2.3.38}$$

The value $\Delta s(x)$ is increasing in $w(x)$ and decreasing in Δ_0. Thus,

- the smallest value $\underline{\Delta s(x)}(\alpha)$ is attained when $w(x)$ is the smallest and Δ_0 is the largest, and
- the largest value $\overline{\Delta s(x)}(\alpha)$ is attained when $w(x)$ is the largest and Δ_0 is the smallest:

$$\underline{\Delta s(x)}(\alpha) = \max\left(\frac{1}{k} \cdot \ln(\underline{W(x)}(\alpha)) - \overline{\Delta}_0(\alpha), 0\right); \tag{2.3.39}$$

$$\overline{\Delta s(x)}(\alpha) = \max\left(\frac{1}{k} \cdot \ln(\overline{W(x)}(\alpha)) - \underline{\Delta}_0(\alpha), 0\right). \tag{2.3.40}$$

The resulting recommendations can be used either as a guidance for a human controller, or – by using fuzzy control – in the design of the automatic controller.

Comment. Fuzzy techniques can be similarly used in other problems related to security, e.g., in finding optimal placement for bio-weapon detectors [49] as described in Section 2.1.

2.4 Efficient Algorithms for Optimizing Sensor Use: Case Study of Security Problems

In this section, we analyze the problem of designing efficient algorithms for optimizing sensor use, on the example of security problems.

The results from this section first appeared in [59].

Formulation of the problem. Security problems typically involve making strategic resource allocation decisions in order to prevent or mitigate attacks. Game theory has been used to model decision-making in a variety of security situations, including the protection of critical infrastructure from terrorist attacks [93, 11], computer network security [4, 72, 96], robot patrolling [27, 3, 34], and scheduling [90]. Recently, research on security games has been deployed to make real-world homeland security decision, including the ARMOR system in use at the LAX airport [83], the IRIS system used by the Federal Air Marshals Service [102], and the GUARDS system developed for the Transportation Security Administration [84].

A key research direction has been the development of faster algorithms to scale to increasingly large and complex instances of security games [17, 79, 47, 42]. Faster algorithms that exploit the structure of security games have been key in enabling new applications of these methods. We present new algorithms for one of the most basic classes of security games: Stackelberg security games with multiple, identical defender resources. This class of games was described by Kiekintveld et al. [47], which also gave a polynomial-time ($O(n^2)$) algorithm for computing Stackelberg equilibrium of these games.

In this section, we present two new algorithms for Stackelberg security games with identical resources. The first solves a special case in worst-case linear time ($O(n)$), and the second solves the general case in $O(n \cdot \log(n))$. In addition to improving on the theoretical complexity of the best known methods for this class of security games, our algorithms are based on a detailed analysis of the structure of the solutions for these games, which may lead to faster algorithms or heuristics for more complex variants of security games.

Security game model: general case. In this section, we adopt the general model of security games described in [47]. A security game has two players, a *defender*, Θ, and an *attacker*, Ψ. There is a set of n targets $t_i \in T$ that the attacker wishes to attack and the defender wishes to protect. In our model, the attacker can choose to attack exactly one target from this set. The defender has a limited number of resources, $m < n$, that can be deployed to protect the targets. We assume throughout that these resources are identical, and that at most one resource can be used to protect each target.

If the attacker chooses to attack target t_i, we call the attack *successful* if the target is left uncovered by a defender, and *unsuccessful* if the target is covered by a defender. The defender's payoff for an uncovered attack is denoted $U^u_\Theta(t)$, and for a covered attack $U^c_\Theta(t)$. Similarly, $U^u_\Psi(t)$ and $U^c_\Psi(t)$ denote the attacker's payoffs in each case. We will make the standard assumptions for security games that $U^u_\Theta(t) < U^c_\Theta(t)$ and $U^u_\Psi(t) > U^c_\Psi(t)$ for all targets t. In other words, the attacker receives a higher payoff for attacking an undefended target than a defended one, and vice versa for the defender. Note that this does not imply that the games are zero-sum (or even strategically zero-sum).

The attacker's possible strategies consist of attacking each of the n targets. The defender's space of possible strategies consists of all possible ways to assign the m resources to the n targets. However, we can conveniently summarize the defenders strategy by defining the *coverage vector* which gives the probability that there is a defender resource assigned to each individual target. Denote these probabilities by c_i, so that $\sum_{i=1}^{n} c_i = m$. The attacker's expected utility for an attack on target t_i can then be written as $(1-c_i) \cdot U^u_\Psi(t_i) + c_i \cdot U^c_\Psi(t_i)$,

and similarly for the defender. Because of our assumptions, for each target t_i, the defender's expected payoff decreases when the probability c_i of defending this target increases. We also assume that all defender resources are identical and can be deployed to any target.

We model the game as a *Stackelberg game* [103] in which the attacker can observe the defender's strategy $(c_1, \ldots, c_n)$ before planning an attack (modeling the capability of attackers to use surveillance to learn security policies). The standard solution concept for these games is Strong Stackelberg Equilbrium (SSE) [57, 8]. In an SSE, the leader first selects a mixed strategy, and then the follower chooses an optimal pure strategy in response, breaking ties in favor of the leader. This behavior can be induced by the leader selecting a strategy arbitrarily close to the equilibrium that causes the the follower to strictly prefer the desired strategy [104], but in practice we compute the limit point where ties are broken in favor of the leader.

Case of fully protective resources: description and analysis of the problem. Let us first consider a practically important case of fully protective resources. When a single resource is deployed at a target, the target is fully protected. For now, we restrict the attacker's payoff for attacking a covered resource to 0: $U^c_\Psi(t) = 0$. We begin with a basic analysis that describes the structure of the solution.

According to the analysis in [47], in our game, the objective of the defender is equivalent to minimizing the expected utility of the attacker, as long as the solution has the largest possible set of targets that are optimal for the attacker to select. Using the Strong Stackelberg Equilibrium assumption, the attacker will select breaking ties in favor of the defender. Therefore, in most cases we will not need to take into account the defender's payoffs directly; the defender's payoff will be maximized implicitly by finding a set of coverage probabilities c_i so as to minimize the expected payoff of the attacker.

The attacker seeks to maximize the expected value of a successful attack:

$$\arg\max_i \ (1 - c_i) \cdot U^u_\Psi(t_i), \tag{2.4.1}$$

while the defender chooses a coverage vector to minimize the attacker's expected payoff.

Let t_{i_o} denote the optimal target to attack, so we have for every target t_i:

$$(1 - c_{i_o}) \cdot U^u_\Psi(t_{i_o}) \geq (1 - c_i) \cdot U^u_\Psi(t_i). \tag{2.4.2}$$

Now, assume that for some i this inequality is strict and that $c_i > 0$. In this case we could decrease c_i and increase the probability c_j for all j such that

$$(1 - c_j) \cdot U^u_\Psi(t_j) = (1 - c_{i_o}) \cdot U^u_\Psi(t_{i_o}), \tag{2.4.3}$$

thus decreasing the expected payoff of the attacker.

Therefore, for the minimizing coverage vector, all targets can be divided into two groups:

- either the expected value for attacking the target is equal to the optimal value,
- or the expected value is less than the optimal value and the coverage probability assigned to the target is 0.

In other words, the optimal solution will have the property that the attacker's expected value for all targets with positive coverage probability is equal to some constant q:

$$(1 - c_i) \cdot U^u_\Psi(t_i) = q. \tag{2.4.4}$$

For any target t_i with $c_i > 0$ we can thus calculate the necessary value of c_i as:

$$c_i = 1 - \frac{q}{U^u_\Psi(t_i)}. \tag{2.4.5}$$

For all other targets $U^u_\Psi(t_i) < q$, and therefore

$$1 - \frac{q}{U^u_\Psi(t_i)} < 0. \tag{2.4.6}$$

Summarizing: once we know q, we can find all the probabilities c_i by using the formula

$$c_i = \max\left(1 - \frac{q}{U^u_\Psi(t_i)}, 0\right). \tag{2.4.7}$$

For each target t_i, this formula requires a constant number of computational steps. Therefore, after q is computed, we can therefore compute all the probabilities c_i in time $O(n)$.

So, to find the optimal covering vector, it is sufficient to find the constant q. This constant can be found from the condition that $\sum\limits_{i=1}^{n} c_i = m$, i.e., that

$$\sum_{i=1}^{n} \max\left(1 - \frac{q}{U_\Psi^u(t_i)}, 0\right) = m. \tag{2.4.8}$$

The left-hand side of this equality decreases as q increases. So, if for some q, the resulting sum is smaller than m, this means that the optimal value q_o is smaller than q: $q_o < q$; similarly, if for some q, the resulting sum is larger than m, this means that the optimal value q_o is larger than q: $q_o > q$.

The structure of the optimal covering vector can be clarified if we sort the targets in order of descending attacker payoffs for successful attacks, so that:

$$U_\Psi^u(t_{(1)}) \geq \ldots \geq U_\Psi^u(t_{(n-1)}) \geq U_\Psi^u(t_{(n)})). \tag{2.4.9}$$

We can also add $U_\Psi^u(t_{(0)}) \stackrel{\text{def}}{=} +\infty$ and $U_\Psi^u(t_{(n+1)}) \stackrel{\text{def}}{=} 0$, then

$$U_\Psi^u(t_{(0)}) \geq \ldots \geq U_\Psi^u(t_{(n)}) \geq U_\Psi^u(t_{(n+1)}). \tag{2.4.10}$$

The values $U_\Psi^u(t_{(i)})$ divide the real line into intervals $[U_\Psi^u(t_{(i+1)}), U_\Psi^u(t_{(i)})]$, so the threshold constant q must be in one of these intervals, i.e., between $U_\Psi^u(t_{(k+1)})$ and $U_\Psi^u(t_{(k)})$ for some k. In this case, according to the above formula for c_i, all targets with a value greater than q (i.e., the targets $t_{(1)}$, $t_{(2)}$, $\ldots$, $t_{(k)}$ in the above ordering) will be protected with positive probability, and all targets with value smaller than q (i.e., targets $t_{(k+1)}$, $t_{(k+2)}$, $\ldots$) are left unprotected. Let k denote the index of last target that has positive probability. Given the constraint that the coverage probabilities add to m, we can write:

$$\sum_{i=1}^{k} \left(1 - \frac{q}{U_\Psi^u(t_{(i)})}\right) = m, \tag{2.4.11}$$

hence

$$k - m = q \cdot \sum_{i=1}^{k} \frac{1}{U_\Psi^u(t_{(i)})}, \tag{2.4.12}$$

and

$$q = \frac{k - m}{\sum\limits_{i=1}^{k} \frac{1}{U^u_\Psi(t_{(i)})}}. \tag{2.4.13}$$

So, instead of selecting q, we can simply select a threshold value k.

Once we have found this k, we can then compute the threshold value q by using the above formula and then use this q to find the optimal coverage probabilities.

For the optimal value $k = k_o$, the corresponding value q is located in the interval $[U^u_\Psi(t_{(k+1)}), U^u_\Psi(t_{(k)})]$. If for some k, the value q computed by the above formula is smaller than $U^u_\Psi(t_{(k+1)})$, this means that we are trying to cover too few targets, with the same q, we can cover more, so the optimal value k_o should be larger: $k > k_o$.

Similarly, if for some k, the value q computed by the above formula is larger than $U^u_\Psi(t_{(k)})$, this means that we are trying to cover too many targets, so the optimal value k_o should be smaller: $k_o < k$.

Let us show that this argument can lead to a linear-time algorithm for finding the optimal coverage vector.

Case of fully protective resources: linear-time algorithm. On each stage of this iterative algorithm, we have three lists of targets:

- the list T^c of the targets t_i about which we are sure that in the optimal coverage, these targets will be covered with a positive probability $c_i > 0$;
- the list T^u of the targets t_i about which we are sure that in the optimal coverage, these targets will not be covered ($c_i = 0$);
- the list $T^?$ of the targets t_i about which we have not yet found out whether they will be covered or not in the optimal coverage.

In the beginning, we set $T^c = T^u = \emptyset$ and

$$T^? = \{t_1, t_2, \ldots, t_n\}. \tag{2.4.14}$$

At each stage, we will also update the value

$$S^c = \sum_{t_i \in T^c} \frac{1}{U^u_\Psi(t_i)}. \tag{2.4.15}$$

In the beginning, since $T^c = \emptyset$, we take $S^c = 0$.

At each iteration, we do the following:

- First, we compute the median m of the values $U^u_\Psi(t_i)$ corresponding to all "undecided" targets $t_i \in T^?$.

- Then, by analyzing the elements of the undecided set $T^?$ one by one, we divide them into two subsets

$$T^+ = \{t_i : U^u_\Psi(t_i) \geq m\}, T^- = \{t_i : U^u_\Psi(t_i) < m\}. \tag{2.4.16}$$

  In the set T^+, we find the target t_{k^+} with the smallest value of $U^u_\Psi(t_i)$; in the set T^-, we find the target t_{k^-} with the largest value of $U^u_\Psi(t_i)$.

- We then compute

$$S^+ = \sum_{t_i \in T^+} \frac{1}{U^u_\Psi(t_i)}, \tag{2.4.17}$$

  $s^+ = S^c + S^+$, and $q = \dfrac{k-m}{s^+}$.

- If $q < U^u_\Psi(t_{k^-})$, then, as we have argued in our analysis, this means that we are trying to cover too few targets, so definitely all the elements from the set T^+ should be covered. Thus, we replace T^c with $T^c \cup T^+$, $T^?$ with T^-, and S^c with s^+.

- If $q > U^u_\Psi(t_{k^+})$, this means that we are trying to cover too many targets, so definitely all the elements from the set T^- should not be covered. Thus, we replace T^u with $T^u \cup T^-$ and $T^?$ with T^+ (and keep S^c unchanged).

- Finally, if $U^u_\Psi(t_{k^-}) \leq q \leq U^u_\Psi(t_{k^+})$, this means that this q is optimal.

Iterations continue until we find the optimal value q. Once we get the optimal value q, we can then find the optimal covering probabilities as $c_i = \max\left(1 - \dfrac{q}{U^u_\Psi(t_i)}, 0\right)$.

Let us prove that this algorithm indeed takes linear time. Indeed, at each iteration, we can compute the median in linear time [18], and all other operations with the set $T^?$ also take time $\mathcal{T}$ linear in the number of elements $|T^?|$ of this set $T^?$: $\mathcal{T} \le C \cdot |T^?|$ for some C. We start with the set $T^?$ of size n. On the next iteration, we have a set of size $n/2$, then $n/4$, etc. Thus, the overall computation time is $\le C \cdot (n + n/2 + n/4 + \ldots) \le C \cdot 2n$, i.e., linear in n.

General case: analysis of the problem. Let us now go back to the general case, when defense resources are not necessarily fully protective. In this general case, the attacker seeks to maximize the expected value of a successful attack:

$$\arg\max_\ell e_\ell(c_\ell), \tag{2.4.18}$$

where

$$e_\ell(c_\ell) \stackrel{\text{def}}{=} (1 - c_\ell) \cdot U^u_\Psi(t_\ell) + c_\ell \cdot U^c_\Psi(t_\ell), \tag{2.4.19}$$

while the defender chooses a coverage vector to minimize the attacker's expected payoff

$$e(c) = \max_\ell e_\ell(c_\ell). \tag{2.4.20}$$

Once we select a coverage vector, we thus divide all the targets into three groups:

- the first group is formed by targets t_i for which $c_i = 1$; these targets that will be guarded with certainty;
- the second group is formed by targets t_j for which $0 < c_j < 1$; these targets with some probability will be guarded and with some probability will not be guarded;
- the third group is formed by targets t_k for which $c_k = 0$; these targets will be not guarded.

Intuitively, this division makes sense:

- the most important targets must be guarded no matter what,
- the least valuable targets will not be guarded at all if we do not have enough resources, and
- intermediate targets will be guarded with some probability.

Let us prove that this intuitive meaning is indeed true. To be more precise, let us prove that in this game, there exists a minimizing vector $(c_1, \ldots, c_n)$ that has the following properties:

- The expected payoff $e_(c_i)$ of each target t_i of the first group (with $c_i = 1$) is larger than or equal to the expected payoff $e_j(c_j)$ of each target t_j of the second group (with $0 < c_j < 1$):
$$e_i(c_i) \geq e_j(c_j). \tag{2.4.21}$$
- The expected payoff $e_j(c_j)$ of all target $t_j, t_{j'}$ from the second group (with $0 < c_j < 1$) is the same:
$$e_j(c_j) \geq e_{j'}(c_{j'}). \tag{2.4.22}$$
- The expected payoff $e_j(c_j)$ of each target t_j from the second group (with $0 < c_j < 1$) is larger than or equal to the expected payoff of each target t_k from the third group (with $c_k = 0$):
$$e_j(c_j) \geq e_k(c_k). \tag{2.4.23}$$

Intuitively, this makes sense: if the attacker's expected payoff from a target t_i that we guard absolutely is smaller than the expected payoff from some other target t_j that we guard with a certain probability, then it makes sense to switch some probability from target t_i to target t_j. In this case, the attacker's expected value for t_j decreases; for t_i it somewhat increases, but since it was smaller than for the target t_j, it remains smaller, and the maximum of these values $e_i(c_i)$ does not increase.

To prove this result more formally, let us start with any minimizing vector and show that by appropriate transformations it can be transformed into a minimizing vector with the desired properties.

First, let us show how we can satisfy the first property. For that, let us show that we can decrease the number of targets t_i for which $c_i = 1$ and for which, for some j, we have $0 < c_j < 1$ and $e_i(c_i) < e_j(c_j)$. Indeed, out of all such targets, let us pick a target for which the value $e_i(c_i)$ is the smallest, and let j be the corresponding target from the second group. Then, for some $\Delta > 0$, we replace c_i with $c_i' = c_i - \Delta$ and c_j with $c_j' = c_j + \Delta$. When Δ is small enough, we have $c_i' > 0$, $c_j' < 1$, and $e_i(c_i')$ is still smaller than all the values $e_\ell(c_\ell)$ for which we had $e_i(c_i) < e_\ell(e_\ell)$.

Let us keep all the other probabilities the same: $e_\ell' = c_\ell$ for all $\ell \neq i, j$. This replacement does not change the sum $\sum c_i$, so while $c_i' \geq 0$ and $c_j' \leq 1$, we still get a coverage vector. As we have mentioned, the expected value of a target decreases with the increase in the probability that this target will be guarded. Thus, when Δ increases, the value $e_i(c_i - \Delta)$ increases while the value $e_j(c_j + \Delta)$ decreases. So, while $e_i(c_i - \Delta) \leq e_j(c_j + \Delta)$, we have $e_i(c_i) < e_i(c_i - \Delta) \leq e_j(c_j + \Delta) < e_j(c_j)$. Thus, $e_i(c_i') < e_j(c_j) \leq e(c) = \max\limits_\ell c_\ell(e_\ell)$ and similarly $e_j(c_i') < e_j(c_j) \leq e(c) = \max\limits_\ell c_\ell(e_\ell)$. For all other targets ℓ, we have $c_\ell' = c_\ell$ hence $e_\ell(c_i') = e_\ell(c_\ell) \leq e(c)$. Thus,

$$e(c') = \min\left(e_i(c_i'), e_j(c_j'), \min_{\ell \neq i,j} e_\ell(c_\ell')\right) \leq e(c). \tag{2.4.24}$$

Since the original vector c is a minimizing vector, the value $e(c)$ is the smallest possible value, we conclude that c' is also a minimizing vector.

Let us show that in the new minimizing vector, the number of targets ℓ from the first group for which the expected value is smaller than for some target from the second group is smaller than the same number computed based on the original minimizing vector. Indeed, in the new minimizing vector, the target t_i is no longer from group one, it is now from group two, so it is sufficient to check that this addition of a new group-two target does not lead to the appearance of a new "wrong-order" target of group one. Indeed, if for some target $t_{i'}$

from group one, we have $e_{i'}(c_{i'}) < e_i(c_i')$, then we could not have $e_i(c_i) < e_{i'}(c_{i'})$ – because we selected Δ so small that all such inequalities remain. Thus, we have $e_{i'}(c_{i'}) \le e_i(c_i)$ but in this case $e_i(c_i) < e_j(c_j)$ implies that $e_{i'}(c_{i'}) < e_j(c_j)$ – and thus, $t_{i'}$ was the wrong-order target already in the original minimizing vector.

By applying this procedure again and again, we arrive at the new minimizing vector for which the number of wrong-order targets of group one is 0, i.e., in which the expected payoff for every target from group one is larger than or equal to the expected payoff for every target from group two.

Similarly, we can get a new minimizing vector in which the expected payoff for every target from group two is larger than or equal to the expected payoff of every target of group three.

Let us now show that we can arrive at the minimizing vector for which for all targets from group two, the expected payoff is the same. Let us show how an appropriate procedure can minimize the number of pairs $(t_j, t_{j'})$ of targets from group two for which $e_j(c_j) < e_{j'}(c_{j'})$. Indeed, let us sort all the corresponding values $e_j(c_j)$ into an increasing sequence, and let us take two neighboring values from this sequence. Similarly to the above case, we replace c_j with $c_j' = c_j - \Delta$ and $c_{j'}$ with $c_{j'}' = c_{j'} + \Delta$. Both expected values $e_j(c_j - \Delta)$ and $e_{j'}(c_{j'} + \Delta)$ linearly depend on Δ, so, by solving the corresponding linear equation, we can find Δ for which $e_j(c_j - \Delta) = e_{j'}(c_{j'} + \Delta)$. If this value Δ satisfies the conditions $c_j' = c_j - \Delta \ge 0$ and $c_{j'}' = c_{j'} + \Delta \le 1$, we get a new minimizing vector in which strict inequality holds for one fewer pair of targets from group two. If this value Δ does not satisfy one of these inequalities, this means that for some smaller value $\Delta' < \Delta$, we have either $c_j' = 0$ or $c_{j'}' = 1$. In both cases, the pairs stops being a wrong-order pair of targets from group two. One can check that no other wrong-order pairs appear after this transformation.

Let us now take the minimizing vector with the desired properties. In particular, this means that for all targets from group two, the attacker's expected value is the same. Let us denote this common value by q. Then, for every target t_j with $0 < c_j < 1$, we have

$$(1 - c_j) \cdot U_\Psi^u(t_j) + c_\ell \cdot U_\Psi^c(t_j) = q. \tag{2.4.25}$$

So, we can calculate c_j as

$$c_j = \frac{U^u_\Psi(t_j) - q}{U^u_\Psi(t_j) - U^c_\Psi(t_j)}. \tag{2.4.26}$$

For targets for which $U^u_\Psi(t_k) < q$, we have $c_k = 0$ – and the above ratio is negative. For targets for which $U^c_\Psi(t_i) > q$, we have $c_i = 1$ – and the above ratio is larger than 1. Thus, if the ratio is smaller than 0, we take $c_i = 0$, and if the ratio is larger than 1, we take $c_i = 1$.

So, once we know q, for all targets t_i, we can find all the covering probabilities c_i by using the following formula:

$$c_i = \min\left(\max\left(\frac{U^u_\Psi(t_i) - q}{U^u_\Psi(t_i) - U^c_\Psi(t_i)}, 0\right), 1\right). \tag{2.4.27}$$

For each target t_i, this formula requires a constant number of computational steps. Therefore, after q is computed, we can therefore compute all the probabilities c_i in time $O(n)$.

So, to find the optimal covering vector, it is sufficient to find the constant q. This constant can be found from the condition that $\sum\limits_{i=1}^{n} c_i = m$, i.e., that

$$\sum_{i=1}^{n} \min\left(\max\left(\frac{U^u_\Psi(t_i) - q}{U^u_\Psi(t_i) - U^c_\Psi(t_i)}, 0\right), 1\right) = m. \tag{2.4.28}$$

The left-hand side of this equality decreases as q increases. So:

- If for some q, the resulting sum is smaller than m, this means that the optimal value q_o is smaller than q: $q_o < q$.
- Similarly, if for some q, the resulting sum is larger than m, this means that the optimal value q_o is larger than q: $q_o > q$.

Here, the target t_i is covered with probability $c_i > 0$ if and only if $q < U^u_\Psi(t_i)$, and the target t_i is covered with probability $c_i = 1$ if and only if $U^c_\Psi(t_i) \geq q$. Thus, the above formula for determining q can be rewritten as follows:

$$k(q) + \sum_{i: U^c_\Psi(t_i) < q \leq U^u_\Psi(t_i)} \frac{U^u_\Psi(t_i) - q}{U^u_\Psi(t_i) - U^c_\Psi(t_i)} = m, \tag{2.4.29}$$

where

$$k(q) \stackrel{\text{def}}{=} \#\{i : U_\Psi^c(t_i) \geq q\}. \tag{2.4.30}$$

Thus, if we know the place of q with respect to all the values $U_\Psi^u(t_i)$ and $U_\Psi^c(t_i)$, we can determine q by explicitly solving the above linear equation.

If we sort all $2n$ values $U_\Psi^u(t_i)$ and $U_\Psi^c(t_i)$ into a decreasing sequence

$$z_0 = +\infty \geq z_1 \geq z_2 \geq \ldots \geq z_{2n-1} \geq z_{2n} \geq z_{2n+1} = 0, \tag{2.4.31}$$

we thus subdivide the real line into $2n+1$ zones $[z_{k+1}, z_k]$, within each of which the relation between q and the values $U_\Psi^u(t_i)$ and $U_\Psi^c(t_i)$ is fixed. Thus, within each zone, we can find the corresponding q and check whether this value is indeed within the corresponding zone. As a result, in order to find q, it is sufficient to find the corresponding value k.

Since the order is decreasing, the larger k, the smaller q, and the more targets we cover. The selection of the zone means that we select which targets we cover fully, and which targets we cover with a positive probability. Similar to the case of fully protective resources:

- If based on this selection, we need more than m resources – i.e., if the value q obtained from solving the above linear equation is smaller than all the values from this zone – this means that we are trying to cover too many targets, so we need to decrease k.
- Similarly, if it turns out that based on this selection, we need fewer than m resources – i.e., that the value q obtained from solving the above linear equation is larger than all the values from this zone – this means that we are trying to cover too few targets, so we can increase k.

Thus, we can use bisection to find the appropriate zone, and we arrive at the following algorithm.

General case: $O(n \cdot \log(n))$ algorithm. First, we sort all $2n$ values $U_\Psi^u(t_i)$ and $U_\Psi^c(t_i)$ into a decreasing sequence:

$$z_1 \geq z_2 \geq \ldots \geq z_{2n-1} \geq z_{2n}. \tag{2.4.32}$$

We then take $z_0 = +\infty$ and $x_{2n+1} = 0$, so that we get:

$$z_0 \geq z_1 \geq z_2 \geq \ldots \geq z_{2n-1} \geq z_{2n} \geq z_{2n+1}. \tag{2.4.33}$$

Then, we use bisection to find the value k for which $z_k \geq q \geq z_{k+1}$. At each stage of this bisection procedure, we keep two values ℓ and u such that $z_\ell \geq q \geq z_u$. In the beginning, we have $\ell = 0$ and $u = 2n + 1$. At each iteration, we do the following:

- First, we compute the midpoint $m = (\ell + u)/2$.
- Then, under the assumption that $q \in [z_{m+1}, z_m]$, we compute

$$k_m = \#\{i : U_\Psi^c(t_i) \geq z_{m+1}\}, \tag{2.4.34}$$

  then $m_0 = m - k_m$, and find q from the resulting linear equation

$$\sum_{i: U_\Psi^c(t_i) \leq z_m \leq z_{m+1} \leq U_\Psi^u(t_i)} \frac{U_\Psi^u(t_i) - q}{U_\Psi^u(t_i) - U_\Psi^c(t_i)} = m_0. \tag{2.4.35}$$

- If the resulting value q is smaller than z_m, then, according to our analysis, this means that the optimal k is larger than m, so we replace the original value ℓ with m.
- If the resulting value q is larger than z_{m+1}, then, according to our analysis, this means that the optimal k is smaller than m, so we replace the original value u with m.

The algorithm stops when $u = \ell + 1$, in which case we have the desired q. Based on this q, we can compute all coverage probabilities by using the above formula

$$c_i = \min\left(\max\left(\frac{U_\Psi^u(t_i) - q}{U_\Psi^u(t_i) - U_\Psi^c(t_i)}, 0\right), 1\right). \tag{2.4.36}$$

There is one more special case the must be considered to ensure that this solution is in fact a Strong Stackelberg Equilibrium. This case occurs when at least one target has coverage $c_i = 1$. In this case, we must ensure that the target that gives maximum payoff for the defender has an optimal payoff for the attacker (so far, we have considered only

the payoffs for the attacker). This can be done by first finding the maximal covered payoff for the attacker $U_{\Psi}^{c}(t)$ for any target that has coverage probability 1. Denote this target by t_{max}. We then loop through each of the targets to determine whether the defender would achieve a higher payoff if the coverage probability was reduced so that the attackers expected payoff was equal to $U_{\Psi}^{c}(t_{max})$. We can compute the necessary coverage for each target using the equation:

$$c_i = \frac{U_{\Psi}^{c}(t_{max}) - U_{\Theta}^{u}(t_i)}{U_{\Theta}^{c}(t_i) - U_{\Theta}^{u}(t_i)} \tag{2.4.37}$$

If the defender's expected payoff for target t_i is greater than $U_{\Psi}^{c}(t_{max})$ given c_i, then we reduce the coverage probability to this new value c_i for target t_i. Note that this can only reduce the total coverage probability required. The additional coverage can either be left unallocated or assigned arbitrarily to any target for which the attacker has an expected payoff less than $U_{\Psi}^{c}(t_{max})$.

Let us prove that this algorithm indeed takes time

$$O(n \cdot \log(n)). \tag{2.4.38}$$

Indeed, sorting can be done in time $O(n \cdot \log(n))$ [18]. At each stage of the bisection method, we handle each target once, so each stage takes $O(n)$ computational steps. We start with an interval $[\ell, u]$ of size $2n$. At each stage, we replace it with a half-size interval $[\ell, m]$ or $[m, u]$. Thus, after the first iteration, we get an interval of size n, after the second, of size $n/2$, ..., and after k-th iteration, an interval of size $(2n)/2^k$. Thus, this procedure stops after $\log_2(2n)$ iterations. So, the overall computation time is indeed

$$O(n \cdot \log(n)) + O(n) \cdot \log(2n) = O(n \cdot \log(n)). \tag{2.4.39}$$

The final stage or analysis for the special case where at least one target coverage $c_i = 1$ requires two loop through each target. The first identifies the fully-covered target with maximum payoff for the attacker $U_{\Psi}^{c}(t_{max})$. The second calculates the required reduction in coverage probability to make the attacker indifferent between t_{max} and any other target, and replaces the coverage probability if a reduction is beneficial for the defender. Since this requires time $O(2 \cdot n)$, the overall complexity remains $O(n \cdot \log(n))$.

Summary of the results. In this section, we have presented two new algorithms for a fundamental class of Stackelberg security games. These algorithms operate in linear time for a restricted case, and $O(n \cdot \log(n))$ for the general case, both improvements over the best known algorithms for this class of games. The algorithms are based on new analysis of the structure of the game-theoretic solutions of these games, which may provide insights to improve the efficiency of algorithms for additional classes of security games.

Chapter 3

Data and Knowledge Processing

Once the data is collected, we need to process this data. For processing, we need to use computers – and the more data we collect, the more computer power we need. It is therefore important to optimally distribute this computing power. This is the problem that we will analyze in Section 3.1.

In many cases, data processing is a creative process, it goes beyond simple application of known algorithms. To come up with the best ways of processing data, of extracting knowledge from the data, we need creative teams – teams combining domain expertise and computer expertise. Just like with computers, simply bringing people together does not always improve their efficiency. It is therefore important to make sure that people collaborate in the most efficient way. This aspect of data and knowledge processing is analyzed in Section 3.2.

The results from this chapter were first published in [58] (Section 3.1) and [98] (Section 3.2).

3.1 Data and Knowledge Processing: How to Best Organize Computing Power

Towards the most efficient way of organizing computing power: enter cloud computing. In many application areas (bioinformatics, geosciences, etc.) we need to process large amounts of data, which requires fast computers and fast communication. Historically, there have been limits on the amount of the information that can be trans-

mitted at a high speed, and these limits affected information processing.

A few decades ago, we could only send the results of data processing fast. As a result, the best strategy to speed up computations was to move all the data into a central location, close to the high performance computers for processing this data.

In the last decades, it became equally fast to move big portions of databases needed to answer a certain query. This enabled the users to switch to a *cyberinfrastructure* paradigm, when there is no longer need for time-consuming moving of data to a central location: the data is stored where it was generated, and when needed, the corresponding data is moved to processing computers; see, e.g., [26, 45, 64, 82, 95] and references therein.

Nowadays, moving the whole databases becomes almost as fast, so there is no longer need to store the data where it was produced – it is possible to store the data where it will be best for future data processing. This idea underlies the paradigm of *cloud computing*.

What is the most efficient way of cloud computing. The main advantage of cloud computing is that we can make computations more efficient by finding optimal placement of the servers that store and/or process the corresponding data. So, in developing cloud computing schemes, it is important to be able to solve this optimization problem. In this chapter, we consider the corresponding problem of optimal data storage in cloud computing.

Comment. This server placement problem is very similar to the type of problems faced by Akamai and other companies that do web acceleration via caching; we therefore hope that our solution can be of help in web acceleration as well.

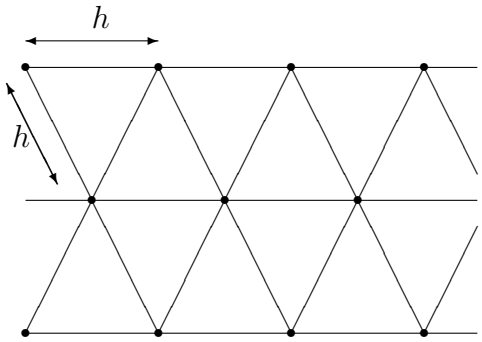
Towards a precise formulation of the problem: first approximation. We usually *know* the geographic density $\rho_u(x)$ describing possible users of this particular database (e.g., a database containing geophysical data), and we know the number of duplicates D that we can afford to store. We *need to determine* the storage density $\rho_s(x)$, i.e., number of copies per geographic region, so as to minimize the average communication delay.

First approximation model: main assumption. In the first approximation, we can measure the travel delay by the average travel distance.

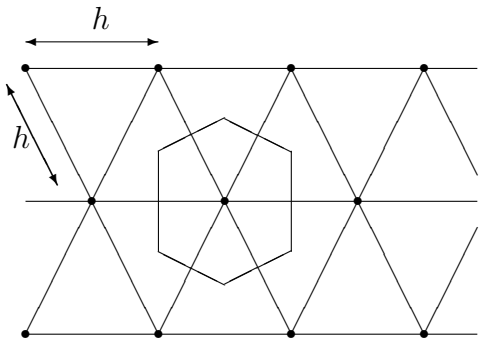
Derivation of the corresponding model. How can we describe this distance in terms of the density $\rho_s(x)$? When the density is constant, we want to place the servers in such a way that the largest distance r to a sensor is as small as possible. (Alternatively, if r is fixed, we want to minimize the number of servers for which every point is at a distance $\leq r$ from one of the servers. In geometric terms, this means that every point on a plane belongs to a circle of radius r centered on one the sensors – and thus, the whole plane is covered by such circles. Out of all such coverings, we want to find the covering with the smallest possible number of sensors.

As we have mentioned in Chapter 2, it is known that the smallest such number is provided by an equilateral triangle grid, i.e., a grid formed by equilateral triangles; see, e.g., [46, 49].

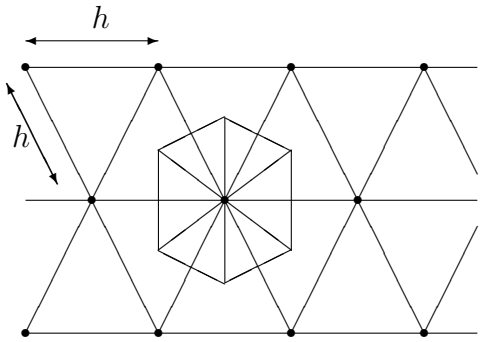
Let us assume that we have already selected the server density function $\rho_s(x)$. Within a small region of area A, we have $A \cdot \rho_s(x)$ servers. Thus, if we, e.g., place these servers on a grid with distance h between the two neighboring ones in each direction, we have:



For this placement, the set of all the points which are closest to a given detector forms a hexagonal area:



This hexagonal area consists of 6 equilateral triangles with height $h/2$:



In each triangle, the height $h/2$ is related to the size s by the formula

$$\frac{h}{2} = s \cdot \cos(60^\circ) = s \cdot \frac{\sqrt{3}}{2},$$

hence

$$s = \frac{h}{\sqrt{3}} = h \cdot \frac{\sqrt{3}}{3}.$$

Thus, the area A_t of each triangle is equal to

$$A_t = \frac{1}{2} \cdot s \cdot \frac{h}{2} = \frac{1}{2} \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{2} \cdot h^2 = \frac{\sqrt{3}}{12} \cdot h^2.$$

So, the area A_s of the whole set is equal to 6 times the triangle area:

$$A_s = 6 \cdot A_t = \frac{\sqrt{3}}{2} \cdot h^2.$$

Each point from the region is the closest to one of the points from the server grid, so the region of area A is thus divided into $A \cdot \rho_s(x)$ (practically) disjoint sets of area $\frac{\sqrt{3}}{2} \cdot h^2$. So, the area of the region is equal to the sum of the areas of these sets:

$$A = (A \cdot \rho_s(x)) \cdot \frac{\sqrt{3}}{2} \cdot h^2.$$

Dividing both sides of this equality by A, we conclude that

$$1 = \rho_s(x) \cdot \frac{\sqrt{3}}{2} \cdot h^2,$$

and hence, that

$$h = \frac{c_0}{\sqrt{\rho_s(x)}},$$

where we denote

$$c_0 \stackrel{\text{def}}{=} \sqrt{\frac{2}{\sqrt{3}}}.$$

The largest distance r to a server is thus equal to

$$\frac{h}{2} = \frac{c_0}{2 \cdot \sqrt{\rho_s(x)}}.$$

The average distance $\overline{\rho}$ is proportional to r – since when we re-scale the picture, all the distances –including the average distance – increase proportionally. Since the distance r is proportional to $(\rho_s(x))^{-1/2}$, the average distance near the location x is thus also proportional to this same value: $\overline{\rho}(x) = \text{const} \cdot (\rho_s(x))^{-1/2}$ for some constant.

At each location x, we have $\sim \rho_u(x)$ users. Thus, the total average distance – the value that we would like to minimize – is equal to $\int \overline{\rho}(x) \cdot \rho_u(x)\, dx$ and is, thus, proportional to

$$\int (\rho_s(x))^{-1/2} \cdot \rho_u(x)\, dx.$$

So, minimizing the average distance is equivalent to minimizing the value of the above integral.

We want to find the server placement $\rho_s(x)$ that minimizes this integral under the constraint that the total number of server is D, i.e., that $\int \rho_s(x) = D$.

Resulting constraint optimization problem. Thus, we arrive at the following optimization problem:

- We know the density $\rho_u(x)$ and an integer D;
- under all possible functions $\rho_s(x)$ for which $\int \rho_s(x)\,dx = D$, we must find a function that minimizes the integral $\int (\rho_s(x))^{-1/2} \cdot \rho_u(x)\,dx$.

Solving the constraint optimization problem. A standard way to solve a constraint optimization problem of optimizing a function $f(X)$ under the constraint $g(X) = 0$ is to use the Lagrange multiplier method, i.e., to apply unconstrained optimization to an auxiliary function $f(X) + \lambda \cdot g(X)$, where the parameter λ (called *Lagrange multiplier*) is selected in such a way so as to satisfy the constraint $g(X) = 0$.

With respect to our constraint optimization problem, this means that we need to select a density $\rho_s(x)$ that optimizes the following auxiliary expression:

$$\int (\rho_s(x))^{-1/2} \cdot \rho_u(x)\,dx + \lambda \cdot \left(\int \rho_s(x)\,dx - D \right).$$

Having an unknown function $\rho_s(x)$ means, in effect, that we have infinitely many unknown values $\rho(x)$ corresponding to different locations x. Optimum is attained when the derivative with respect to each variable is equal to 0. Differentiating the above expression with respect to each variable $\rho_s(x)$, and equating the result to 0, we get the equation

$$-\frac{1}{2} \cdot (\rho_s(x))^{-3/2} \cdot \rho_u(x) + \lambda = 0,$$

hence $\rho_s(x) = c \cdot (\rho_u(x))^{2/3}$ for some constant c.

The constant c can be determined from the constraint $\int \rho_s(x)\,dx = D$, i.e., that

$$\int c \cdot (\rho_u(x))^{2/3}\,dx = c \cdot \int (\rho_u(x))^{2/3}\,dx = D.$$

Thus,

$$c = \frac{D}{\int (\rho_u(x))^{2/3}\,dx},$$

and we arrive at the following solution.

Solution to the problem. Once we know the user density $\rho_u(x)$ and the total number of servers D that we can afford, the optimal server density $\rho_s(x)$ is equal to

$$\rho_s(x) = D \cdot \frac{(\rho_u(x))^{2/3}}{\int (\rho_u(y))^{2/3}\, dy}.$$

Discussion. In line with common sense, the optimal server density increases when the user density increases, i.e.:

- in locations where there are more users, we place more servers, and
- in locations where there are fewer users, we place fewer servers.

However, when the user density decreases, the server density decreases slower – because otherwise, if we took the server density simply proportional to the user density, the delays in areas with few users would have been huge.

Comment. From the mathematical viewpoint, this analysis is similar to the analysis of a security-related optimization problem, in which, instead of placing servers, we need to place sensors; see [49].

Towards a more realistic model: first idea. In the above first approximation, we only took into account the time that it takes to move the data to the user. This would be all if the database was not changing. In real life, databases need to be periodically updated. Updating also takes time. Thus, when we find the optimal placement of servers, we need to take into account not only expenses on moving the data to the users, but also the expenses of updating the information.

Towards a precise formulation of this idea. How do we estimate these expenses? In a small area, where the user distribution is approximately uniform, the servers are also uniformly distributed, i.e., they form a grid with distance $h = 2r$ between the two neighboring servers [46, 49]. Within a unit area, there are $\sim 1/r^2$ servers, and reaching

each of them from one of its neighbors requires time proportional to the distance $\sim r$. The overall effort of updating all the servers can be obtained by multiplying the number of servers by an effort needed to update each server, and is thus proportional to $1/r^2 \cdot r \sim 1/r$. We already know that $r \sim (\rho_s(x))^{-1/2}$, thus, the cost of updating all the servers in the vicinity of a location x is proportional to $(\rho_s(x))^{1/2}$. The overall update cost can thus be obtained by integrating this value over the whole area. Thus, we arrive at the following problem.

Resulting optimization problem:

- We know the density $\rho_u(x)$, an integer D, and a constant C that is determined by the relative frequency of updates in comparison with frequency of normal use of the database;
- under all possible functions $\rho_s(x)$ for which $\int \rho_s(x)\,dx = D$, we must find a function that minimizes the expression

$$\int (\rho_s(x))^{-1/2} \cdot \rho_u(x)\,dx + \int C \cdot (\rho_s(x))^{1/2}\,dx.$$

Solving the problem. To solve the new optimization problem, we can similarly form the Lagrange multiplier expression

$$\int (\rho_s(x))^{-1/2} \cdot \rho_u(x)\,dx + \int C \cdot (\rho_s(x))^{1/2}\,dx + \lambda \cdot \left(\int \rho_s(x)\,dx - D \right),$$

differentiate it with respect to each unknown $\rho_s(x)$, and equate the resulting derivative to 0. As a result, we get an equation

$$-\frac{1}{2} \cdot (\rho_s(x))^{-3/2} \cdot \rho_u(x) + \frac{1}{2} \cdot C \cdot (\rho_s(x))^{-1/2} + \lambda = 0.$$

This is a cubic equation in terms of $(\rho_s(x))^{-1/2}$, so while it is easy to solve numerically, there is no simple analytical expression as in the first approximation case.

The resulting solution $\rho_s(x)$ depends on the choice of the Lagrange multiplier λ, i.e., in effect, we have $\rho_s(x) = \rho_s(x, \lambda)$. The value λ can be determined from the condition that $\int \rho_s(x, \lambda)\,dx = D$.

Second idea. The second idea is that usually, a service provides a time guarantee, so we should require that no matter where a user is located, the time for this user to get the desired information from the database should not exceed a certain value. In our model, this means that a distance r from the user to the nearest server should not exceed a certain given value r_0. Since $r \sim (\rho_s(x))^{-1/2}$, this means, in turn, that the server density should not decrease below a certain threshold ρ_0.

This is an additional constraint that we impose on $\rho_s(x)$. In the first approximation model, it means that instead of the formula $\rho_s(x) = c \cdot (\rho_u(x))^{2/3}$ – which could potentially lead to server densities below ρ_0 – we should have $\rho_s(x) = \max(c \cdot (\rho_u(x))^{2/3}, \rho_0)$.

The parameter c can be determined from the constraint

$$\int \rho_s(x)\,dx = \int \max(c \cdot (\rho_u(x))^{2/3}, \rho_0)\,dx = D.$$

Since the integral is an increasing function of c, we can easily find the solution c of this equation by bisection (see, e.g., [18]).

Combining both ideas. If we take both ideas into account, then we need to consider only those roots of the above cubic equation which are larger than or equal to ρ_0; if all the roots are $< \rho_0$, we take $\rho_s(x) = \rho$.

The resulting solution $\rho_s(x)$ depends on the choice of the Lagrange multiplier λ, i.e., in effect, we have $\rho_s(x) = \rho_s(x, \lambda)$. The corresponding value λ can also be similarly determined from the equation $\int \rho_s(x, \lambda)\,dx = D$.

3.2 Data and Knowledge Processing: How to Best Organize Research Teams

How to best organize research teams? To solve this problem, we propose a natural model describing competition between two research groups of the same average research strength.

The analysis of this model enables us to conclude that a more diverse group has an advantage: namely, the more diverse the group, the higher the average quality of its publications.

Diversity is beneficial. Experiments and simulation have shown that, in general, more diverse groups have an advantage over less diverse ones; see, e.g., [37, 38, 68, 78].

What we do in this section. In this section, we provide an additional quantitative argument in favor of diversity of research groups. Namely, we show that if we have two competing research groups with the same average strength, then the more diverse research group has a clear advantage.

Natural assumption: strength is normally distributed. Normal distributions are ubiquitous, they appear in many real-life situations; in particular, they describe the distribution of many characteristics of a human being such as height, weight, blood pressure, or IQ. The ubiquity of normal distribution can be explained by the fact that in many cases, the value of the quantity is caused by many independent factors, and the known Central Limit Theorems states, crudely speaking, that the distribution of the sum of large number small independent factors is close to normal; see, e.g., [94].

It is therefore reasonable to assume that within each of the two competing research groups, strength is normally distributed. In general, a normal distribution is uniquely determined by its mean μ and its standard deviation σ. In terms of strength, the mean is the average strength, while the standard deviation describe diversity: the larger the standard deviation, the more diverse the group.

We assume that both groups have the same average strength μ, but that the first group is more diverse: $\sigma_1 > \sigma_2$.

How the groups compete: a description. We assume that each group coordinate the research efforts of its members, so there is no unnecessary competition within each group; the only competition is between the two groups. Once a member of one of the research

groups selects a problem – a problem that people in the field consider to be important – it is highly probably that the same problem will be picked up by some member of another research group.

The groups (being competitors) do not coordinate their research efforts with each other. As a result, the corresponding member of another research group is randomly selected from that group. If two researchers of different research strength $s_1 > s_2$ work on the same problem, it is reasonable to expect that the stronger researcher will get the results first – and this will result in a publication of quality corresponding to this higher strength s_1.

Let us analyze the resulting model.

Analysis of the model. Under the above assumptions, let us see which of the two groups has an advantage. Intuitively, the answer is not clear:

- on the one hand, the more diverse research group has a larger number of stronger researchers, which gives this group an *advantage* over the less diverse group;
- on the other hand, the more diverse research group also has a larger number of weaker researchers, which gives this group a *disadvantage* over the less diverse group.

At first glance, diversity brings no advantage. In the above competition, which of the two groups will be more successful? Let us first consider the simplest measure of success: the resulting number of publications.

The first group gets a publication is a value s_1 randomly selected from the first group exceeds a value s_2 randomly selected from the second group: $s_1 > s_2$. Thus, the number of publications produced by the first group is proportional to the probability that for randomly selected values s_1 and s_2, we have $s_1 - s_2 > 0$, i.e., equivalently, that $s_1 - s_2 > 0$. The two independent random variables x_1 and x_2 are normally distributed with the same mean μ. It is known that the difference of two independent normally distributed random variables is also normally distributed. The mean of the difference $s_1 - s_2$ is equal to the difference

of the means, i.e., to $\mu - \mu = 0$. Thus, $s_1 - s_2$ is a normally distributed random variable with 0 mean. For such random variable, the probability of it being positive is exactly $\frac{1}{2}$.

Thus, when the two research groups have the same average strength, in half of the cases, the first group will succeed, in half of the cases, the second group will succeed. So, both groups will generate, on average, the same number of publications.

Towards a deeper analysis. In terms of *number* of published papers, diversity does not bring any advantage. However, different publications have different *quality*. What if, instead of simply counting the number of publications, we would instead estimate the average quality of a publication?

According to our model, the first group succeeds if $s_1 > s_2$ and produces a paper of quality s_1. Thus, the average quality q_1 of papers produced by the first research group is equal to the conditional expectation $q_1 = E[s_1 \,|\, s_1 > s_2]$. Similarly, the the average quality q_2 of papers produced by the second research group is equal to the conditional expectation $q_2 = E[s_2 \,|\, s_1 > s_2]$.

Let us estimate these two quantities.

Estimating the desired quantities. The first research group produces a paper of quality s_1:

- if there is a person of strength s_1 in this group and
- if this person was stronger than the competitor, i.e., a person with a (randomly selected) strength s_2 from the second research group.

The probability of the first research group having a member of strength s_1 is determined by the normal distribution, i.e., has the form

$$f_1(s_1) = \frac{1}{\sigma_1} \cdot \phi\left(\frac{s_1 - \mu}{\sigma_1}\right),$$

where

$$\phi(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)$$

is the probability density of the standard normal distribution (i.e., a normal distribution with mean 0 and standard deviation 1).

The probability that s_1 will win over the competitor is equal to

$$\text{Prob}(s_2 < s_1).$$

By definition of the cumulative distribution function (cdf) $F_2(x)$ of the random variable s_2, this probability is equal to $\text{Prob}(s_2 < s_1) = F_2(s_1)$. Since the variable s_2 is normally distributed, this probability has the form $F_2(s_1) = \Phi\left(\dfrac{s_1 - \mu}{\sigma_2}\right)$, where $\Phi(x)$ is the cdf of the standard normal distribution.

Since s_1 and s_2 are independent, the probability distribution function $f(s_1)$ for the publication quality s_1 is proportional to the product of the two probabilities, i.e., has the form

$$f(s_1) = \text{const} \cdot \phi\left(\frac{s_1 - \mu}{\sigma_1}\right) \cdot \Phi\left(\frac{s_1 - \mu}{\sigma_2}\right).$$

Such a distribution is known: it is a *skew-normal* distribution; see, e.g., [6, 63, 105] and references therein. To be more precise, the usual formula for the skew-normal distribution has the form

$$f(s_1) = \text{const} \cdot \phi\left(\frac{s_1 - \mu}{\sigma_1}\right) \cdot \Phi\left(\alpha \cdot \left(\frac{s_1 - \mu}{\sigma_1}\right)\right),$$

which coincides with the above form for $\alpha = \dfrac{\sigma_1}{\sigma_2}$.

It is known that the mean value of the skew-normal random variable is equal to $q_1 = \mu + \sqrt{\dfrac{2}{\pi}} \cdot \sigma_1 \cdot \dfrac{\alpha}{\sqrt{1+\alpha^2}}$. Substituting $\alpha = \dfrac{\sigma_1}{\sigma_2}$ into this formula and multiplying both the numerator and the denominator of the corresponding fraction by σ_2, we conclude that

$$q_1 = \mu + \frac{\sigma_1^2}{\sqrt{\sigma_1^2 + \sigma_2^2}}.$$

Similarly, the average quality of papers published by the second research group is equal to

$$q_2 = \mu + \frac{\sigma_2^2}{\sqrt{\sigma_1^2 + \sigma_2^2}}.$$

Summary of the results. From the above formulas, we can see that the larger the standard deviation σ_i, the larger the average quality q_i of the corresponding publications. Thus, while a diverse group produces, on average, the same number of publications, the average quality of these publications is higher – and the more diverse the group, the higher the quality.

Chapter 4

Knowledge Propagation and Resulting Knowledge Enhancement

Once we have transformed data into knowledge, we need to propagate this knowledge – so that other researchers can use and enhance this knowledge. For that, first, we need to motivate people to learn the new knowledge, we need to make sure that the idea is propagated to more and more people. To ensure that, we need to analyze the process of idea propagation; this is done in Section 4.1.

Once a person is willing to learn the corresponding techniques and ideas, we can start the actual learning. For this learning to be successful, we need to get a good understanding of where the person stands now, what is his/her level of knowledge in the corresponding areas. This assessment problem is analyzed in Section 4.2.

Once this information is known, we need to actually present this information to the interested folks – and use appropriate feedback to modify (if needed) the speed with which this knowledge is presented. Issues related to the material's presentation are analyzed in Sections 4.3 and 4.4. Specifically, in Section 4.3, we consider the problem from the global viewpoint: e.g., in what order we should present different parts of the material. In Section 4.4, we consider this problem from the local viewpoint: what is the best way to present different items. Finally, in Section 4.5, we analyze the problems related to feedback.

4.1 Analyzing the Early Stages of Idea Propagation

New good ideas sometimes propagate too slowly. To speed up their propagation, we need to have a quantitative understanding of how ideas propagate. An intuitive understanding of ideas propagation has led to several reasonable first-approximation mathematical models. These models provide a good description of idea propagation on the later stages, when the ideas have already been adopted by a reasonably large number of people. However, at the critically important early stages, these models are not perfect: these models predict a linear growth with time, while empirical growth data is often better described by a power law. In this section, we provide an intuitive theoretical explanation of the observed power-law growth.

The results from this section were first published in [60].

Propagation of new tools and new ideas – one of the main ways science and technologies progress. Science and technology are progressing at an enormous speed. New ideas appear all the time, new tools are being designed all the time that enable us to do things that we could not do before – and do them faster, more reliably, and more efficiently.

It is extremely important to come up with new ideas, to design new tools, but mere design is not enough: it is important to make sure that these ideas and tools do not stay with their inventors, that they are widely adopted and thus propagate.

Current first approximation model of ideas propagation. We would like to know how ideas propagate, i.e., how the number $n(t)$ of people who use the new idea grows with time t.

The main current model of idea propagation (see, e.g., [2, 9, 10, 28, 29, 41, 69, 89, 101, 106, 109] and references therein) is as follows. For the idea to spread, people who have not yet adopted the idea must learn about it – either from the original announcement or from people who already use this idea. The probability that a new person will learn

this new idea can thus be estimated as $a + b \cdot n(t)$, where a is the probability to learn this idea from the original announcement, and b is the probability to encounter one of the followers. Out of the total population of N people, $N - n(t)$ not-yet-users are exposed to this learning. Since the probability of each of them learning about the new idea is proportional to $a + b \cdot n(t)$, the total number of people who learn about the new idea is proportional to $(a + b \cdot n(t)) \cdot (N - n(t))$. Thus, we arrive at the differential equation

$$\frac{dn}{dt} = c \cdot (a + b \cdot n(t)) \cdot (N - n(t)),$$

where c is the corresponding proportionality coefficient. Thus, we get

$$\frac{dn}{dt} = (A + B \cdot n(t)) \cdot (N - n(t)), \tag{4.1.1}$$

where $A \stackrel{\text{def}}{=} c \cdot a$ and $B \stackrel{\text{def}}{=} c \cdot b$.

Many refinements of this model have been proposed (see, e.g., [69]), but the model (4.1.1) remains the main first approximation model of knowledge propagation.

Solution to the first approximation model. By moving all the terms containing n to the left-hand side and all the terms containing time t to the right-hand side, we conclude that

$$\frac{dn}{(A + B \cdot n) \cdot (N - n)} = dt. \tag{4.1.2}$$

Here, as one can easily check,

$$\frac{1}{(A + B \cdot n) \cdot (N - n)} = \frac{1}{A + B \cdot N} \cdot \left(\frac{B}{A + B \cdot n} + \frac{1}{N - n} \right). \tag{4.1.3}$$

Thus, the left-hand side of the formula (4.1.2) takes the form

$$\frac{1}{A + B \cdot N} \cdot \left(\frac{B \cdot dn}{A + B \cdot n} + \frac{dn}{N - n} \right) =$$

$$\frac{1}{A + B \cdot N} \cdot \left(\frac{d(A + B \cdot n)}{A + B \cdot n} - \frac{d(N - n)}{N - n} \right). \tag{4.1.4}$$

So, the integral of this left-hand side takes the form

$$\frac{1}{A + B \cdot N} \cdot (\ln(A + B \cdot n) - \ln(N - n)) = \frac{1}{A + B \cdot N} \cdot \ln \left(\frac{A + B \cdot n}{N - n} \right). \tag{4.1.5}$$

Hence, integrating both sides of the equation (4.1.2), we get

$$\frac{1}{A+B\cdot N}\cdot\ln\left(\frac{A+B\cdot n}{N-n}\right)=t+C, \tag{4.1.6}$$

where C is the integration coefficient. Therefore,

$$\ln\left(\frac{A+B\cdot n}{N-n}\right)=k\cdot t+c, \tag{4.1.6}$$

where $k\stackrel{\text{def}}{=}A+B\cdot N$ and $c\stackrel{\text{def}}{=}k\cdot C$. Raising e to the (equal) left- and right-hand sides of the equation (4.1.6), we get

$$\frac{A+B\cdot n}{N-n}=C'\cdot\exp(k\cdot t), \tag{4.1.7}$$

where $C'\stackrel{\text{def}}{=}\exp(c)$. Multiplying both sides of this equation by $N-n$, we get

$$A+B\cdot n=N\cdot C'\cdot\exp(k\cdot t)-n\cdot C'\cdot\exp(k\cdot t), \tag{4.1.8}$$

hence

$$n\cdot(B+C'\cdot\exp(k\cdot t))=C'\cdot\exp(k\cdot t)-A, \tag{4.1.9}$$

and, thus,

$$n(t)=\frac{C'\cdot\exp(k\cdot t)-A}{B+C'\cdot\exp(k\cdot t)}. \tag{4.1.10}$$

If we start measuring time from the moment the idea was launched, so that $n(0)=0$, then we conclude that $C'=A$ and thus, the formula (4.1.10) takes the form

$$n(t)=\frac{A\cdot(\exp(k\cdot t)-1)}{B+A\cdot\exp(k\cdot t)}. \tag{4.1.11}$$

Initial propagation of a new idea: asymptotic description. Once the idea has spread, it will continue spreading; the most critical period is right after the idea's appearance, when $t\approx 0$. For such t, asymptotically, the first approximation model (4.1.11) implies that

$$n(t)\approx c\cdot t, \tag{4.1.12}$$

where $c\stackrel{\text{def}}{=}\dfrac{A\cdot k}{B+A}$ i.e., that $n(t)$ linearly grows with time t.

Empirical data seems to be inconsistent with this asymptotics. While for medium and large times t, the first approximation model (4.1.11) is in a reasonably good accordance with data, for small t, the empirical data shows a clearly non-linear behavior (see, e.g., [9, 43]), a behavior which is better described by a power function

$$n(t) \approx c \cdot t^a \tag{4.1.13}$$

for some $a \neq 1$.

What we do in this section. In this section, we provide a simple intuitive model which explains such power-law growth.

Main idea behind our explanation: a qualitative description. To describe how a new idea propagates, let us consider one specific tool (or idea) aimed at solving problems from a specific class. For example, this tool may be a new (e.g., more efficient) software for solving large systems of linear equations.

Any person who sometimes solves the problem from the given class is a potential user of this tool. We start with the initial situation, in which only the author of the tool knows it and uses it. Eventually, other potential users start learning and using this tool.

When does a potential user start learning the new tool? On the one hand, there are clear benefits in learning a new tool: once a person learns the new tool, he or she can solve the problems from the corresponding class more efficiently.

- For example, efficiency may mean faster computations. In this case, the user will be able to solve large systems of linear equations faster. This will save the time needed to solve such systems, and enable the user to get the results faster.

- Alternatively, efficiency may mean that the user may be able to use fewer processors of a multi-processor computer system to solve the same problem – so, if the user pays for the computer time, he or she will be able to save some money by using this new software.

On the other hand, new tools, new ideas are not always easy to learn. One needs to invest some effort – e.g., time – into learning the new tool. A potential user will start learning the new tool only if the expected benefits exceed the investment needed to learn this tool. So, to figure out when a particular user will start learning the tool, we need to be able to estimate both potential benefits and the required investment.

The potential benefits of using a tool depend on how often it will be used.

- Some potential users solve the corresponding problems very frequently. For such users, a potential benefit may be large.
- Other users encounter the corresponding problems rarely. For such users, the potential benefit of learning the new tool may be small.

In our analysis, we need to take this difference into account.

To estimate the difficulty of learning the tool, we need to take into account that this difficulty depends on how many people have already learned it. If a tool is currently used only by a few folks, it is more difficult to learn it: if there is a question about this tool, it is not so easy to find someone who knows the answer. On the other hand, if the tool is widely used, learning this tool is much easier: when there is a question, one of the nearby colleagues who is already using this tool can answer.

For example, in a Computer Science department, it is easy for someone to learn one of the widely used languages such as C++ or Java: whatever question may arise, there are plenty of people around who know these languages already. On the other hand, a new operating system – e.g., a new version of Windows – may be simpler to use than C++, but in the beginning, it is not so easy to learn – since in the beginning, very few people have an experience of using it and therefore, it is difficult to find help if a problem arises.

Let us show how these qualitative ideas can be translated into a quantitative model.

Heavy users vs. light users. As we have mentioned, a user will start learning the new tool only if the expected benefit of its use exceeds the expenses needed to learn this tool.

For each user, the expected benefit b of using the tool is proportional to the number x of the corresponding problems (per unit time) that this user encounters: $b = C \cdot x$, for some proportionality constant C (that describes the benefit of using the tool to solve a single problem). From this viewpoint, each user can be characterized by the corresponding value x.

Let $L(t)$ describe the cost of learning the tool at moment t. In this notation, at each moment of time t, a potential user – characterized by the value x – will start learning the tool if the benefit $C \cdot x$ exceeds the cost $L(t)$: $C \cdot x > L(t)$. This condition can be equivalently described as $x > x_0(t)$, where we denoted $x_0(t) \stackrel{\text{def}}{=} \dfrac{L(t)}{C}$. This ratio $x_0(t)$ serves as a threshold:

- "heavy users", i.e., users for which $x > x_0(t)$, will start adopting the tool, while
- "light users", i.e., users for which $x < x_0(t)$, will continue using previous tools.

Thus, at each moment of time, the state of propagation can be characterized by a single value – this threshold value $x_0(t)$.

Distribution of users. To describe how knowledge propagates, we need to know how many users are there with different levels of usage x. In many practical problems, the distribution is described by the power law (see, e.g., [16, 86]), in which the proportion $P(x \geq X)$ of objects x for which x exceeds a given threshold X is determined by a formula

$$P(x \geq X) = C_0 \cdot X^{-\alpha} \tag{4.1.14}$$

for some constants C_0 and α. The ubiquitous character of power laws was popularized by Benoit Mandelbrot in his fractal theory; see, e.g., [66].

How easier is to to learn a new tool when we already have a given number n of users. In the beginning, learning a new tool is not very easy, but ultimately, tools and techniques become relatively easy to learn. For example:

- calculus used to be a great 17 century achievement, accessible only to a few great minds;
- however, nowadays, many kids study elements of calculus already in high school.

The reason why, in the beginning, learning a new tool is not easy is that a person learning the new tool *can* go astray (and *goes* astray). The more advice we get, the more accurately we understand what needs to be done – i.e., crudely speaking, the more accurate is the direction in which we are going – and thus, the smaller amount of effort will be wasted. The resulting amount of effort can be viewed as proportional to the inaccuracy with which we know the direction in which to go in learning the tool.

To find this direction, we can use the advice and expertise of the existing users. If we have n users that we can consult, this means that we have n estimates for the desired direction. In general, according to statistics (see, e.g., [94]), if we have n similar independent estimates of the same quantity, then, by taking their average, we can get a combined estimate which is $\sqrt{n}$ times more accurate than each of the individual estimates. Thus, it is reasonable to assume that when we have n users, the amount of effort needed to learn the tool is (approximately) equal to $\dfrac{b}{\sqrt{n}}$ for some constant b.

Resulting dynamics of propagation. As we have mentioned, a person starts learning a new tool if the expected benefit of its use exceeds the cost of learning. Once we have n users, the cost of learning is equal to $\dfrac{b}{\sqrt{n}}$. The expected benefit of leaning the tool is proportional to the average number of problems encountered by the potential user, i.e., to the number x; in other words, this benefit can be described as $a \cdot x$ for some constant a. So, at this stage, only persons for which $a \cdot x \geq \dfrac{b}{\sqrt{n}}$ have an incentive to study this tool. This condition can be described equivalently as $x \geq X_0$, where we denoted $X_0 \stackrel{\text{def}}{=} \dfrac{b}{a \cdot \sqrt{n}}$. According to the power-law distribution, out of N who may be potentially interested in this tool, the total number of persons who have an incentive to study this tool is equal to

$$N \cdot P(X > X_0) = N \cdot C_0 \cdot X_0^{-\alpha} = N \cdot C_0 \cdot \left(\frac{b}{a \cdot \sqrt{n}}\right)^{-\alpha} = c_1 \cdot n^{\alpha/2}, \tag{4.1.15}$$

for an appropriate constant $c_1 \stackrel{\text{def}}{=} N \cdot C_0 \cdot \left(\frac{b}{a}\right)^{-\alpha}$.

The rate $\frac{dn}{dt}$ with which the number of users n increases is proportional to the number of potential users who study the new tool, i.e., to the number of persons who have an incentive to study this tool. Thus, we conclude that

$$\frac{dn}{dt} = c_2 \cdot n^{\alpha/2} \tag{4.1.16}$$

for some constant c_2.

Moving terms containing n to the left-hand side and terms containing t to the right-hand side, we get

$$n^{-\alpha/2} \cdot dn = c_2 \cdot dt. \tag{4.1.17}$$

Now, we can integrate both sides. The result of this integration depends on the value $\alpha/2$.

When $\alpha/2 = 1$, integration leads to

$$\ln(n) = c_2 \cdot t + C, \tag{4.1.18}$$

where C is the integration constant. We want to describe the starting period of idea propagation, when $n(t) = 0$ for $t = 0$. For $n = 0$, however, the left-hand side of (4.1.18) is infinite, while the right-hand side is finite. Thus, we cannot have $\alpha/2 = 1$.

When $\alpha/2 \neq 1$, integration of (4.1.17) leads to

$$\frac{1}{1-\alpha/2} \cdot n^{1-\alpha/2} = c_2 \cdot t + C. \tag{4.1.18}$$

We want to satisfy the requirement that $n(t) = 0$ when $t = 0$. When $t = 0$, the right-hand side of the formula (4.1.18) is equal to C. When $n = 0$, the value $n^{1-\alpha/2}$ is equal to 0 when $1 - \alpha/2 > 0$ and to ∞ when $1 - \alpha/2 < 0$. Since $C < \infty$, the condition that $n(t) = 0$ when $t = 0$ can only be satisfied when $1 - \alpha/2 > 0$. In this case, for $t = 0$, the formula (4.1.18) takes the form $0 = C$. Substituting $C = 0$ into the formula (4.1.18), we conclude that

$$\frac{1}{1-\alpha/2} \cdot n^{1-\alpha/2} = c_2 \cdot t, \tag{4.1.19}$$

hence

$$n^{1-\alpha/2} = (c_2 \cdot (1 - \alpha/2)) \cdot t. \tag{4.1.20}$$

Raising both side by the power $a \stackrel{\text{def}}{=} 1/(1 - \alpha/2)$, we conclude that

$$n(t) = c \cdot t^a, \tag{4.1.21}$$

where $c \stackrel{\text{def}}{=} (c_2 \cdot (1 - \alpha/2))^a$. This is exactly the formula that we wanted to explain.

Summary of the results. So, the above light user–heavy users model indeed explains the observed power-law growth of the number of adoptees of a new idea.

4.2 Analyzing the Assessment of the Students' Initial Knowledge Level

Once a person is willing to learn the corresponding techniques and ideas, we can start the actual learning. For this learning to be successful, we need to get a good understanding of where the person stands now, what is his/her level of knowledge in the corresponding areas.

To find the current level of a student's knowledge, a sequence of problems of increasing complexity if normally used; if a student can solve a problem, the system generates a more complex one; if a student cannot solve a problem, the system generates an easier one. To find a proper testing scheme, we must take into account that every time a student cannot solve a problem, he/she gets discouraged. To take this into account, in this section, we define an overall effect on a student by combining "positive" and "negative" problems with different weights, and we design a testing scheme which minimizes this effect.

Need for a placement test. Computers enable us to provide individualized learning, at a pace tailored to each student. In order to start the learning process, it is important

to find out the current level of the student's knowledge, i.e., to place the student at an appropriate level.

Usually, such placement tests use a sequence of N problems of increasing complexity; if a student is able to solve a problem, the system generates a more complex one; if a student cannot solve a problem, the system generates an easier one – until we find the exact level of this student. After this, the actual learning starts.

A seemingly natural idea. A natural tendency is to speed up this preliminary stage and to get to actual leaning as soon as possible, i.e., to minimize the number of problems given to a student.

Resulting solution: bisection. The solution to the corresponding optimization problem is a well-known bisection procedure; see, e.g., [18]. To describe this procedure, let us add, to the problems of levels 1 though N, two fictitious "problems":

- a trivial problem that everyone can solve – which will be called level 0; and
- a very complex problem that no one can solve – which will be called level $N + 1$.

In the beginning, we know that a student can solve a problem at level 0 (since everyone can solve a problem at this level) and cannot solve a problem of level $N + 1$ (since no one can solve problems at this level).

After the tests, we may know that a student can or cannot solve some problems. Let i be the highest level of problems that a student has solved, and let j be the lowest level of problems that a student cannot solve. If $j = i + 1$, then we know exactly where the student stands: he or she can solve problems of level i but cannot solve problems of the next complexity level $i + 1$.

If $j > i = 1$, we need further testing to find out the exact level of knowledge of this student. In the bisection method, we give the student a problem on level $m \stackrel{\text{def}}{=} (i + j)/2$. Depending on whether a student succeeded in solving this problem or not, we either increase i to m or decrease j to m.

In both cases, we decrease the interval by half. We started with an interval $[0, N+1]$. After s steps, we get an interval of width $2^{-s} \cdot (N+1)$. Thus, when $2^{-s} \cdot (N+1) \le 1$, we get an interval of width 1, i.e., we have determined the student's level of knowledge. This requires $s = \lceil \log_2(N+1) \rceil$ steps.

The problem with bisection. The problem with bisection is that every time a student is unable to solve a problem, he/she gets discouraged; in other words, such problems have a larger effect on the student than problems which the student can solve. For example, if a student is unable to solve any problem already on level 1, this students will get a negative feedback on all $\approx \log_2(N+1)$ problems – and will be thus severely discouraged.

How to solve this problem: an idea. To take the possible discouragement into account, let us define an overall effect on a student by combining "positive" and "negative" problems with different weights.

In other words, we will count an effect of a positive answer as one, and the effect of a negative answer as $w > 1$. For positive answers, the student simply gets tired, while for negative answers, the student also gets stressed and frustrated. The value w can be determined for each individual student.

Resulting optimization problem. For each testing scheme, the resulting effect on each student can be computed as the number of problems that this student solved plus w multiplied by the number of problems that this student did not solve. This effect depends on a student: for some students it may be smaller, for other students it may be larger. As a measure of quality of a testing scheme, let us consider the worst-case effect, i.e., the largest effect over all possible students.

Our objective is to find a testing scheme which places all the students while leading to the smallest effect on a student, i.e., for which the worst-case effect is the smallest possible.

Testing scheme: a general description. A general testing scheme works as follows. First, we ask a student to select a problem of some level n. Depending on whether a student succeeds or not, we ask the student to solve a problem of some other level $n' > n$ or $n' < n$, etc.

As a result, we get the knowledge level of a student, i.e., we get the level i for which the student can solve the problems on this level but cannot solve problems on the next level $i + 1$. This level i can take any of the $N + 1$ values from 0 to N.

Deriving the main formula. Let $e(x)$ denote the smallest possible effect needed to find out the knowledge level of a student in a situation with $x = N + 1$ possible student levels.

In the beginning, we know that a student's level is somewhere between 0 and N. In the optimal testing scheme, we first ask a student to solve a problem of some level n. Let us consider both possible cases: when the student succeeds in solving this problem and when the student doesn't.

If the student successfully solved the level n problem, this means that after providing a 1 unit of effect on the student, we know that this student's level is somewhere between n and N. In this case, we must select among $N - n + 1 = x - n$ possible student levels. By definition of the function $e(x)$, the remaining effect is equal to $e(x - n)$. Thus, in this case, the total effect on a student is equal to $1 + e(x - n)$.

If the student did not solve the problem of level n, this means that after producing w units of effect on the student, we learn that the student's level is somewhere between 0 and $n - 1$. The remaining effect to determine the student's level is equal to $e(n)$. Thus, the total effect on the student is equal to $w + e(n)$.

The worst-case effect $e(x)$ is, by definition, the largest of the two effects $1 + e(x - n)$ and $w + e(n)$: $e(x) = \max(1 + e(x - n), w + e(n))$. In the optimal method, we select n (from 1 to $N = x - 1$) for which this value is the smallest possible. Thus, we conclude that

$$e(x) = \min_{1 \le n < x} \max(1 + e(x - n), w + e(n)). \tag{4.2.1}$$

The value $n(x)$ corresponding to x can be determined as the value for which the right-hand side of the expression (4.2.1) attains its minimum.

Comment. It is worth mentioning that similar formulas appear in other situations; see, e.g., [77, 100]. Because of this similarity, in this section, we have used – after a proper modification – s ome of the mathematics from [77, 100].

Towards the optimal testing scheme. For $x = 1$, i.e., for $N = 0$, we have $e(1) = 0$. We can use the formula (1) to sequentially compute the values $e(2)$, $e(3)$, ..., $e(N + 1)$ by using formula (1); while computing these values, we also compute the corresponding minimizing values $n(2)$, $n(3)$, ..., $n(N + 1)$.

In the beginning, we know that a student's level ℓ is between 0 and N, i.e., that $0 \le \ell < N + 1$. At each stage of the testing scheme, we know that the student's level ℓ is between some numbers i and j: $i \le \ell < j$, where i is the largest of the levels for which the student succeeded in solving the problem, and j is the smallest level for which the student was unable to solve the corresponding problem. In this case, we have $j - i$ possible levels i, $i + 1$, ..., $j - 1$. In accordance with the above algorithm, we should thus ask a question corresponding to the $n(j - i)$-th of these levels. If we count from 0, this means the level $i + n(j - i)$. Thus, we arrive at the following algorithm.

Resulting optimal testing scheme. First, we take $e(1) = 0$, and sequentially compute the values $e(2)$, $e(3)$, ..., $e(N + 1)$ by using the main formula (1), while simultaneously recording the corresponding minimizing values $n(2)$, ..., $n(N + 1)$.

At each stage of testing, we keep track of the bounds i and j for the student's level. In the beginning, $i = 0$ and $j = N + 1$. At each stage, we ask the student to solve a problem at level $m = i + n(j - i)$.

- If the student succeeds in solving this problem, we replace the original lower bound i with the new bound m.

- If the student did not succeed in solving the problem on level m, we replace the original upper bound j with the new bound m.

We stop when $j = i + 1$; this means that the student's level is i.

Example 1. Let us consider an example when $N = 3$ and $w = 3$. In this example, we need to compute the values $e(2)$, $e(3)$, and $e(4)$.

- We take $e(1) = 0$.
- When $x = 2$, the only possible value for n is $n = 1$, so

$$e(2) = \min_{1 \le n < 2} \{\max\{1 + e(2 - n), 3 + e(n)\}\} =$$

$$\max\{1 + e(1), 3 + e(1)\} = \max\{1, 3\} = 3.$$

  Here, $e(2) = 3$, and $n(2) = 1$.

- To find $e(3)$, we must compare two different values $n = 1$ and $n = 2$:

$$e(3) = \min_{1 \le n < 3} \{\max\{1 + e(3 - n)), 3 + e(n)\}\} =$$

$$\min\{\max\{1 + e(2), 3 + e(1)\}, \max\{1 + e(1), 3 + e(2)\}\} =$$

$$\min\{\max\{4, 3\}, \max\{1, 6\}\} = \min\{4, 6\} = 4.$$

  Here, the minimum is attained when $n = 1$, so $n(3) = 1$.

- To find $e(4)$, we must consider three possible values $n = 1$, $n = 2$, and $n = 3$, so

$$e(4) = \min_{1 \le n < 4} \{\max\{1 + e(4 - n), 3 + e(n))\}\} =$$

$$\min\{\max\{1 + e(3), 3 + e(1)\}, \max\{1 + e(2), 3 + e(2)\},$$

$$\max\{1 + e(1), 3 + e(3)\}\} =$$

$$\min\{\max\{5, 3\}, \max\{4, 6\}, \max\{1, 7\}\} =$$

$$\min\{5, 6, 7\} = 5.$$

  Here, the minimum is attained when $n = 1$, so $n(4) = 1$.

So here, the optimal testing procedure is as follows. First, we have $i = 0$ and $j = N+1 = 4$, so we ask a student to solve a problem of level $m = i + n(j - i) = 1$.

If a student did not succeed in solving this level 1 problem, we replace the original upper bound j with the new value $j = 1$. Now, $j = i + 1$, so we conclude that the student is at level 0.

If the student succeeds in solving the level 1 problem, we take $i = 1$ (and keep $j = 4$ the same). In this case, the next problem is of level $m = i + n(j - i) = 2$.

If the student fails to solve the level 2 problem, then we replace the original upper bound j with the new value $j = m = 2$. Here, $j = i + 1$, so we conclude that the student is at level 1.

If the student succeeds is solving the problem at level 2, then we replace the previous lower bound i with the new bound $i = m = 2$. Now, we give the student the next problem of level $i + n(j - i) = 2 + n(4 - 2) = 2 + 1 = 3$.

If the student fails to solve this problem, then we replace the original upper bound j with the new value $j = m = 3$. Here, $j = i + 1$, so we conclude that the student is at level 2.

If the student succeeds in solving the problem at level 3, then we replace the previous lower bound i with the new bound $i = m = 3$. Here, $j = i + 1$, so we conclude that the student is at level 2.

Comment. In this case, the optimal testing scheme is the most cautious one, when we increase the level by one every time. This way, we are guaranteed that a tested student has no more than one negative experience.

Example 2. Let us now consider an example when $N = 3$ and $w = 1.5$.

- We take $e(1) = 0$.

- When $x = 2$, then

$$e(2) = \min_{1 \le n < 2} \{\max\{1 + e(2 - n), 3 + e(n)\}\} =$$

$$\max\{1 + e(1), 1.5 + e(1)\} = \max\{1, 1.5\} = 1.5.$$

  Here, $e(2) = 1.5$, and $n(2) = 1$.

- To find $e(3)$, we must compare two different values $n = 1$ and $n = 2$:

$$e(3) = \min_{1 \le n < 3} \{\max\{1 + e(3 - n)), 1.5 + e(n)\}\} =$$

$$\min\{\max\{1 + e(2), 1.5 + e(1)\}, \max\{1 + e(1), 1.5 + e(2)\}\} =$$

$$\min\{\max\{2.5, 1.5\}, \max\{1, 3\}\} = \min\{2.5, 3\} = 2.5.$$

  Here, the minimum is attained when $n = 1$, so $n(3) = 1$.

- To find $e(4)$, we must consider three possible values $n = 1$, $n = 2$, and $n = 3$, so

$$e(4) = \min_{1 \le n < 4} \{\max\{1 + e(4 - n), 1.5 + e(n))\}\} =$$

$$\min\{\max\{1 + e(3), 1.5 + e(1)\}, \max\{1 + e(2), 1.5 + e(2)\},$$

$$\max\{1 + e(1), 1.5 + e(3)\}\} =$$

$$\min\{\max\{3.5, 1.5\}, \max\{2.5, 3\}, \max\{1, 4\}\} =$$

$$\min\{3.5, 3, 4\} = 3.$$

  Here, the minimum is attained when $n = 2$, so $n(4) = 2$.

So here, the optimal testing procedure is as follows. First, we have $i = 0$ and $j = N+1 = 4$, so we ask a student to solve a problem of level $m = i + n(j - i) = 2$.

If a student did not succeed in solving the level 2 problem, we replacing the original upper bound j with the new value $j = 2$. Now, we ask the student to solve a problem on level $m = i + n(j - i) = 1$. If a student succeeds, his/her level is 1; if the student does not succeed, his/her level is 0.

If the student succeeds in solving the level 2 problem, we take $i = 2$ (and keep $j = 4$ the same). In this case, the next problem is of level $m = i + n(j - i) = 3$. If a student succeeds, his/her level is 3; if the student does not succeed, his/her level is 2.

Comment. In this case, the optimal testing scheme is the bisection.

Computational complexity. For each n from 1 to N, we need to compare n different values. So, the total number of computational steps is proportional to $1 + 2 + \ldots + N = O(N^2)$.

Additional problem. When N is large, N^2 may be too large. In some applications, the computation of the optimal testing scheme may takes too long. For this case, we have developed a faster algorithm for producing a testing scheme which is only asymptotically optimal.

A faster algorithm for generating an asymptotically optimal testing scheme: description. First, we find the real number $\alpha \in [0, 1]$ for which $\alpha + \alpha^w = 1$. This value α can be obtained, e.g., by applying bisection [18] to the equation $\alpha + \alpha^w = 1$.

Then, at each step, once we have the lower bound i and the upper bound j for the (unknown) student level ℓ, we ask the student to solve a problem at the level

$$m = \lfloor \alpha \cdot i + (1 - \alpha) \cdot j \rfloor.$$

Comments. This algorithm is similar to bisection, expect that bisection corresponds to $\alpha = 0.5$. This makes sense, since for $w = 1$, the equation for α takes the form $2\alpha = 1$, hence $\alpha = 0.5$. For $w = 2$, the solution to the equation $\alpha + \alpha^2 = 1$ is the well-known golden ratio $\alpha = \dfrac{\sqrt{5} - 1}{2} \approx 0.618$.

Computational complexity. At each step, we end up with either an interval $[i, m]$ whose width is $1 - \alpha$ from the original size, or with the interval $[m, j]$ whose width is α

from the original size. Since $\alpha \geq 1-\alpha$, the worst-case decrease is decrease by a factor of α. In k steps, we decrease the width N to $\leq N \cdot \alpha^k$. Thus, we stop for sure when $N \cdot \alpha^k \leq 1$, i.e., after $k = O(\log(N))$ problems.

At each level, we need a constant number of computation steps to compute the next level, so the overall computation time is $O(\log(N))$.

In what sense the resulting testing scheme is asymptotically optimal. We will prove that for this scheme, there is a constant C such that for every N, the worst-case effect from this scheme differs from the worst-case effect of the optimal testing scheme by no more than C.

Proof that the resulting testing scheme is indeed asymptotically optimal. Let us denote the optimal effect by $e(N)$ and the worst-case effect corresponding to our procedure by $e_0(N)$. Let us also denote $K = 2^{-\alpha}$. To prove our result, we will prove that there exist constants $C > 0$ and $C_1 > 0$ such that for every N, we have

$$K \cdot \log_2(N) \leq e(N)$$

and

$$e_0(N) \leq K \cdot \log_2(N) + C - \frac{C_1}{N}.$$

By definition, $e(N)$ is the smallest worst-case effect of all possible testing schemes, thus, $e(N) \leq e_0(N)$. So, if we prove the above two inequalities, we will indeed prove that our algorithm is asymptotically optimal.

Proof of the first inequality. Let us first prove the first inequality by induction over N. The value $N = 1$ represents the induction base. For this value, $K \cdot \log_2(1) = 0 = e(1)$, so the inequality holds.

Let us now describe the induction step. Suppose that we have already proved the inequality $K \cdot \log_2(n) \leq e(n)$ for all $n < N$. Let us prove that $K \cdot \log_2(N) \leq e(N)$.

Due to our main formula, $e(N)$ is the smallest of the values

$$\max\{1 + e(x - n), w + e(n)\}$$

over $n = 1, 2, \ldots, N - 1$. So, to prove that $K \cdot \log_2(N)$ is indeed the lower bound for $e(N)$, we must prove that $K \cdot \log_2(N)$ cannot exceed each of these values, i.e., that

$$K \cdot \log_2(N) \le \max\{1 + e(N - n), w + e(n)\}$$

for every $n = 1, 2, \ldots, N - 1$. For these n, we have $n < N$ and $N - n < N$, so for all these values, we already know that $K \cdot \log_2(n) \le e(n)$ and

$$K \cdot \log_2(N - n) \le e(N - n).$$

Therefore,

$$1 + K \cdot \log_2(N - n) \le 1 + e(N - n),$$

$$w + K \cdot \log_2(n) \le w + e(n),$$

and

$$\max\{1 + K \cdot \log_2(N - n), w + K \cdot \log_2(n)\} \le$$

$$\max\{1 + e(N - n), w + e(n)\}.$$

So, to prove the desired inequality, it is sufficient to prove that

$$K \cdot \log_2(N) \le$$

$$\max\{1 + K \cdot \log_2(N - n), w + K \cdot \log_2(n)\}.$$

We will prove this inequality by considering two possible cases: $n \le (1 - \alpha) \cdot N$ and $n \ge (1 - \alpha) \cdot N$.

- When $n \le (1 - \alpha) \cdot N$, we have $N - n \ge \alpha \cdot N$ and therefore,

$$1 + K \cdot \log_2(N - n) \ge z,$$

where

$$z \stackrel{\text{def}}{=} 1 + K \cdot \log_2(\alpha \cdot N) = 1 + K \cdot \log_2(N) + K \cdot \log_2(\alpha).$$

Here, by definition of $K = 2^{-\alpha}$, we have $\log_2(\alpha) = -1/K$, hence

$$1 + K \cdot \log_2(\alpha) = 0,$$

and so $z = K \cdot \log_2(N)$. In this case,

$$K \cdot \log_2(N) \le z = 1 + K \cdot \log_2(N - n) \le$$

$$\max\{1 + K \cdot \log_2(N - n), w + K \cdot \log_2(n)\}.$$

- When $n \ge (1 - \alpha) \cdot N$, we have $w + K \cdot \log_2(n) \ge t$, where

$$t \stackrel{\text{def}}{=} w + K \cdot \log_2((1 - \alpha) \cdot N) = w + K \cdot \log_2(N) + K \cdot \log_2(1 - \alpha).$$

By definition of α, we have $1 - \alpha = \alpha^w$, so $\log_2(1 - \alpha) = w \cdot \log_2(\alpha)$ and thus, $w + K \cdot \log_2(1 - \alpha) = w \cdot (1 + K \cdot \log_2(\alpha)) = 0$. Hence, $t = K \cdot \log_2(N)$. So, in this case,

$$K \cdot \log_2(N) \le t = w + K \cdot \log_2(n) \le$$

$$\max\{1 + \log_2(N - n), w + K \cdot \log_2(n)\}.$$

In both cases, we have the desired inequality. The induction step is proven, and so, indeed, for every N, we have

$$K \cdot \log_2(N) \le e(N).$$

Proof of the second inequality. Let us now prove that there exist real numbers $C > 0$ and $C_1 > 0$ for which, for all N,

$$e_0(N) \le K \cdot \log_2(N) + C - \frac{C_1}{N}.$$

To prove this inequality, we will pick a value N_0, prove that this inequality holds for all $N \le N_0$, and then use mathematical induction to show that it holds for all $N > N_0$ as well.

Induction basis. Let us first find the conditions on C, C_1, and N_0 under which for all $N \leq N_0$,

$$e_0(N) \leq K \cdot \log_2(N) + C - \frac{C_1}{N}.$$

Subtracting $K \cdot \log_2(N)$ and adding $\frac{C_1}{N}$ to both sides of the this inequality, we get

$$C \geq \frac{C_1}{N} + e_0(N) - K \cdot \log_2(N)$$

for all N from 1 to N_0. So, to guarantee that this inequality holds, if we have already chosen N_0 and C_1, we can choose

$$C = \max_{1 \leq N \leq N_0} \left(\frac{C_1}{N} + e_0(N) - K \cdot \log_2(N) \right).$$

Induction step. Let us assume that for all $n < N$ (where $N > N_0$), we have proven that

$$e_0(n) \leq K \cdot \log_2(n) + C - \frac{C_1}{n}.$$

We would like to conclude that

$$e_0(N) \leq K \cdot \log_2(N) + C - \frac{C_1}{N}.$$

According to the definition of $e_0(N)$, we have

$$e_0(N) = \max\{1 + e_0(N - n), w + e_0(n))\},$$

where $n = \lfloor (1 - \alpha) \cdot N \rfloor$. Due to induction hypothesis, we have

$$e_0(n) \leq K \cdot \log_2(n) + C - \frac{C_1}{n}$$

and

$$e_0(N - n) \leq K \cdot \log_2(N - n) + C - \frac{C_1}{N - n}.$$

Therefore,

$$e_0(N) \leq \max \left\{ 1 + K \cdot \log_2(N - n) + C - \frac{C_1}{N - n}, \right.$$

$$w + K \cdot \log_2(n) + C - \frac{C_1}{n}\Big\}.$$

Thus, to complete the proof, it is sufficient to conclude that this maximum does not exceed

$$K \cdot \log_2(N) + C - \frac{C_1}{N}.$$

In other words, we must prove that

$$1 + K \cdot \log_2(N-n) + C - \frac{C_1}{N-n} \le K \cdot \log_2(N) + C - \frac{C_1}{N}$$

and that

$$w + K \cdot \log_2(n) + C - \frac{C_1}{N-n} \le K \cdot \log_2(N) + C - \frac{C_1}{N}. \tag{4.2.2}$$

Without losing generality, let us show how we can prove the second of these two inequalities. Since $n = \lfloor (1-\alpha) \cdot N \rfloor$, the left-hand side of the inequality (2) can be rewritten as

$$W_1 + K \cdot \log_2((1-\alpha) \cdot N) + K \cdot (\log_2(n) - \log_2((1-\alpha) \cdot N)) + C - \frac{C_1}{n}.$$

We already know that $w + K \cdot \log_2((1-\alpha) \cdot N) = K \cdot \log_2(N)$. Thus, the left-hand side of (2) takes the simpler form

$$K \cdot \log_2(N) + K \cdot (\log_2(n) - \log_2((1-\alpha) \cdot N)) + C - \frac{C_1}{n}.$$

Substituting this expression into (2) and canceling the terms $K \cdot \log_2(N)$ and C in both sides, we get an equivalent inequality

$$K \cdot (\log_2(n) - \log_2((1-\alpha) \cdot N)) - \frac{C_1}{n} \le -\frac{C_1}{N}. \tag{4.2.3}$$

Let us further simplify this inequality. We will start by estimating the difference $\log_2(n) - \log_2((1-\alpha) \cdot N)$. To estimate this difference, we will use the intermediate value theorem, according to which, for every smooth function $f(x)$, and for arbitrary two values a and b, we have $f(a) - f(b) = (a-b) \cdot f'(\xi)$ for some $\xi \in [a, b]$. In our case,

$$f(x) = \log_2(x) = \frac{\ln(x)}{\ln(2)},$$

$a = n$, and $b = (1-\alpha)\cdot N$. Here,

$$f'(\xi) = \frac{1}{\xi \cdot \ln(2)},$$

so

$$f'(\xi) \leq \frac{1}{n \cdot \ln(2)};$$

also, $|a-b| \leq 1$, so, the difference $\log_2(n) - \log_2((1-\alpha)\cdot N)$ can be estimated from above by:

$$\log_2(n) - \log_2((1-\alpha)\cdot N) \leq \frac{1}{n \cdot \ln(2)}.$$

Hence, the above inequality holds if the following stronger inequality holds:

$$\frac{K}{n \cdot \ln(2)} - \frac{C_1}{n} \leq -\frac{C_1}{N},$$

or, equivalently,

$$\frac{C_1}{N} \leq \frac{C_1 - K/\ln(2)}{n}. \tag{4.2.4}$$

Here, $n \geq (1-\alpha)\cdot N - 1$, i.e.,

$$\frac{n}{N} \geq (1-\alpha) - \frac{1}{n}.$$

When $N \to \infty$, we have $n \to \infty$ and $\dfrac{1}{n} \to 0$. Thus, for every $\varepsilon > 0$, there exists an N_0 starting from which $\dfrac{1}{n} \leq \varepsilon$ and hence, $n \geq (1-\alpha-\varepsilon)\cdot N$. For such sufficiently large N, the inequality (4.2.4) can be proven if we have

$$\frac{C_1}{N} \leq \frac{C_1 - K/\ln(2)}{(1-\alpha-\varepsilon)\cdot N},$$

i.e., if we have

$$C_1 \leq \frac{C_1 - K/\ln(2)}{1-\alpha-\varepsilon}. \tag{4.2.5}$$

Since $0 \leq \alpha \leq 1$, for sufficiently large C_1, this inequality is true. For such C_1, therefore, the induction can be proven and thus, the second inequality is true.

The statement is proven.

What if we also know probabilities: formulation of the problem. In some cases, we also know the frequencies p_0, p_1, ..., p_N with which students are at the corresponding levels. These frequencies can be alternatively described by the corresponding cumulative distribution function $F(i) \stackrel{\text{def}}{=} \text{Prob}(\ell < i) = p_0 + p_1 + \ldots + p_{i-1}$. In this situation, instead of the *worst-case* effect, we can alternatively consider the *average* effect – and look for a testing scheme which minimizes the average effect.

Towards a scheme which minimizes the average effect. Let $e(i, j)$ be the smallest possible conditional average effect under the condition that a student's actual level is between i and j, i.e., that the student has successfully solves a problem at level i and was unable to solve the problem at level j. (The original situation corresponds to $i = 0$ and $j = N + 1$.)

In this situation, we ask the student to solve a problem at some level $n \in (i, j)$. Let us consider both possible cases: when the student was able to solve this problem, and when a student was unable to solve this problem.

If a student was able to solve the problem at level n, this means that the student's level is in between n and j. By definition of a function $e(\cdot, \cdot)$, the expected remaining effect is equal to $e(n, j)$. Thus, in this case, the overall expected effect on the student is equal to $1 + e(n, j)$. The conditional probability of this case can be obtained by dividing the probability $F(j) - F(n)$ that the student's level is between n and j by the original probability $F(j) - f(i)$ that the student's level is between i and j. Thus, this probability is equal to $\dfrac{F(j) - F(n)}{F(j) - F(i)}$.

If a student was unable to solve the problem at level n, this means that the student's level is in between i and n. By definition of a function $e(\cdot, \cdot)$, the expected remaining effect is equal to $e(i, n)$. Thus, in this case, the overall expected effect on the student is equal to $w + e(i, n)$. The conditional probability of this case can be obtained by dividing the probability $F(n) - F(i)$ that the student's level is between i and n by the original probability $F(j) - f(i)$ that the student's level is between i and j. Thus, this probability

is equal to $\dfrac{F(n)-F(i)}{F(j)-F(i)}$.

Thus, we have the expected effect $1+e(n,j)$ with probability $\dfrac{F(j)-F(n)}{F(j)-F(i)}$, and the expected effect $w+e(i,n)$ with probability $\dfrac{F(n)-F(i)}{F(j)-F(i)}$. So, the overall expected effect is equal to

$$\frac{F(j)-F(n)}{F(j)-F(i)}\cdot(1+e(n,j))+\frac{F(n)-F(i)}{F(j)-F(i)}\cdot(w+e(i,n)).$$

Since we want to minimize the average effect, we select n for which this value is the smallest possible. Thus, we arrive at the following formula:

Main formula: average case.

$$e(i,j)=$$
$$\min_{i\le n<j}\left(\frac{F(j)-F(n)}{F(j)-F(i)}\cdot(1+e(n,j))+\frac{F(n)-F(i)}{F(j)-F(i)}\cdot(w+e(i,n))\right). \tag{4.2.6}$$

Towards the optimal testing scheme. When $j=i+1$, we know that the student's level is i, so no additional testing is needed and the effect is 0: $e(i,i+1)=0$. We can start with these values and sequentially use the formula (4.2.6) to compute the values $e(i,i+2)$, $e(i,i+3)$, etc. In each case, we find $n(i,j)$ for which the minimum is attained.

Resulting optimal testing scheme. First, we take $e(i,i+1)=0$ for all i, and use the formula (4.2.6) to sequentially compute the values $e(i,i+2)$, $e(i,i+3)$, ..., until we cover all possible values $e(i,j)$. For each i and j, we record the value $n(i,j)$ for which the right-hand side of the formula (4.2.6) is the smallest.

At each stage of the testing, we keep track of the bounds i and j for the student's level. In the beginning, $i=0$ and $j=N+1$. At each stage, we ask the student to solve a problem at level $m=n(i,j)$.

- If the student succeeds in solving this problem, we replace the original lower bound i with the new bound m.

- If the student did not succeed in solving the problem on level m, we replace the original upper bound j with the new bound j.

We stop when $j = i + 1$; this means that the student's level is i.

Computational complexity. For each of $O(N^2)$ pairs $i < j$ of numbers from 0 to N, we need to compare $j - i = O(N)$ different values. So, the total number of computational steps is proportional to $O(N^2) \cdot O(N) = O(N^3)$.

Comment. For large N, this computation time may be too large. It would be nice – similarly to the worst-case optimization – to come up with a faster algorithm even if it generates only an asymptotically optimal testing scheme.

4.3 Analyzing the Way the Material Is Presented: Global Aspects

Once this information is known, we need to actually present this information to the interested folks – and use appropriate feedback to modify (if needed) the speed with which this knowledge is presented. Issues related to the material's presentation are analyzed in this section and in the following section. In this section, we consider the problem from the global viewpoint: e.g., in what order we should present different parts of the material.

In the traditional approach to learning, if we want students to learn how to solve different types of problems, we first teach them how to solve problems of the first type, then how to solve problems of the second type, etc. It turns out that we can speed up learning if we interleave problems of different types. In particular, it has bene empirically shown that interleaving problems of four different types leads to a double speed-up. In this section, we provide a possible explanation for this empirical fact.

Traditional approach to learning several skills. Traditionally, when students need to learn several skills, they learn them one by one:

- first, they learn the skill a;
- once they have mastered skill a, they start learning skill b,
- etc.

For example, in a geometry class, students need to learn how to solve several different types of problems. For that purpose:

- they first spend several class periods learning how to solve problems of type a,
- then they spend several class periods learning how to solve problems of type b,
- etc.

Interleaving: an alternative approach. An alternative approach is *interleaving*, when students learn several skills at the same time. For example, instead of first solving several problems of type a, then several problem of type b, etc., they solve a problem of type a, then a problem of type b, etc., then again a problem of type a, then again b, etc.

In other words, instead of a usual sequence of problem types

$$aaa \ldots bbb \ldots ccc \ldots$$

we use an interleaving sequence

$$abc \ldots abc \ldots abc \ldots$$

Interleaving enhances learning. Several studies show that interleaving enhances different types of learning, from learning to play basketball [31, 55] to learning art [51] to learning mathematics [56, 88, 99]; see also [12].

Quantitative fact. In particular, in [99], it is shown that interleaving of four different types of geometric problems increases the average number of correct answers on the test twice, from 38% to 77%.

In other words, interleaving of four different types of problems doubles the learning speed.

What we do in this section. In this section, we provide a possible explanation to the above enhancement, an explanation based on a simple geometric model.

A simple geometric model. Let us describe traditional and interleaved approaches in geometric terms. We want students to learn to solve four different types of problems. In the beginning, the students do not know how to solve any of these problems. The objective is for them to be able to solve all four types of problems.

We can represent the state of the students at each moment of time by the percentage (x_1, x_2, x_3, x_4) of problems of each type that a student can solve.

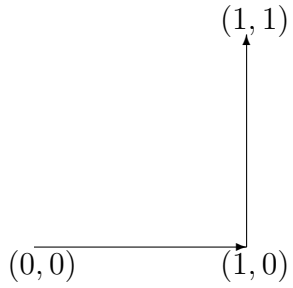
- In the beginning, the students are in the state $(0, 0, 0, 0)$.
- Our objective is to reach the state $(1, 1, 1, 1)$.

How traditional approach is represented in this geometric model. In the traditional approach, the students first learn to solve problems of the first type, then they learn how to solve problems of the second type, etc. In other words:

- the students first move from the state $(0, 0, 0, 0)$ to the state $(1, 0, 0, 0)$,
- then they move to the state $(1, 1, 0, 0)$,
- after that, they move to the state $(1, 1, 1, 0)$,
- and, finally, they move to the desired state $(1, 1, 1, 1)$.

At each stage of this process, we can assume that the students follow the shortest path – a straight line – to get to the corresponding state.

Each stage has length 1, so the total length of all four stages is equal to 4.



How the interleaved approach is represented in this geometric model. In the interleaved approach, at each moment of time, the students have spent equal time on problems of all fours types and thus, their skills in solving problems of all four types are equal.

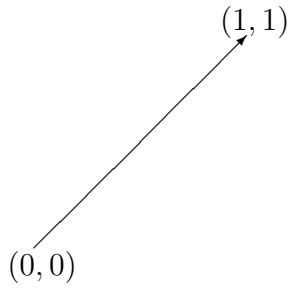
In geometric terms, this means that their state is described by a tuple (x, x, x, x). Thus, for this approach, learning follows the diagonal path

$$\{(x, x, x, x) : x \in [0, 1]\}.$$

This diagonal is the straight line segment connecting the original state $(0, 0, 0, 0)$ with the desired state $(1, 1, 1, 1)$.

The length of this path is equal to the distance between these two states $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$, i.e., to the value

$$\sqrt{(1-0)^2 + (1-0)^2 + (1-0)^2 + (1-0)^2} = \sqrt{4} = 2.$$



Resulting explanation of the empirical fact. We see that in the interleaved approach, the path to the desired state is *twice shorter* than in the traditional approach. This may explain why, when we interleave four different types of problems, learning becomes *twice faster*.

4.4 Analyzing the Way the Material Is Presented: Local Aspects

In this section, we continue the analysis of the problems related to the material's presentation. In the previous section, we considered the problem from the global viewpoint: e.g., in what order we should present different parts of the material. In this section, we consider this problem from the local viewpoint: what is the best way to present different items.

As we mentioned earlier, these issues are especially important in computerized education, when the material is presented by an automated system, without a detailed supervision from a human teacher.

To design such systems, practitioners use empirical rules and laws derived from the experience of computerized information access. One of the most well-known of these laws is an empirical Fitts's Law, according to which the average time T of accessing an icon of size w at a distance d from the center of the screen is proportional to the logarithm of the ratio w/d. There exist explanations for this law, but these explanations have gaps.

In this section, we show that these gaps can be explained if we analyze this problem from the geometric viewpoint. Thus, we get a possible theoretical explanation of the Fitts's Law.

What is Fitts's Law. The efficiency of computer-based systems for education, information, commerce, etc., strongly depends on the user-friendliness of the corresponding interfaces, in particular, on the location and size of the appropriate icones. When deciding the location and size of different icons on a computer screen, designers use the Fitts's Law

[24, 25]. This law describe how the average time T of accessing an icon depends on the distance d from the center of the screen to the icon and on the linear size w of this icon: $T = a + b \cdot \ln\left(\dfrac{d}{w}\right)$, for some constants a and b.

How Fitts's Law is used in interface design. The use of Fitts's Law started with the very first mouse-accessible interfaces; see, e.g., [14]. It is based on the following idea.

Each icon corresponds to a specific task or group of tasks. Some tasks are more frequent, some are rarer: for example, editing is a frequent task, while logging off is a rarer task. For each task, we can empirically determine the frequency f_i with which this task is performed. We can therefore gauge the user-friendliness of the interface by the average time

$$\sum_i f_i \cdot T_i = \sum_i f_i \cdot \left(a + b \cdot \ln\left(\frac{d_i}{w_i}\right)\right)$$

needed to access the required icon. Out of several possible interfaces, we select the one for which this average time is the smallest.

Fitts's Law: qualitative aspects. From the qualitative viewpoint, the Fitts's Law says that T decreases when d decreases and/or w increases. In other words:

- the closer the icon to the center, the easier it is to find this icon, and
- the larger the icon size, the easier it is to find it.

From this viewpoint, Fitts's Law is simply common sense.

Quantitative aspects of the Fitts's Law need explanation. That the time T should monotonically depend on the distance d and on the size w is clear, but there are many different monotonic functions. The fact that overwhelming majority of experimentally results is in very good accordance with one type of monotonic dependence – the logarithmic law – needs explanation.

Current explanation of Fitts's Law. A current explanation of Fitts's Law [15] is is based on the fact that our motions are not perfect. For simplicity, this explanation assumes that each movement aiming at reaching an object at distance d actually only follows a slightly smaller distance $(1-\varepsilon)\cdot d$, for some accuracy $\varepsilon < 1$. Thus, after the original movement, we are still a distance ($\varepsilon \cdot d$ away from the desired object. We therefore need the next movement to reach this object.

This second movement brings us to the distance $\varepsilon \cdot (\varepsilon \cdot d) = \varepsilon^2 \cdot d$ to the target. In general, after k movements, we are at a distance $\varepsilon^k \cdot d$ from the target. If we aim at the center of an icon, then we reach a point within the icon when this distance is smaller than or equal to the icon's half-size $\frac{w}{2}$, i.e., when $\varepsilon^k \cdot d \le \frac{w}{2}$. From the condition that $\varepsilon^k \cdot d \approx \frac{w}{2}$, we can determine the number of iterations k as $k \approx \frac{1}{\ln(\varepsilon)} \cdot \ln\left(\frac{2d}{w}\right)$. One can easily check that we thus get $k \approx a + b \cdot \ln\left(\frac{d}{w}\right)$, where $a = \frac{\ln(2)}{\ln(\varepsilon)}$ and $b = \frac{1}{|\ln(\varepsilon)|}$.

The overall time needed to reach the icon consists of the time of the smooth motions and the time needed to switch from one motion to another. Usually, the switch time is much larger. So, in first approximation, we can simply ignore the time of the smooth movements and conclude that the time T is proportional to the number of switches k. Thus, we arrive at the Fitts's formula.

This explanation has some gaps. As noted, e.g., in [67], the above explanation is not perfect, it has two gaps. The first gap is not critical: the above derivation assumes that for the same distance d, the motor error is always the same, while in reality, in repeated experiments, we may get different values of the motor error. This gap is not critical, because the above derivation will not change is we take into account that after the first iteration, the distance to the target is only *approximately* equal to $\varepsilon \cdot d$.

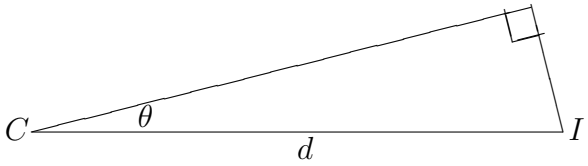
The second gap is more serious. The above derivation is based on the assumption that if we want to move to a distance d, then the accuracy with which we can perform this movement is equal to $\varepsilon \cdot d$. In other words, this derivation is based on the assumption that

the *relative* accuracy $\dfrac{\varepsilon \cdot d}{d}$ is the same for all the distances. If the relative accuracy depends on the distance d, i.e., if the accuracy is equal to $\varepsilon(d) \cdot d$ for some function $\varepsilon(d) \neq \text{const}$, then, instead of the Fitts's Law, we would get a different formula.

What we do. To come up with a more convincing explanation of the Fitts's Law, we therefore need to explain why the relative accuracy does not depend on the distance. This is what we do in this section.

Our explanation of Fitts's Law. Let us assume that the cursor (controlled, e.g., by a mouse) is currently located at the center C of the screen, and we want to move it to the location of the icon I. The shortest way from one point to another is a straight line, so naturally, we start a straight line in the direction of the icon. To be more precise, we select an angle leading us to the icon, and we follow a straight line in the direction of this angle.

If we could set up the angle exactly, we would then follow the straight line to the desired icon and reach this icon in one movement. In practice, of course, there is a motor error; we cannot set the angle of our movement exactly, we can only set up this angle with some accuracy θ. Because of this accuracy, the straight line that we actually follow is at an angle of order θ from the line connecting the center of the screen with the target icon.



As a result of this motion inaccuracy, we do not reach the desired point I, the closest we get to I is at a distance $\approx d \cdot \sin(\theta)$. As a result of a movement, we get from the location at a distance d from the target point I to a new location whose distance to I is approximately equal to $\varepsilon \cdot d$, where $\varepsilon \stackrel{\text{def}}{=} \sin(\theta)$.

To reach the desired location I, starting from this new point, we again aim at I. As a result, we get from the point at a distance $\approx \varepsilon \cdot d$ to I to a new point whose distance from

I is approximately equal to $\varepsilon \cdot (\varepsilon \cdot d) = \varepsilon^2 \cdot d$. After k iterations, we reach a point at a distance $\approx \varepsilon^k \cdot d$ to the target point I. We reach the icon if this distance does not exceed the icon's half-width $\frac{w}{2}$, i.e., when $\varepsilon^k \cdot d \approx \frac{w}{2}$.

As we have mentioned, the resulting number of iterations is $k \approx a + b \cdot \ln\left(\frac{w}{d}\right)$. Under a natural assumption that the average time T needed to reach an icon is proportional to this number of iterations, we get the desired Fitts's Law.

Comment. It is worth mentioning that a similar geometric argument describes how the number of corrections needed for inter-stellar travel depends on the travel distance d; see, e.g., [53].

4.5 Analyzing the Effect of Feedback

The efficiency of presenting information to students depends on how we present the material and on whether we use the appropriate feedback – and how exactly we use it. In the previous two sections, we analyzed different ways of presenting the material. In this section, we analyze the effect of feedback.

A recent study [19] published in the *Notices of American Mathematical Society* showed among many factors which could potentially affect the students' success, only one factor determines the success of a technique: the presence of immediate feedback. On average, students who receive immediate feedback learn twice faster than students who are taught in a more traditional way, with a serious feedback only once or twice a semester (after a test).

The very fact that immediate feedback is helpful is not surprising: it helps the student clear misconceptions and avoid the wrong paths. However, the fact that different techniques involving feedback lead to practically the same learning speed-up is intriguing. To explain this speed-up, we provide a simplified first-order description of a learning process in simple geometric terms. We show that already in this first approximation, the geometric

description leads to the observed two-fold speed-up in learning.

Student understanding is extremely important. One of the main objectives of a course – whether it is calculus or physics or any other course – is to enable students to understand the main concepts of this course. Of course, it is also desirable that the students learn the corresponding methods and algorithms, but understanding is the primary goal. If a student does not remember by heart how to compute the derivative of a product, he or she can look up the formula on the web or even derive the formula – and so, most probably, this student will succeed in the following classes which depend on the use of derivatives. However, if a student does not have a good understanding of what is a derivative, then even if this student remembers some formulas, the student will probably not be able to decide which formula to apply in what situation.

How to gauge student understanding. To properly gauge student's understanding, several disciplines have developed *concept inventories*, sets of important basic concepts and questions testing the students' understanding of these concepts. The first such concept inventory was developed in physics, to gauge the students' understanding of the basic concepts of Newtonian mechanics such as the concept of force; the corresponding Force Concept Inventory (FCI) is described in [30, 32, 33, 35, 36]. A similar Calculus Concept Inventory (CCI) is described in [20, 21].

A student's degree of understanding is measured by the percentage of the questions that are answered correctly. The class's degree of understanding is measured by averaging the students' degrees. An ideal situation is when everyone has a perfect 100% understanding; in this case, the average score is 100%. In practice, the average score is smaller than 100%.

How to compare different teaching techniques. To gauge how successful is a given teaching technique, we can measure the average score μ_0 before the class and the average score μ_f after the class. A perfect class is when the whole difference $100 - \mu_0$ disappeared, i.e., the students' average score went from μ_0 to $\mu_f = 100$. In practice, of course, the

students' gain $\mu_f - \mu_0$ is somewhat smaller than the ideal gain $100 - \mu_0$. It is reasonable to measure the success of a teaching method by describing which portion of the ideal gain is covered, i.e., by the ratio

$$g \stackrel{\text{def}}{=} \frac{\mu_f - \mu_0}{100 - \mu_0}.$$

Empirical results. It turns out that for different teaching methods, the normalized gain g does not depend on the initial level μ_0, does not depend on the textbook used or on the teacher. Only one factor determines the value g: the absence or presence of immediate feedback.

In the traditionally taught classes, where the students get their major feedback only after their first midterm exam, the normalized gain g is consistently smaller than in the classes where the students got immediate feedback during every class period.

Specifically, for traditionally taught classes, the average value of the gain is $g \approx 0.23$, while for the classes with an immediate feedback, the average value of the gain is $g \approx 0.48$; see, e.g., [19, 30].

In other words, students who receive immediate feedback, on average, learn twice faster than students who are taught by traditional methods.

Natural question. The consistent appearance of the doubling of the rate of learning seems to indicate that there is a fundamental reason behind this empirical result.

What we do in this section. In this section, we provide a possible geometric explanation for the above empirical result.

Why geometry. Learning means that the student – who did not originally know the material – becomes knowledgeable of this material. To check how well a student learned, we can apply different tests. Based on the results of these tests, we can determine the current state of the student knowledge. In other words, at any given moment of time, the

state of the student's knowledge can be characterized by several numbers $(x_1, \ldots, x_n)$ – the student's scores on different parts of the test.

Each such state can be naturally represented as a point in the n-dimensional space – namely, a point with coordinates $x_1, \ldots, x_n$. In the starting state S, the student does not know the material; the desired state D describes the situation when a student has the desired knowledge. When a student learns, the student's state of knowledge changes continuously forming a (continuous) trajectory γ which starts at the starting state S and ends up at the desired state D.

First simplifying assumption: all students learn at the same rate. Some students learn faster, others learn slower. The above empirical fact is not about their individual learning rates, it is about the *average* rates of student learning, averaged over all kinds of students. From this viewpoint, it makes sense to simplify the complex actual situation – in which different students have different learning rates – with a simplified model, in which all the students have the same average learning rate.

Let us give an example of why such a replacement makes sense when we only consider averages:

- if we are want to study the difference between people's appetites, it makes sense to keep their differing heights intact;
- however, if we are planning to serve a meal to a large group of people, it makes perfect sense, when ordering food ingredients, to ignore the individual differences and assume that everyone has an average appetite.

In geometric terms, the rate of learning corresponds to the rate with which the student's state changes, i.e., corresponds to how far the student's state of knowledge changes in a given period of time. In these terms, the assumption that all the students have the same learning rate means that the states corresponding to different students change with the same rate. In other words, in this geometric model, the time that it takes for a student to

get from the initial state S to the desired state D is proportional to the total length of the corresponding curve γ.

In these terms, to explain the fact that students who receive instant interaction learn twice faster means that on average, we need to show that their learning trajectories are, on average, twice shorter.

Second simplifying assumption: the shape of the learning trajectories. In the beginning, a student may be eager to study, but often, he/she is not sure which direction to go. A student usually has misconceptions about physics and/or calculus, misconception that may lead the student in a wrong direction. We can describe this by assuming that when a student starts at the starting point S, he/she moves in a random direction.

In situations when the student deviated from the direction towards the desired state D, a feedback enables the student understand that he/she is going in the wrong direction. After the feedback, the student corrects his/her trajectory.

In the case of immediate feedback, this correction comes right away, so, in effect, the student immediately starts following the right direction. In other words, in learning with immediate feedback, the student's learning trajectory is a straight line which goes directly from S to D.

In the traditional learning, feedback comes only with midterm exams. Usually, there are two midterm exams, and they are scheduled in such a way that between themselves, they cover all the material studied in the course, and each covers approximately the same amount of material. Thus, the first midterm exam usually covers half of the material. In geometric terms, it means that this exam is given once the student covered half of the distance between S and D. This exam checks whether the student has correctly reached the midpoint $M \stackrel{\text{def}}{=} \dfrac{S+D}{2}$ between S and D. Once the student has covered the half-distance $d/2$ in the originally selected direction, the results of the first midterm exam provides a necessary correction, and the student starts going straight towards the correct midpoint M. After that, the same process starts again: the student goes for $d/2$ in the random

direction, and then comes back to D.

Resulting geometric description of leaning with and without immediate feedback. In learning with immediate feedback, a student follows a straight line from S to D. The length of the corresponding trajectory is equal to the distance $d \stackrel{\text{def}}{=} \rho(S, D)$ between the states S and D.

In learning without immediate feedback, a student first follows a straight line of length $d/2$ which goes in a random direction, then goes straight to the midpoint M, then again follows a straight line of length $d/2$ in a random direction, and finally takes a straight like to D.

Third simplifying assumption: the state space is 1-D. While in general, we can think of different numerical characteristics describing different aspects of student knowledge, in practice, we are pretty comfortable using a single number – usually, an overall grade for the course – to characterize the student's state of knowledge. It is therefore reasonable to make one more simplifying assumption: that the state of a student is characterized by only one parameter x_1.

Let us compare the lengths of the corresponding trajectories. Under our simplifying assumption, the learning time is proportional to the length of the corresponding trajectory. Thus, to compare the learning rates, we need to compare the lengths of the corresponding trajectories.

In case of immediate feedback, the learning trajectory has length d. So, to make a comparison, we must estimate the length of a trajectory corresponding to the traditional learning.

This trajectory consists of two similar parts: the part connecting S and M and the part connecting M and D. Hence, to estimate the total average length, it is sufficient to estimate the average length from S to M and then multiply the result by two.

Analysis of the model. In case of immediate feedback, the learning trajectory has length d.

In the case of traditional learning, under the 1-D assumption, a student initially goes either in the correct direction or in the opposite (wrong) direction; the idea that the direction is chosen randomly can be naturally formalized as an assumption that both directions occur with equal probability 1/2.

If the student's trajectory initially moves in the correct direction, then after traveling the distance $d/2$, the state gets exactly into the desired midpoint D; so, the overall length of the S-to-M of the trajectory is exactly $d/2$.

If the student's trajectory initially goes in the wrong direction, then the student ends up at a point at the same distance $d/2$ from S but on the wrong side of S. Getting back to M then means first going back to S and then going from S to M. The overall length of this trajectory is thus $3d/2$.

With probability 1/2, the length is $d/2$, with probability 1/2, the length is $3d/2$. So, the average length of the S-to-M part of the learning trajectory is equal to

$$\frac{1}{2} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{3d}{2} = d.$$

The average length of the whole trajectory is double that, i.e., $2d$ – twice larger than the length corresponding to immediate feedback.

Since we assumed that the learning time is proportional to the length of the learning trajectory, we can thus draw the following conclusion.

Summary of the results. In this 1-D model, a student following an instant feedback trajectory reaches the desired state, on average twice faster than a student following the traditional-learning trajectory. This is exactly what we wanted to explain.

Chapter 5

Knowledge Use

How can we use the acquired knowledge? In many practical situations, we have a well-defined problem, with a clear well-formulated objective. Such problems are typical in engineering: we want a bridge which can withstand a given load, we want a car with a given fuel efficiency, etc. There exist many techniques for solving such well-defined optimization problems.

However, in many practical situations, it is important to also take into account subjective user preferences. This subjective aspect of decision making is known as *Kansei engineering*. This aspect is what we analyze in this chapter.

The results from this chapter first appeared in [40].

Need for Kansei Engineering. Traditional engineering deals with objective characteristics of a design: There may be several different designs with the given ranges on characteristics, e.g., we may have different car designs within the given price range, efficiency range, size restrictions, etc. Different people make different choices between these designs based on their subjective preferences.

This is how people select cars, this is how people select chairs, etc. Engineering that takes such subjective preference into account is known as *Kansei Engineering*; see, e.g., [39, 71, 91, 107, 108].

Need to select designs. Different people have different preferences. Thus, to satisfy customers, we must produce several different designs: a car company produces cars of several different designs, a furniture company produces chairs of several different designs,

etc.

The creation of each new design is often very expensive and time-consuming. As a result, the number of new designs is usually limited. The question is: once we know what customers want, and once we know how many different designs we can afford, how should we select these designs?

What we do in this chapter. In this chapter, we describe a reasonable mathematical model within which we can find an optimal collection of design.

Towards a mathematical model. Let us denote the number of parameters needed to describe different designs by n. Then, each design can be characterized by an n-dimensional vector $x = (x_1, \ldots, x_n)$. Let us assume that the unit of different parameters are selected in such a way that a unit of each parameter represents the same difference for the user. Under this selection, it is reasonable to assume that the user's difference between two designs can be described by the Euclidean distance $d(x, y)$ between the corresponding vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$:

$$d(x, x') = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

We have a large number of potential users. For each user, some design is ideal, and the farther we are from this ideal design, the less desirable this design is. For our purposes, we can simply identify each user with this ideal vector x.

There are usually very many users, each of these users can be characterized by a vector x. Ideally, we should record all these vectors, but in practice, it is reasonable to describe how many users are in different zones. In other words, a reasonable way to describe the users is to provide the distribution on the set of all possible designs that characterizes how popular different designs are. A natural way to describe a distribution of customers is to provide the population density $\rho_u(x)$ at different points x from the corresponding

n-dimensional region. For this function, $\rho_u(x) \geq 0$ and the integral $\int \rho_u(x)\,dx$ is equal to the total number of potential customers.

Similarly, we can have a large number of engineered designs. So, instead of explicitly listing these designs, we can simply describe how many different designs are manufactured in different zones. Let us describe the corresponding design density by $\rho_m(x)$. Here, $\rho_m(x) \geq 0$ and

$$\int \rho_m(x)\,dx = D, \tag{5.1}$$

where D denotes the total number of designs.

If a manufacturer produces an ideal design, then the potential customer will buy it for sure. The larger the distance between the ideal and the actual designs, the less probable it is that the customer will purchase this design. Let $p(r)$ be the probability that a customer will purchase a design at distance r from the ideal one.

When the average density of the actual designs is $\rho_m(x)$, this means that in an area of linear size r and volume $V = r^n$, we have $\rho_m(x) \cdot r^n$ designs. So, we have one design in the area of size r for which $\rho_m(x) \cdot r^n = 1$. This equality leads to $r = \dfrac{1}{\sqrt[n]{\rho_m(x)}}$. So, around the point x, the probability that a customer buys a design is equal to $p(r) = p\left(\dfrac{1}{\sqrt[n]{\rho_m(x)}}\right)$. In the area of volume dx around the point x, there are $\rho_u(x)\,dx$ customers. Since the proportion $p(r)$ of them buys the design, the total number of customers in this area who purchased some design is equal to

$$\rho_u(x) \cdot p(r)\,dx = \rho_u(x) \cdot p\left(\frac{1}{\sqrt[n]{\rho_m(x)}}\right)\,dx.$$

Thus, the total number C of customers who bought our designs is equal to

$$C = \int \rho_u(x) \cdot p\left(\frac{1}{\sqrt[n]{\rho_m(x)}}\right)\,dx. \tag{5.2}$$

Our objective is to maximize the overall profit. Let s be our gain from selling a single unit. Then, by selling units to C customers, we gain the amount $C \cdot s$. Let d be the cost

of generating one design; then, by producing D designs, we spend the amount $D \cdot d$. If we subtract the expenses from the gain, we get the profit

$$M = C \cdot s - D \cdot d. \tag{5.3}$$

Resulting optimization problem. We are given the functions $\rho_u(x)$ and $p(r)$ and the values s and d. We need to select a function $\rho_m(x)$ for which the profit (5.3) is the largest possible, where the values C and D by using formulas (5.1) and (5.2). In other words, we need to optimize the following expression:

$$M = s \cdot \int \rho_u(x) \cdot p\left(\frac{1}{\sqrt[n]{\rho_m(x)}}\right) dx - d \cdot \int \rho_m(x)\, dx. \tag{5.4}$$

Towards a solution. To solve the above optimization problem, we differentiate the objective function M by each unknown $\rho_m(x)$ and equate the resulting derivative to 0. Thus, we get

$$s \cdot \frac{1}{n} \cdot p'\left(\frac{1}{\sqrt[n]{\rho_m(x)}}\right) \cdot \frac{1}{\sqrt[n]{\rho_m(x)} \cdot \rho_m(x)} \cdot \rho_u(x) - d = 0, \tag{5.5}$$

where $p'(r)$ is the derivative of $p(r)$. By moving d to the right-hand side, we get an equivalent formula

$$s \cdot \frac{1}{n} \cdot p'\left(\frac{1}{\sqrt[n]{\rho_m(x)}}\right) \cdot \frac{1}{\sqrt[n]{\rho_m(x)} \cdot \rho_m(x)} \cdot \rho_u(x) = d. \tag{5.6}$$

By dividing both sides by $s \cdot \dfrac{1}{n} \cdot \rho_u(x)$, we keep all the terms depending on the unknowns in the left-hand side and move all the known terms to the right-hand side:

$$p'\left(\frac{1}{\sqrt[n]{\rho_m(x)}}\right) \cdot \frac{1}{\sqrt[n]{\rho_m(x)} \cdot \rho_m(x)} = \frac{d \cdot n}{s \cdot \rho_u(x)}. \tag{5.7}$$

Thus, for $z \stackrel{\text{def}}{=} \dfrac{1}{\sqrt[n]{\rho_m(x)}}$, we get an equation

$$p'(z) \cdot z^{n+1} = \frac{d \cdot n}{s \cdot \rho_u(x)}. \tag{5.8}$$

Thus, if we denote by i the function which is inverse to $p'(z) \cdot z^{n+1}$, we get, for z, an explicit formula

$$z = i\left(\frac{d \cdot n}{s \cdot \rho_u(x)}\right) \tag{5.9}$$

Once we know $z = \dfrac{1}{\sqrt[n]{\rho_m(x)}}$, we can the reconstruct the desired density $\rho_m(x)$ as $\rho_m(x) = \dfrac{1}{z^n}$, i.e., as

$$\rho_m(x) = \frac{1}{\left(i\left(\dfrac{d \cdot n}{s \cdot \rho_u(x)}\right)\right)^n}. \tag{5.10}$$

So, we arrive at the following solution to our original problem.

Solution. Let us form an auxiliary function $p'(z) \cdot z^{n+1}$, where $p'(z)$ denotes a derivative, and then form an inverse function $i(z)$ to this auxiliary function. In other words, we define $i(z)$ in such a way that $i(p'(z) \cdot z^{n+1}) = z$ for all z. Then, the optimal distribution $\rho_m(x)$ of designs can be described by the formula (5.10).

Comment. Similar arguments are used to select optimal sensor placements (Section 2.1) and in the optimal setting of cloud computing (Section 3.1).

Chapter 6

Future Work

Main theoretical activity. Recently, a seminal book appeared [80], a book that describes the successful results of using simulations and models to enhance knowledge propagation. This book describes many well-justified results, and it also describes many interesting empirical observations and empirical dependencies for which no well-established theoretical explanations are available yet.

Our plan is to look into these observations and results – as described in this book and as described in the corresponding papers – and try to see if some of them can be theoretically explained.

Auxiliary theoretical activity. The book [80] covers many aspects of knowledge propagation, but there are some aspects – with which I am familiar from my pedagogical experience – which this book does not cover.

Specifically, the book views knowledge propagation from the viewpoint of mathematical optimization, where we try to find the best values of the parameters of the corresponding knowledge propagation process. In this model, all the decisions are centralized.

Educational practice, however, shows that in many cases, efficiency can be drastically improved by decentralization, when instead of all the decisions made in a centralized manner, we allow teachers – and even students themselves – select between several different ways of propagating knowledge.

There have been several empirical studies of this phenomenon; see, e.g., [54, 81, 85]. We plan to analyze this data and see if we can come up with a theoretical explanation for the known empirical results – explanation which can help decide how much autonomy leads to

the best learning and best knowledge propagation.

Practical applications. Of course, while theoretical research is interesting, the ultimate goal of the theoretical research is to enhance actual knowledge propagation. As part of our research, we have already developed some practical recommendations.

We plan to test these recommendations on the actual processes of teaching and knowledge propagation. In particular, we plan to test these recommendations on data acquisition, processing, and propagation related to cyberinfrastructure. These applications are what motivated our research, so we hope that our recommendations will be useful for cyberinfrastructure-related applications.

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Curriculum Vitae

Leonardo Octavio Lerma was born in Parral, Chihuahua, México in November 6, 1965. He obtained his Bachelor's degree in Industrial Engineering from the Chihuahua Institute of Technology in 1988.

Between 1988 and 1992, he attended the University of Texas at El Paso, where he obtained a Master's degree in Electrical Engineering. Upon graduation and until 1996, he worked first as a manufacturing and product engineer, and later as an engineering team leader for Delphi Automotive Systems, in Ciudad Júarez, México.

During this time, he obtained his residency in the United States and was hired as an Electrician apprentice with Funk and Company. A year later, he was recruited by a subsidiary of Yazaki Northamerica where he worked as a manufacturing and process engineer. While working for Yazaki, he obtained an MBA from the University of Phoenix at Santa Teresa in Management of Technology.

In 2001, he started a small company specialized in machine design and systems integration. The company operated until 2010 in El Paso, Texas. During the Summer of 2010, he obtained an alternative teacher certification to pursue his passion for teaching and, on a part-time basis, pursue a doctoral program at the University of Texas at El Paso in Computational Science.

He currently teaches science courses at Coronado High School. His long term interests are to restart a technical company or work as an investigator/researcher at a National Laboratory.

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