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# How to Define “and”- and “or”-Operations for Intuitionistic and Picture Fuzzy Sets

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## Abstract

The traditional fuzzy logic does not distinguish between the cases when we know nothing about a statement  $S$  and the cases when we have equally convincing arguments for  $S$  and for its negation  $\neg S$ : in both cases, we assign the degree 0.5 to such a statement  $S$ . This distinction is provided by *intuitionistic fuzzy logic*, when to describe our degree of confidence in a statement  $S$ , we use two numbers  $a_+$  and  $a_-$  that characterize our degree of confidence in  $S$  and in  $\neg S$ . An even more detailed distinction is provided by *picture fuzzy logic*, in which we differentiate between cases when we are still trying to understand the truth value and cases when we are no longer interested. The question is how to extend “and”- and “or”-operations to these more general logics. In this paper, we provide a general idea for such extension, an idea that explain several extensions that have been proposed and successfully used.

## 1 Formulation of the Problem

**Traditional fuzzy logic: brief reminder.** In the traditional fuzzy logic (see, e.g., [2, 4, 5, 6, 7, 8]), we describe our degree of confidence in a statement  $A$  by a number  $a \in [0, 1]$ , so that:

- 0 means no confidence,
- 1 means full confidence, and
- intermediate values describe intermediate degrees of confidence.

In practice, there is often a need to estimate the degree of confidence in composite statements like  $A \& B$  and  $A \vee B$ . There are many such statements, so it is not feasible to ask the experts about all of them. Instead, we must

estimate our degree of confidence in  $A \& B$  and  $A \vee B$  based on our degrees of confidence  $a$  and  $b$  in the statements  $A$  and  $B$ . These estimates  $f_{\&}(a, b)$  and  $f_{\vee}(a, b)$  are known as “and”-operation (*t-norm*) and “or”-operation (*t-conorm*).

- The most widely used operations are  $f_{\&}(a, b) = \min(a, b)$  and  $f_{\vee}(a, b) = \max(a, b)$ .
- Another very popular choice is  $f_{\&}(a, b) = a \cdot b$  and  $f_{\vee}(a, b) = a + b - a \cdot b$ .

**Need for intuitionistic fuzzy logic.** In the traditional fuzzy logic, we do not distinguish between:

- the cases when we know nothing about  $A$  and
- the cases when we have equally strong arguments for and against  $A$ .

In both types of cases, we assign  $a = 0.5$ . To make this distinction, we can use two degrees:

- the degree  $a_+$  of confidence in  $A$  and
- the degree of confidence  $a_-$  in  $\neg A$ ,

the degrees for which  $a_+ + a_- \leq 1$ . In this *intuitionistic fuzzy* approach (see, e.g., [1]):

- in the first case, we have  $a_+ = a_- = 0$ , and
- in the second case, we have  $a_+ = a_- = 0.5$ .

To define “and”- and “or”-operations to intuitionistic fuzzy sets, we can find  $f(a, b)$  for which  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$  always imply  $f_{\&}(a_+, b_+) + f(a_-, b_-) \leq 1$ .

**Picture fuzzy sets.** If  $a_+ + a_- < 1$ , this means that we do not have enough evidence for or against  $A$ .

- This may mean that we are still trying.
- This may mean that we are not interested in  $A$  at all.
- This may also mean that we are interested to some degree  $a_0$ , for which  $a_+ + a_- + a_0 \leq 1$ .

How do we define “and”- and “or”-operations on such *picture sets* (see, e.g., [3])? Again, a similar idea is to find a function  $f_0(a, b)$  for which  $a_+ + a_- + a_0 \leq 1$  and  $b_+ + b_- + b_0 \leq 1$  always imply  $f_{\&}(a_+, b_+) + f(a_-, b_-) + f_0(a_0, b_0) \leq 1$ .

**Problem.** What operations should we define? In this paper, we use the above idea to describe possible “and”- and “or”-operations for intuitionistic and picture sets. These turn out to be the same operations that have been shown to be practically successful – but now we have a theoretical explanation for these heuristic operations.

## 2 Case of Intuitionistic Fuzzy Sets

**Definition 2.1.** Let  $f_{\&}(a, b)$  be an “and”-operation. We say that  $f(a, b)$  is an intuitionistic operation corresponding to  $f_{\&}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , then

$$f_{\&}(a_+, b_+) + f(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_-, b_-)$ , there exist values  $a_+$  and  $b_+$  for which  $a_+ + a_- \leq 1$ ,  $b_+ + b_- \leq 1$ , and

$$f_{\&}(a_+, b_+) + f(a_-, b_-) = 1.$$

**Proposition 2.1.** For each “and”-operation  $f_{\&}(a, b)$ , the corresponding intuitionistic operation has the form

$$f(a, b) = 1 - f_{\&}(1 - a, 1 - b).$$

**Corollary 2.1** For each “and”-operation ( $t$ -norm)  $f_{\&}(a, b)$ , the corresponding intuitionistic operation is an “or”-operation ( $t$ -conorm).

**Corollary 2.2.** For  $f_{\&}(a, b) = \min(a, b)$ , the corresponding intuitionistic operation is  $f(a, b) = \max(a, b)$ .

**Corollary 2.3.** For  $f_{\&}(a, b) = a \cdot b$ , the corresponding intuitionistic operation is  $f(a, b) = a + b - a \cdot b$ .

**Definition 2.2.** Let  $f_{\vee}(a, b)$  be an “or”-operation. We say that  $f(a, b)$  is an intuitionistic operation corresponding to  $f_{\vee}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_- \leq 1$  and  $b_+ + b_0 \leq 1$ , then

$$f_{\vee}(a_+, b_+) + f(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_0, b_0)$ , there exist values  $a_+$ ,  $b_+$ ,  $a_-$ , and  $b_-$  for which  $a_+ + a_- \leq 1$ ,  $b_+ + b_0 \leq 1$ , and

$$f_{\vee}(a_+, b_+) + f(a_-, b_-) = 1.$$

**Proposition 2.2.** For each “or”-operation  $f_{\vee}(a, b)$ , the corresponding intuitionistic operation has the form

$$f(a, b) = 1 - f_{\vee}(1 - a, 1 - b).$$

**Corollary 2.4.** For each “or”-operation ( $t$ -conorm)  $f_{\vee}(a, b)$ , the corresponding intuitionistic operation is an “and”-operation ( $t$ -norm).

**Corollary 2.5.** For  $f_{\vee}(a, b) = \max(a, b)$ , the corresponding intuitionistic operation is  $f(a, b) = \min(a, b)$ .

**Corollary 2.6.** For  $f_{\vee}(a, b) = a + b - a \cdot b$ , the corresponding intuitionistic operation is  $f(a, b) = a \cdot b$ .

### 3 Case of Picture Fuzzy Sets

*Comment.* Based on the results of Section 2, we conclude that in intuitionistic fuzzy logic:

- if we start with an “and”-operation, we end up with an “or”-operation, and
- vice versa, if we start with an “or”-operation, we end up with an “and”-operation.

For each “and”-operation, we have a very specific “or”-operation and vice versa. However, for the purpose of generality, it makes sense to also consider the case when instead of a related pair of “and”- and “or”-operations, we have a general pair of such operations, with the only condition that whenever  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , then we have

$$f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-) \leq 1.$$

The dual “and”- and “or”-operations – as described by Propositions 2.1 and 2.2 – always satisfy this condition.

**Definition 3.1.** We say that an “and”-operation  $f_{\&}(a, b)$  and an “or”-operation are compatible if whenever  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , we have

$$f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-) \leq 1.$$

**Definition 3.2.** Let  $f_{\&}(a, b)$  be an “and”-operation, and let  $f_{\vee}(a, b)$  be a compatible “or”-operation. We say that  $f_0(a, b)$  is a picture “and”-operation corresponding to  $f_{\&}(a, b)$  and  $f_{\vee}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_0 + a_- \leq 1$  and  $b_+ + b_- + b_0 \leq 1$ , then

$$f_{\&}(a_+, b_+) + f_0(a_0, b_0) + f_{\vee}(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_0, b_0)$ , there exists values  $a_+$ ,  $b_+$ ,  $a_-$ , and  $b_-$  for which  $a_+ + a_0 + a_- \leq 1$ ,  $b_+ + b_- + b_0 \leq 1$ , and

$$f_{\&}(a_+, b_+) + f_0(a_0, b_0) + f_{\vee}(a_-, b_-) = 1.$$

**Proposition 3.1.** For each “and”-operation  $f_{\&}(a, b)$  and each compatible “or”-operation  $f_{\vee}(a, b)$ , the corresponding picture “and”-operation has the form

$$f_0(a, b) = 1 - \max_{a_+ \leq 1-a, b_+ \leq 1-b} (f_{\&}(a_+, b_+) + f_{\vee}(1-a-a_+, 1-b-b_+)).$$

**Proposition 3.2.** For  $f_{\&}(a, b) = \min(a, b)$  and  $f_{\vee}(a, b) = \max(a, b)$ , the corresponding picture “and”-operation is  $f_0(a, b) = \min(a, b)$ .

**Proposition 3.3.** For  $f_{\&}(a, b) = a \cdot b$  and  $f_{\vee}(a, b) = a+b-a \cdot b$ , the corresponding picture “and”-operation is  $f_0(a, b) = a \cdot b$ .

**Definition 3.3.** Let  $f_{\vee}(a, b)$  be an “or”-operation, and let  $f_{\&}(a, b)$  be a compatible “and”-operation. We say that  $f_0(a, b)$  is a picture “or”-operation corresponding to  $f_{\vee}(a, b)$  and  $f_{\&}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_0 + a_- \leq 1$  and  $b_+ + b_- + b_0 \leq 1$ , then

$$f_{\vee}(a_+, b_+) + f_0(a_0, b_0) + f_{\&}(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_0, b_0)$ , there exists values  $a_+$ ,  $b_+$ ,  $a_-$ , and  $b_-$  for which  $a_+ + a_0 + a_- \leq 1$ ,  $b_+ + b_- + b_0 \leq 1$ , and

$$f_{\vee}(a_+, b_+) + f_0(a_0, b_0) + f_{\&}(a_-, b_-) = 1.$$

**Proposition 3.4.** For each “and”-operation  $f_{\&}(a, b)$  and each compatible “or”-operation  $f_{\vee}(a, b)$ , the corresponding picture “or”-operation has the form

$$f_0(a, b) = 1 - \max_{a_+ \leq 1-a, b_+ \leq 1-b} (f_{\vee}(a_+, b_+) + f_{\&}(1-a-a_+, 1-b-b_+)).$$

The Propositions 3.1 and 3.4 leads to a somewhat unexpected conclusion:

**Corollary 3.1.** For each pair of compatible “and”- and “or”-operations, the corresponding picture “or”-operation is the same as the corresponding picture “and”-operation.

## 4 Possible Extensions: Idea

**Fuzzy logic  $\rightarrow$  intuitionistic logic  $\rightarrow$  picture logic: a brief reminder.** We started with the usual fuzzy logic, in which, for each statement  $A$  with degree of confidence  $a_+$ , the degree of confidence in its negation  $\neg A$  is equal to  $a_- = 1 - a_+$ , i.e., for which  $a_+ + a_- = 1$ .

We then took into account that we may be uncertainty about both  $A$  and  $\neg A$ , i.e., we may have  $a_+ + a_- \leq 1$ . This is the case of intuitionistic fuzzy logic. In intuitionistic fuzzy logic, we can consider the remaining degree

$$a_0 \stackrel{\text{def}}{=} 1 - (a_+ + a_-)$$

that describes our uncertainty. In this case, we have  $a_+ + a_0 + a_- = 1$ .

The next natural step was to take into account that not all the difference  $1 - (a_+ + a_-)$  may be due to uncertainty, and that we may have situations in which, for the corresponding uncertainty degree  $a_0$ , we have  $a_0 < 1 - (a_+ + a_-)$ . In this case, in general, we have  $a_0 \leq 1 - (a_+ + a_-)$ . This is the case of picture fuzzy logic. In picture fuzzy logic, we can consider the remaining degree  $a_r \stackrel{\text{def}}{=} (1 - (a_+ + a_-)) - a_0$ . In this case, we have  $a_+ + a_0 + a_- + a_r = 1$ .

**Natural next steps.** It seems reasonable to apply the same idea again, and consider the cases in which  $a_+ + a_0 + a_- + a_r \leq 1$ . For this new extension of fuzzy logic, we can use the same idea as above to extend operations on  $a_+$ ,  $a_-$ , and  $a_0$  components to the operations of the remainders  $a_r$ . For example, for “and”-operations, we require that every time we have  $a_+ + a_0 + a_- + a_r \leq 1$  and  $b_+ + b_0 + b_- + b_r \leq 1$ , we should have

$$f_{\vee}(a_+, b_+) + f_0(a_0, b_0) + f_{\&}(a_-, b_-) + f_r(a_r, b_r) \leq 1,$$

and for each pair  $(a_r, b_r)$ , we should have at least one case when the above inequality becomes equality.

From this requirement, we can also extract an explicit formula for  $f_r(a, b)$ : for example, for  $f_{\&}(a, b) = \min(a, b)$ ,  $f_{\vee}(a, b) = \max(a, b)$ , and  $f_0(a, b) = \min(a, b)$ , we get  $f_r(a, b) = \min(a, b)$ .

For such extended logic, we can define a new remainder

$$a_n \stackrel{\text{def}}{=} 1 - (a_+ + a_0 + a_- + a_r)$$

for which  $a_+ + a_0 + a_- + a_r = 1$ . We can now again apply the same idea and consider cases for which  $a_+ + a_0 + a_- + a_r \leq 1$ , etc. Our approach enables us to define “and”- and “or”-operations for all such extensions.

## 5 Proofs

**Proof of Proposition 2.1.** The first condition implies that

$$f(a_-, b_-) \leq 1 - f_{\&}(a_+, b_+)$$

for all  $a_+$  and  $b_+$ , and  $b_-$  for which  $a_+ \leq 1 - a_-$  and  $b_+ \leq 1 - b_-$ . This means that  $f(a_-, b_-)$  should be smaller than or equal to the smallest of the values  $1 - f_{\&}(a_+, b_+)$  for such  $a_+$  and  $b_+$ , i.e., equivalently, that is smaller than or equal to 1 minus the largest possible value of  $f_{\&}(a_+, b_+)$ .

Since “and”-operation is non-strictly increasing in both variables, its largest value is attained when both  $a_+$  and  $b_+$  attain their largest values  $1 - a_-$  and  $1 - b_-$ . Thus, the first condition is equivalent to

$$f(a_-, b_-) \leq 1 - f_{\&}(1 - a_-, 1 - b_-).$$

The second part of the definition implies that we have equality:

$$f(a_-, b_-) = 1 - \max_{a_+ \leq 1 - a_-, b_+ \leq 1 - b_-} f_{\&}(a_+, b_+).$$

The “and”-operation is monotonic in both variables, so, for the given values of  $a_-$  and  $b_-$ , the largest value of  $f_{\&}(a_-, b_-)$  is attained when  $a_-$  and  $b_-$  attains the largest possible values  $a_- = 1 - a_0 - a_+$  and  $b_- = 1 - b_0 - b_+$ . Thus, we get the desired formula. The proposition is proven.

**Proof of Proposition 2.2** is similar.

**Proof of Proposition 3.1.** The first condition implies that

$$f_0(a_0, b_0) \leq 1 - (f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-))$$

for all  $a_+$ ,  $a_-$ ,  $b_+$ , and  $b_-$  for which  $a_+ + a_0 + a_- \leq 1$  and  $b_+ + b_0 + b_- \leq 1$ . Thus, we can conclude that

$$f_0(a_0, b_0) \leq 1 - \max_{a_+ + a_0 + a_- \leq 1, b_+ + b_0 + b_- \leq 1} (f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-)).$$

The second part of the definition implies that we have equality:

$$f_0(a_0, b_0) = 1 - \max_{a_+ + a_0 + a_- \leq 1, b_+ + b_0 + b_- \leq 1} (f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-)).$$

The “or”-operation is monotonic in both variables, so, for the given values of  $a_+$  and  $b_+$ , the largest value of  $f_{\vee}(a_-, b_-)$  (and thus, of the sum  $f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-)$ ) is attained when  $a_-$  and  $b_-$  attains the largest possible values  $a_- = 1 - a_0 - a_+$  and  $b_- = 1 - b_0 - b_+$ . Thus, we get the desired formula. The proposition is proven.

**Proof of Proposition 3.2.** For  $a_+ b_+ = 0$ , we get

$$\begin{aligned} f_{\&}(a_+, b_+) + f_{\vee}(1 - a - a_+, 1 - b - b_-) &= f_{\&}(0, 0) + f_{\vee}(1 - a, 1 - b) = \\ &= \min(0, 0) + \max(1 - a, 1 - b). \end{aligned}$$

For all other values  $a_+ \leq 1 - a$  and  $b_+ \leq 1 - b$ , we have

$$\begin{aligned} v &\stackrel{\text{def}}{=} f_{\&}(a_+, b_+) + f_{\vee}(1 - a - a_+, 1 - b - b_+) = \\ &= \min(a_+, b_+) + \max(1 - a - a_+, 1 - b - b_+). \end{aligned}$$

If we add the same constant  $c$  to two numbers  $p$  and  $q$ , the same number will remain larger, so we get  $\max(p + c, q + c) = \max(p, q)$ . In particular, we get

$$v = \max(1 - a - a_+ + \min(a_+, b_+), 1 - b - b_+ + \min(a_+, b_+)).$$

In the first of the two minimized terms, we can use the fact that  $\min(a_+, b_+) \leq a_+$ , thus

$$1 - a - a_+ + \min(a_+, b_+) \leq 1 - a - a_+ + a_+ = 1 - a.$$

In the second term, we can similarly use the fact that  $\min(a_+, b_+) \leq b_+$ , thus

$$1 - b - b_+ + \min(a_+, b_+) \leq 1 - b - b_+ + b_+ = 1 - b.$$



Hence,  $v \leq \max(1 - a, 1 - b)$ . So, the maximum is indeed  $\max(1 - a, 1 - b)$ .

Thus,  $f_0(a, b) = 1 - \max(1 - a, 1 - b)$ . When the subtracted number is the largest, the difference is the smallest, so we get

$$f_0(a, b) = \min(1 - (1 - a), 1 - (a - b)) = \min(a, b).$$

The proposition is proven.

**Proof of Proposition 3.3.** For  $a_+ = b_+ = 0$ , we get

$$\begin{aligned} f_{\&}(a_+, b_+) + f_{\vee}(1 - a - a_+, 1 - b - b_+) &= f_{\&}(0, 0) + f_{\vee}(1 - a, 1 - b) = \\ 0 \cdot 0 + 1 - a + 1 - b - (1 - a) \cdot (1 - b) &= \\ 1 - a + 1 - b - 1 + a + b - a \cdot b &= 1 - a \cdot b. \end{aligned}$$

For all other values  $a_+ \leq 1 - a$  and  $b_+ \leq 1 - b$ , we have

$$\begin{aligned} f_{\&}(a_+, b_+) + f_{\vee}(1 - a - a_+, 1 - b - b_+) &= \\ a_+ \cdot b_+ + 1 - a - a_+ + 1 - b - b_+ - (1 - a - a_+) \cdot (1 - b - b_+) &= \\ a_+ \cdot b_+ + 1 - a - a_+ + 1 - b - b_+ - 1 + a + a_+ + b + b_+ - a \cdot b - a \cdot b_+ - a_+ \cdot b - a_+ \cdot b_+ &= \\ 1 - a \cdot b - a \cdot b_+ - a_+ \cdot b &\leq 1 - a \cdot b. \end{aligned}$$

Thus, the maximum is indeed  $1 - a \cdot b$ , and 1 minus this maximum is simply  $a \cdot b$ . The proposition is proven.

**Proof of Proposition 3.4** is similar to the proof of Proposition 3.1.

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## References

- [1] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer-Verlag, Berlin, Heidelberg, 1999.
- [2] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.
- [3] Bui Cong Cuong and V. Kreinovich, "Picture Fuzzy Sets - a new concept for computational intelligence problems", *Proceedings of the Third World Congress on Information and Communication Technologies WICT'2013*, Hanoi, Vietnam, December 15–18, 2013, pp. 1–6.

- [4] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
- [5] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.
- [6] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.
- [7] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
- [8] L. A. Zadeh, “Fuzzy sets”, *Information and Control*, 1965, Vol. 8, pp. 338–353.