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Strong Coupled Quark Matter And Drawbacks Of Its BCS-BEC Crossover

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STRONG COUPLED QUARK MATTER AND DRAWBACKS
OF ITS BCS-BEC CROSSOVER

ISRAEL PORTILLO VAZQUEZ

Computational Science Program

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STRONG COUPLED QUARK MATTER AND DRAWBACKS
OF ITS BCS-BEC CROSSOVER

by

ISRAEL PORTILLO VAZQUEZ

THESIS
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Abstract

It was found many years ago, that the ground state wave functions of the Bardeen-Cooper-Schrieffer (BCS) superconductivity in a degenerate Fermi gas and the Bose-Einstein condensation (BEC) of composite molecules are essentially the same, and those two states can be smoothly connected through a crossover. This BCS-BEC crossover has been recently extended to relativistic Fermi System. This thesis is dedicated to exploring the possible crossover from BEC to BCS phase on strong-interacting matter and its implications for the system Equation of State. The study will be concentrated on different phases of dense quark-matter for vanishing temperatures and densities beyond hadronic matter. Under dense conditions a deconfined quark phase is expected as the hadrons are dissociated.
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Chapter 1

Introduction

In the last century, the laws that govern the fundamental particles have been written successfully in the language of quantum field theory (QFT). The electro-weak unification has extended and generalized the quantum theory of electromagnetism (QED) to include the weak nuclear forces responsible for radioactivity. In a similar way, quantum chromodynamics (QCD), the theory of quarks and gluons, proves an analogous theory of the strong nuclear forces. Together they summarize our present knowledge of the basic constituent of matter in The Standard Model of Particles [1].

QFT is equally employed on the study of many-particle systems. The resulting phenomenology of a collective motion of many-particles is not always obtained directly from the natural laws that govern one or a few particles. The correlations of many particles can yield new physical principles and physical laws. The behavior of matter composed of atoms and molecules interacting in the regime of QED is the subject of study of condensed matter physics. This branch of physics has allowed an understanding of almost all forms of matter around us. On one hand, band theory has described semiconductors, metals, magnets, superconductors, and superfluids; on the other, Landau’s symmetry-breaking theory in a statistical context allows an understanding of the different states of matter, and the mechanism that leads to the transitions between them.

However, when matter is under extreme conditions, the atoms and molecules disassociate and new degrees of freedom are exited. The strong nuclear force begins to play an important role on the structure of matter. At high temperatures, the nuclei’s atoms melt. The quarks and gluons confined inside are liberated and matter becomes a mixture of quarks and gluons, a quark-gluon plasma. Similarly, at high densities, the nucleons over-
lap, their constituent quarks are no longer connected to one nucleon and can move around more freely, forming a gigantic nucleus made of exotic particles. At any of those extremes, color and flavors degrees of freedom emerge, giving rise to new and more complex phases of matter. This strong-interacting matter is now in the regime of QCD.

Many phenomena of QCD are found to be analogous to the ones already observed in traditional condensed matter, while many others are new and completely different. The investigation of the possible phases that strong-interacting matter may have is a topic of intensive research. The microscopic theory of QCD is described by a non-Abelian gauge theory, written in terms of quark fields with color charges. In principle, this theory must lead to the description of all these strong interacting phases. However, their application to a many-body system has not been possible except under highly extreme conditions. The non-Abelian nature of the theory leads to a coupling that is strong, but its strength decreases with the distance. For asymptotically short distances, the coupling strength vanishes [2], making the employment of perturbative techniques possible [3]. In those regimes, matter will consist of almost-free quarks and gluons.

Numerical Lattice QCD calculations had successfully explored the perturbative regime of high temperatures and very low densities [4]. Their results have been experimentally confirmed in heavy-ion colliders. Nevertheless, with extreme high densities and low temperatures, matter consists of a highly degenerate Fermi liquid. Quarks at the Fermi surface have very large momenta and their coupling is asymptotically weak, creating the possibility of approximating their interaction by a one-gluon exchange. The attractive channels of this kind of interaction lead to the formation of diquark condensates [5], a non-perturbative phenomenon essentially equivalent to the Cooper instability of the Bardeen-Cooper-and-Schrieffer theory of superconductivity (BCS). The condensate breaks the SU(3) color gauge symmetry of the ground state, producing a color superconductor; therefore, the subject takes the name of color superconductivity (CS), a non-perforative phenomena where lattice simulation are not applicable since the path integral, discretized on the lattice, can no longer be interpreted statistically at finite densities.
CS has been known for a long time [5]. Perhaps the experimental incredibility of former studies or the lack of practical use has caused it to almost be forgotten. Nevertheless, it was not until the late 1990’s that the interest in CS was rediscovered. On the basis of an effective theory for low energy QCD [6], it was shown that CS may already exist at relatively moderate densities of the order of a few times the nuclear matter density, and might be realized in the interiors of compact stars, altering its properties significatively. Thus, CS has become a central subject for astrophysics to understand the structure and evolution of compact stars, and for theoreticians to interpolate current theories to those new regimes of matter with the ultimate goal of finding a prediction that eventually could be tested on such astrophysical objects.

Over the past decades, a rigorous theoretical analysis of QCD using weak-coupling methods of BCS theory [7] has lead to the conclusion that at sufficiently high density and sufficiently low temperature, the most likely QCD phase will be the so-called color-flavor-locking (CFL) [6], characterized by a spin-0 diquark condensate antisymmetric in both color and flavor. In this state, the density energy is such that one can assume the three quarks up (u), down (d), and strange (s) as massless, creating an approximate chiral symmetry. Nevertheless, at moderate densities the mass of the s quark, which is much greater that the u and d quarks, cannot be neglected. The mismatch between the Fermi momenta produced by the strange quark mass and the constraints imposed by electric and color neutralities [8] break the chiral symmetry. As a consequence of the pairings with mismatched Fermi surfaces, the CFL phase exhibits chromomagnetic instabilities [9].

The phase preceding the CFL in density is still a puzzle. Many possibilities have been suggested [10]. One possible scenario where this instability can be avoided occurs if in the region of moderate-low densities, the strong coupling constant becomes sufficiently high [11]. On the other hand, the increase of the coupling constant strength at low density will reduce the Cooper-pair coherence length, which may reach values in the order of the inter-quark spacing [12]. This scenery will certainly modify the properties of the ground state, and as already found in other physical contexts [13]. This fact strongly suggests the
possibility of a crossover from a color-superconducting BCS dynamics to a Bose-Einstein Condensate (BEC) one \[14\] – \[17\]. Even though the same underlying theory, based on the formation of Cooper pairs, can yield both BEC and BCS, and the symmetry breaking order parameter on the BCS-BEC crossover is the same, the quasiparticle spectra in the two regions are completely different. In the BCS region, where the diquark coupling is relatively weak, the energy spectrum of the excitations has a fermionic nature, while in the strong-coupling region, formed by the BEC molecules, the energy spectrum of the quasiparticles is bosonic.

1.1 Objectives of the Thesis and Outline

This thesis is dedicated to exploring the possible crossover from BEC to BCS (BCS-BEC) phase on strong-interacting matter. The study will be concentrated on different phases of quark-matter for vanishing temperatures and densities beyond hadronic matter. Under these conditions, for high enough densities, the emergence of CS phases is expected as the hadrons are dissociated and quarks form Cooper pairs. Firstly, a simplified pure fermion system characterized by a four-fermion interaction is investigated. The thermodynamical potential at finite temperature and density of this system is calculated, and used to find how the superconducting gap and chemical potential change with the interaction strength at zero temperature. An important feature of this work is the incorporation of diquark-diquark interactions. As it has been pointed out in Ref. \[18\], together with the fact that there exists an attractive channel between quarks that favors the diquark formation, there is, as a corollary, a diquark-diquark repulsion. This repulsion is due to unfavorable cross-channel correlations between quarks belonging to different diquarks. Later, a similar study is realized considering a high-strongly-interacting quark matter on the CFL phase. Then, a more realistic model for the interior of a compact start will be consider, the two-flavors color superconducting phase (2SC). Finally, an important goal of this thesis is to develop a methodology to approach the numerical problems that are commonly faced on the area
of thermal-QFT.

The outline of this thesis is as follows. After this introduction (Chapter 1), Chapter 2 is dedicated to a brief theoretical review of the main concepts related to our study. The description of the phenomena of superconductivity, the BCS-BEC crossover, the importance on the astrophysical contest, the QCD phase diagram, as well as the techniques used on thermal QFT, the QCD Lagrangian, and its effective-low-energy theory, the Nambu-Jona-Lassino (NJL) model are reviewed. In Chapter 3, the simple system with four-fermion interactions is studied, their equation of state (EoS) is obtained taking into account self-consistently the diquark-diquark repulsion. This stage will serve as a model to illustrate, in detail, the procedure addressed in the subsequent two chapters. In Chapter 4, superdense matter in the CFL state is consider. In Chapter 5, a more realists case for Astrophysics, i.e. of matter in the 2SC state subject to electric and color charge neutralities will be studied. Chapter 6 describes the methods used on the numerical analysis of the previous chapters. At the end, Chapter 7 is dedicated to the remarks and conclusions of this study.
Chapter 2

Theoretical Background

2.1 Superconductivity and Cooper pair

Surprising and inspiring are words that well describe the phenomena of superconductivity over the last century. Beginning in 1991, Kamerlingh Onnes discovered Superconductivity when observing that the electrical resistance of various metals disappeared when the temperature was lowered below some critical value [19]. Subsequently, in 1933, Meissner discovered perfect diamagnetism in superconductors [20]. The brothers F. and H. London, who first realized the quantum character of the phenomenon [21], were the first to try to explain it. However, it was not until 1950 that the first theory of superconductivity was developed by Ginzburg and Landau (GL) [22]. Their theory provided a description of the macroscopic properties of superconductors, including their transition between the superconducting and the normal phases. Seven years later, Bardeen, Cooper and Schrieffer created the microscopic theory that bears their name (BCS) [23]. The BCS theory was based on the fundamental theorem [24], which states that a superconducting phase can arise in a low-temperature fermionic system, provided that an arbitrarily small attractive interaction exist between a pair of fermions. The paired fermions will form a so-called Cooper pair. As long as the Cooper pair does not evaporate, due to thermal effects, it will behave as a compound boson and may condensate to the ground state since the pair no longer obeys the exclusion Pauli principle. Therefore, the superconducting state observed by Onnes on cool $^3$He could be explained in terms of charged Cooper pairs, formed by the attraction between a pair of electrons produced by the phonon exchange.

In 1959, Gor’kov realized that the GL theory was equivalent to the BCS theory around
the phase transition point [25]. In one sense, the GL theory was the prototype of the modern effective theories, which served as inspiration for the idea of generating elementary particle masses through the mechanism of dynamical symmetry breaking [26]. This idea was a crucial ingredient of the Standard Model of the elementary particles, where the masses are generated by the formation of the Higgs condensate, much in the same way as superconductivity originates from the presence of a gap. Later, in the 70’s, the BCS mechanism was introduced on the framework of QCD. The asymptotic freedom of the theory and the existence of an attractive channel in the color interaction, allows quarks to form Cooper pairs analogous to the formed on superconductivity by the electrons.

The idea behind superconductivity is that the energy of a fermionic system, at low temperature, decreases with the formation of Cooper pairs. This could be seen analyzing the distribution function of a fermion system \( f(p, T) \), at finite temperature, and density

\[
f(p, T) = \frac{1}{e^{(\epsilon_p - \mu)/T} + 1} \tag{2.1}\]

where \( \mu \) is the chemical potential, \( p \), \( m \), and \( \epsilon_p = \sqrt{p^2 + m^2} \) are the momentum, mass and energy respectively of a free fermion. Taking the limit of \( f(p, T) \) when \( T \to 0 \)

\[
\lim_{T \to 0} f(p, T) = f_F(p) = \theta(\mu - \epsilon_p) \tag{2.2}\]

the system forms a Fermi liquid. States with momenta greater than the Fermi momenta \( p_F = \sqrt{\mu^2 - m^2} \) are empty. In the ground state, the fermions occupy all available quantum states with the lower possible energies, up to the Fermi energy \( \epsilon_F = \mu \). Thus, there is no cost in free energy for adding or subtracting a fermion at the Fermi surface

\[
\Omega_N = \epsilon_p - \mu N \\
\Omega_{N \pm 1} = (\epsilon_p \pm \epsilon_F) - \mu (N \pm 1) = \Omega_N \tag{2.3}\]

causin an enormous degeneracy at the Fermi surface. On the other hand, if two fermions bounded with energy \( \epsilon_B \) are added to the system,

\[
\Omega_{N+2} = (\epsilon_p + 2\epsilon_F - \epsilon_B) - \mu (N + 2) = \Omega_N - \epsilon_B \tag{2.4}\]
the free energy decreases. In fact, the system will be more stable if bonded pairs are added to the Fermi surface. Hence, any present attraction would lead to a gain in free energy via pairing, causing an instability of the Fermi surface and condensation will take place. The formation of these Cooper pairs leads to a new ground state with a non-zero energy gap $\Delta$ in the quasiparticle spectrum $E_k$, typically of the form

$$E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta^2}.$$  \hspace{1cm} (2.5)

In summary, an attractive interaction channel on a degenerate fermion system causes the formation of Cooper pairs, and the formed Cooper pairs causes a gap in the single-particle excitation spectrum. This is due to the binding energy associated with a Cooper pair, since an interaction with an energy lower than this binding energy will not cause a single-particle excitation. For low temperatures, the Cooper pairs will resist temperature fluctuations producing a condensation that will modify the ground state of the system.

### 2.2 BCS-BEC Crossover

After many years of its discovery, the study of superconducting phases has become an increasingly active area of research occurring in physics on all scales, from condense matter to nuclear elementary particles and astrophysics. Recent research on the topic deals with systems, such as excitons in semiconductors, spin-polarized hydrogen, laser-cooled atoms, high temperature superconductors, and subatomic matter [27]. Bosonic condensates with non-vanishing expectation values of quark operators are believed to be fundamental features of the vacuum and the structure of elementary particles, such as the nucleon, underlying the spontaneous breaking of the chiral symmetry of the strong interaction [28]. The Higgs boson, which gives rise to the masses of elementary particles, has been suggested as a condensation of top (t) and antitop ($\bar{t}$) quarks [29]. Therefore, in high energy physics, the normal state of the vacuum contains quark condensates. On the other hand, mesonic and baryonic condensation may be an important feature in compact stars. The condensate may
form superfluid and semiconducting phases, which will influence the EoS of the stars and consequently shapes their structure.

The same underlying theory, based on the formation of Cooper pairs, can yield both BEC and BCS, in different limits: BEC in the limit of particle-particle correlation length very short compared to the average inter-particle spacing, and BCS in the opposite limit. The distance of the Cooper pairs on the BCS is much larger than the average separation of the particles. On the other hand, BEC correlation length is very short, forming bound states with bosonic degrees of freedom, which generally, but not always, are composites originating from underlying fermionic degrees of freedom, as in BCS. The phenomenon of BEC is a phase transition in which a macroscopic number of particles all go into the same quantum ground state.

Both theories, the BCS and the BEC are described by the same order parameter $\Delta$. However, there is a key difference between them: the quasiparticle spectra in the two regions are completely different. In the BCS region, where the diquark coupling is relatively weak, the energy spectrum of the excitations has a fermionic nature, while in the strong-coupling region, formed by the BEC molecules, the energy spectrum of the quasiparticles is bosonic. The fermionic spectrum is known to have a minimum at $E = 0$. On the other hand, the bosonic spectrum increases with the momentum $k$.

**2.3 QCD Phase Diagram**

The QCD phase diagram is a mapping of the states of strong-interacting-matter as a function of temperature $T$ and baryon number chemical potential $\mu$. Each phase is characterized by a certain symmetry and a certain order parameter, which characterized the ground state. The structure of the phases depends on the parameters of the model in which it is constructed. It is particularly influenced by the coupling constant and the value of the s quark mass. Fig. 2.1 shows a well known QCD phase diagram with a simple sketch of the global phases of QCD and the regions that are currently being explored [30].
It is well known that at the extremes of the axes, the theory is asymptotically free. At high energies, hadrons dissociate, allowing quarks and gluons to deconfine forming a quark-gluon plasma (QGP). The regime along the temperature axes, at zero baryon density, has been analyzed using lattice QCD calculations. Experimentally, this regime has been the subject of ultra-relativistic heavy-ion colliders. The results of both experiments reveals a crossover transition from QGP to a phase of hadrons, at a temperature of approximately 170 MeV [4]. On the other hand, the measure of the physical values of the quark masses, as increasing the baryon density, implies that the transition from hadronic matter to superdense matter is a first order transition [31]. Consequently, there will be a critical point that delimits the crossover from the phase transition. The location of this critical point has been addressed by ultra-relativistic heavy-ion experiments, performed at CERN and BNL.

The regime along the baryon density axis, on the other hand, at low temperatures, corresponds to CS, specifically the CFL phase. This is a robust and symmetric phase, consisting of three flavors, that is invariant under SU(3) rotational in color and flavor. The energy scale is such that the three quark masses are neglected. All nine quarks, three flavors
and three colors, together form a simple and elegant diquark condensate that breaks the SU(3) color symmetry.

Moving down the chemical potential, at some critical value of the s mass, the CFL phase becomes unstable. The mismatch of the pairing Fermi surfaces of the s quark, with the u and d quarks, produce chromomagnetic instabilities [9]. The next phase down in density is uncertain albeit several phases have been proposed. The phases could be classified according with the paring ansatz [32]

\[
\langle \phi^\alpha_a C \gamma_5 \phi^\beta_b \rangle \sim \Delta_1 \epsilon^{\alpha\beta 1} \epsilon_{ab1} + \Delta_2 \epsilon^{\alpha\beta 2} \epsilon_{ab2} + \Delta_3 \epsilon^{\alpha\beta 3} \epsilon_{ab3}
\]

where \( \phi^\alpha_a \) is a quark of color \( \alpha = \{r, g, b\} \) and flavor \( a = \{u, d, s\} \), and \( C = i \gamma_0 \gamma_2 \) is the charge conjugation matrix. The ansatz on (2.6) implies that the \((rd, gu), (bu, rs), \) and \((gs, bd) \) quarks pair with gap parameter \( \Delta_1, \Delta_2, \) and \( \Delta_3 \) respectively, while \((ru, gd, bs) \) quarks pair each other with all three gaps. Notice the Cooper pairs are symmetric in color and flavor on all phases.

A rich spectrum of phases can be generated depending on the coupling interaction strength. It has been proposed a gapless CFL [32], a metallic CFL, a uSC, a dSC, a sSC [33], as well as the standard 2SC phase [34] and a gapless 2SC [35]. This proposed phases with its corresponding paring are summarized on Table 2.1 (NQ means Normal Quark matter).

Some parameters concerning the superconducting phases, like the quark mass and their coupling, are not accessible to observation, and their values rely on educated guesses. A possible scenario, where the instabilities created by the s-quark mass can be avoided, occurs if in the region of moderate-low densities the strong coupling constant becomes sufficiently high [10, 11]. However, the increase of the coupling constant can drastically modify the phase diagram. This fact is shown in Fig. 2.2 for strong coupling \( (G_D \simeq G_s, \) where \( G_D \) and \( G_s \) denoting the diquark and quark-antiquark coupling constant respectively). There, the CFL phase has moved to chemical potential values of 415 MeV and higher, making possible the existence of matter in CFL state on compacts stars, where the the baryon
Table 2.1: Gaps on Different phases of QCD.

<table>
<thead>
<tr>
<th>Phase</th>
<th>Gaps</th>
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<tbody>
<tr>
<td>NQ</td>
<td>$\Delta_1 = \Delta_2 = \Delta_3 = 0$</td>
</tr>
<tr>
<td>2SC</td>
<td>$\Delta_1 = \Delta_2 = 0$ and $\Delta_3 \neq 0$</td>
</tr>
<tr>
<td>2SCus</td>
<td>$\Delta_1 = \Delta_3 = 0$ and $\Delta_2 \neq 0$</td>
</tr>
<tr>
<td>2SCds</td>
<td>$\Delta_2 = \Delta_3 = 0$ and $\Delta_1 \neq 0$</td>
</tr>
<tr>
<td>uSC</td>
<td>$\Delta_2 \neq 0$ $\Delta_3 \neq 0$ and $\Delta_1 \neq 0$</td>
</tr>
<tr>
<td>dSC</td>
<td>$\Delta_1 \neq 0$ $\Delta_3 \neq 0$ and $\Delta_2 \neq 0$</td>
</tr>
<tr>
<td>sSC</td>
<td>$\Delta_1 \neq 0$ $\Delta_2 \neq 0$ and $\Delta_3 \neq 0$</td>
</tr>
<tr>
<td>CFL</td>
<td>$\Delta_1 \neq 0$ $\Delta_2 \neq 0$ and $\Delta_3 \neq 0$</td>
</tr>
</tbody>
</table>

density may reach up to 500 MeV. At zero temperature, the system directly enters the 2SC phase after the chiral phase transition. The higher diquark coupling strength, which leads to a higher gain in energy for the formation of a Cooper pair and enables pairing even for larger differences between the Fermi momenta of u and d quarks. The 2SC phase also spans a larger region in the temperature direction. At low temperature and low chemical potential the diagram is dominated by the normal phase in which the chiral symmetry is broken ($\chi_{SB}$), and the quarks have relatively large constituent masses [36].

The scenario represented in Fig. 2.2 is the one explored in this thesis. We concentrate on the states of quark-matter for vanishing temperatures and densities beyond hadronic matter: CFL and 2SC. Those regimes are not accessible by lattice QCD calculations nor by direct experimentation at present. Moreover, the possible connections between CS and the observation of compact stars are not yet clear. Nevertheless, a well understanding of the structure and properties of the phases of QCD may provide a clue to link CS with an observable state.
2.4 Astrophysics of Compact Objects

Compact stars, play a unique role in physics and astrophysics. They are the only candidates to contain matter under extreme physical conditions in the Universe. White dwarfs are so greatly compressed that their electrons form a relativistic degenerate gas, neutron stars go even further forming superdense nuclear matter, and black holes simply go outside the domains of general relativity as well as of the well established Standard Model.

Of particular interest, in the search for the QCD phase diagram, are neutron stars, the remanent of a type-II supernova explosion. Matter inside neutron stars are compressed so much, by the gravitational forces, that its density reaches values several times the nuclear density, and their gravitational energy values as high as 10 times its binding energy [37]. In such condition, a deconfined phase of quark matter, made of stable u, d, and possible s quarks, may exist. It is expected that, at zero pressure, the strange quark matter will have a lower energy per baryon than ordinary matter, which make the strange matter the most
stable substance in nature [38].

Compact stars participate in many astronomical phenomena. The investigation of exotic states of matter, and the possible phases like CS in compact stars, are examples of the most exiting topic in the area of high energy physics. A wide range of knowledge on general relativity, astrophysics, particle physics, and high energy physics is necessarily to describe compact stars’ interiors. The EoS of neutron stars is complex and model dependent. Practically, theories explaining the structure and evolution of compact stars are based on extrapolations of reliable physical theories of the structure of matter tested in laboratory. Density plays a key role on the construction of their EoS, which will yield the internal structure of the star. A difficulty to construct a theory on neutron stars’ interiors is to predict the s quark mass. While not accessible in the laboratory, it should lies somewhere between its vacuum constituent mass of more than 500 MeV and its current quark mass of around 100 MeV. Neutrality of electric and color charge, beta equilibrium reactions, hydrostatic equilibrium, supplemented by a microscopic model of many-body nucleon interaction are one of the restriction that can be imposed on their EoS. Nevertheless, giving the combination of high densities and relatively low temperatures, it is natural to imply that a superconducting state can form, and resist the evaporation due to thermal effects, at the moderately high density that the inner core of neutron stars can reach.

The study of neutron star leads to a better understanding of QCD phenomenology including the phase of nuclear matter that we are living, and their observation provides a unique opportunity to improve and test these theories.

### 2.5 Basic Notation

The metric used is $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(+, -, -, -)$. The $\gamma$-matrices are given on the Dirac’s representation

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

(2.7)
which satisfy the clifford algebra \( \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \). With the Pauli matrices \( \vec{\sigma} = \{\sigma^1, \sigma^2, \sigma^3\} \)

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (2.8)

The contravariant four-vector is denoted by

\[
x^\mu = \{x^0, x^1, x^3, x^4\}
\] (2.9)

and the covariant four-vector is

\[
x_\mu = \eta_{\mu\nu}x^\nu = \{x^0, -x^1, -x^3, -x^4\} = \{x_0, x_1, x_3, x_4\}
\] (2.10)

The covariant and contravariant derivatives are given respectively as

\[
\partial_\mu = \begin{bmatrix} \frac{\partial}{\partial x^0}, & \frac{\partial}{\partial x^1}, & \frac{\partial}{\partial x^2}, & \frac{\partial}{\partial x^3} \end{bmatrix} \\
\partial^\mu = \begin{bmatrix} \frac{\partial}{\partial x_0}, & \frac{\partial}{\partial x_1}, & \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_3} \end{bmatrix}
\] (2.11)

## 2.6 Introduction to Thermal Quantum Field Theory

The relativist system of particles, considered in this thesis, are moving in a high-density-medium, which has been modified by different condensates, but are in thermal equilibrium. Hence, its stable state must take into consideration the effects that this medium produces. This scenario is well described in the framework of thermal-QFT. This section is dedicated to the exposition of the essential elements of thermal-QFT. Under these bases, the microscopic theory of superconducting matter will be described in terms of their macroscopic variables.

In equilibrium statistical mechanics, a dynamical system is described by an ensemble. One usually encounters three types of ensemble. The microcanonical ensemble, used to describe an isolated system with a fixed energy \( (E) \), particle number \( (N) \), and volume
(V). The canonical ensemble, used to describe a system in contact with a heat reservoir at fixed temperature \((T)\), with fixed \(N\), and \(V\), while the system \(E\) can freely be exchanged with a heat reservoir. And the grand canonical ensemble, with a fixed \(T\) and \(V\), while the system can exchange \(N\) as well as \(E\) with a heat-particle reservoir, therefore it must have a chemical potential \(\mu\).

The grand canonical ensemble is the more appropriate choice to compute observables when dealing with relativistic quantum systems at finite temperature where particles appear and disappear. In this way, the vacuum can be thought as the heat reservoir interchanging the particles, which are spontaneously created or annihilated.

In the grand canonical ensemble, a dynamical system characterized by a Hamiltonian \(\hat{H}\), with \(n\) difference species of particles \(\hat{N}_i\) with \(i = \{1, \ldots, n\}\) is characterized by the statistical density matrix

\[
\hat{\rho} = \exp\left\{-\beta(\hat{H} - \mu_i\hat{N}_i)\right\} \tag{2.12}
\]

where the chemical potential of the specie \(i (\mu_i)\), and the inverse temperature, \(\beta = 1/T\), may be thought as Lagrange multipliers which determines the mean number of particles and energy respectively.

Knowing the density matrix, the ensemble average of any operator \(\hat{A}\) could be obtained as

\[
\langle A \rangle = \frac{1}{Z} \sum_n \langle n|A|n \rangle e^{-\beta(\hat{H} - \mu_i\hat{N}_i)} = \frac{\text{Tr} \ \hat{\rho} \ \hat{A}}{Z} \tag{2.13}
\]

where the grand canonical partition function, the most important quantity in statistical mechanics, is

\[
Z = \text{Tr} \ \hat{\rho} \tag{2.14}
\]

From the partition function, the free-energy \(\Omega\), could be computed

\[
\Omega = \frac{1}{\beta} \ln Z \tag{2.15}
\]

The free energy allows to derive all other thermodynamical quantities as the pressure \((P)\), number particle density \((N_i)\), entropy \((S)\), and energy density \((E)\),
\[ P = \frac{\partial \Omega}{\partial V} , \quad S = \frac{\partial \Omega}{\partial T} , \quad E = TS - PV + \mu N_i \] (2.16)

The trace operation in Eq. (2.14) represents the sum over all eigen-states that could be described by \( \phi_a \). Then, the partition function, in terms of \( \phi \), is

\[ Z = \text{Tr} \ e^{-\beta (\hat{H} - \mu \hat{N}_i)} = \int d\phi_a \langle \phi_a | e^{-\beta (\hat{H} - \mu \hat{N}_i)} | \phi_a \rangle \] (2.17)

An other way to approach a quantum mechanical system is through the path integrals formalism. In this formalism, the partition function \( Z \), on Eq. (2.17), can be expressed as a functional integral over the fields \( \phi \) and conjugate momenta \( \pi \), introducing the Hamiltonian density \( \mathcal{H} \) and any conserve charge density \( N \)

\[ Z = \int \mathcal{D} \pi \mathcal{D} \phi \ e^{\int_0^\beta d\tau \int d^3x \left( i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu N(\pi, \phi) \right)} \] (2.18)

where the notations \( \mathcal{D} \psi \) and \( \mathcal{D} \pi \) stand for the functional integration over the fields and their conjugate momenta. Both theories, path integral (2.18), and statistical operator representation with (2.14) and (2.17), equally describes a quantum system at zero temperature.

The correlation between a system in the state \( \phi(t) \) at \( t = 0 \) and later time \( t \), with fixed \( \beta \), is given by

\[ \langle \phi(0) \phi(t) \rangle_\beta \] (2.19)

This relation could be written as

\[ \langle \phi(0) \phi(t) \rangle_\beta = \frac{1}{Z} \text{Tr} \left[ e^{-\beta (\hat{H} - \mu \hat{N}_i)} \phi(0) \phi(t) \right] = \frac{1}{Z} \text{Tr} \left[ e^{-\beta (\hat{H} - \mu \hat{N}_i)} \phi(t) e^{i(-i\beta H)} \phi(0) e^{-i(-i\beta H)} \right] = \frac{1}{Z} \text{Tr} \left[ e^{-\beta (\hat{H} - \mu \hat{N}_i)} \phi(-i\beta) \phi(t) \right] = \langle \phi(-i\beta) \phi(t) \rangle_\beta \] (2.20)

It can be seen in Eq. (2.20) that the imaginary temperature plays the role as a time variable. This fact gives rise to the imaginary time formalism where \( t \to -i\tau \). Then,
Eq. (2.20) becomes
\[
\langle \phi(0)\phi(\tau) \rangle_{\beta} = \langle \phi(\beta)\phi(\tau) \rangle_{\beta}
\] (2.21)

This compactification of the space along the temporal direction has as a consequence that the momentum along the temporal direction becomes discrete. This could be explained considering the fact that Eq. (2.26) is describing a system in thermodynamic equilibrium, it follows that the state of the field $\psi$ at time $t = 0$ must be the same state $\psi$ at later time $t$. Then
\[
\phi(x, 0) = \epsilon \phi(x, \beta)
\] (2.22)

where $\epsilon = +1$ whether the fields commute, bosonic field, or $\epsilon = -1$ if anti-commute, fermionic field. Thus, in the energy-momentum space, taking into account the periodicity on the fields, the Fourier expansion of $\phi$ is discrete
\[
\phi(x, \tau) = \sum_{n, p} e^{i(\omega_n \tau + p \cdot x)} \phi(\omega_n, p)
\] (2.23)

where
\[
\omega_n = \frac{2\pi n}{\beta} \quad \text{for bosonic fields}
\]
\[
\omega_n = \frac{2\pi (n + 1)}{\beta} \quad \text{for fermionic fields}
\] (2.24)

with $n = \{-\infty, \ldots , -1, 0, 1, \ldots, \infty\}$, and $\omega_n$ are the Matsubara frequencies.

On the other hand, in QFT, the generating function $Z$ plays the role that the partition function does in statistical mechanics. At zero temperature, the generating function is given by
\[
Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \ e^{\int d^4 x \mathcal{L}}
\] (2.25)

The expression for both, generating function $Z$ and partition function $Z$, have a very similar appearance. This fact make possible to introduce the temperature in QFT employing the imaginary time formalism. The, proceeding in similar way that in statistical mechanics, it is possible to rewritten Eq. (2.25) as
\[
Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \ e^{\int_0^\beta d\tau \int d^3 x \mathcal{L}}
\] (2.26)
where the temporal direction of the field becomes discrete, as consequence of the equilibrium conditions as in (2.23).

Consider the free Dirac field $\psi$, at non-zero temperature and non-zero chemical potential. In the path integral formalism, the generating function $Z$ is given by

$$Z = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \ e^{\int \! d^4 x (\mathcal{L}_{\text{Dirac}} + \mu \psi^\dagger \psi)}$$

where the chemical potential $\mu$ has been introduced as a shift to the energy, and the Dirac Lagrangian density is given by

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

In the imaginary time formalism, the metric changes from Minkowski to Euclidian, $t^2 - x \to - (\tau^2 + x^2)$ and the imaginary time is substituted by $\beta$. Then, using the Fourier transform Eq. (2.23), the generating function Eq. (2.27) can be rewritten as

$$Z = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \ e^{\int \! d^3 p \psi^\dagger G^{-1} \psi}$$

where $G^{-1}$ is the inverse propagator given, in terms of the Matsubara frequencies, by

$$G^{-1} = \sum_n \bar{\psi}_{n,p} \left( i(\omega_n + \mu) + \gamma^0 \gamma^i p^i + m \gamma^0 \right) \psi_{n,p}$$

The functional integrals on (2.29) can be performed exactly

$$Z = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \ e^{\int \! d^3 p \psi^\dagger G^{-1} \psi} = \det [G^{-1}]$$

Then, with $Z$, the free energy is obtained as

$$\Omega = \frac{1}{\beta} \ln Z = \frac{1}{\beta} \ln \det G^{-1} = \frac{1}{\beta} \text{Tr} \ln G^{-1}$$

where the identity

$$\ln \det G = \text{Tr} \ln G$$
was used. The trace (Tr) is a functional trace that sums over all internal degrees of freedom: momentum, Matsubara frequencies, spins, particles, and antiparticles; therefore, it contains all the physics of the system. The free energy, Eq. (2.33), allows to obtain all thermodynamical quantities, through the relations (2.16), making possible the construction of the system’s EoS.

2.7 QCD Formalism

The discovery that at very short distances the nucleon constituents behaved like weakly interacting point particles in the 60’s popularized the quark model of hadrons proposed by Gell-Mann and Zweig. Later on it was shown that the only generalizable field theory of quarks with the mentioned characteristics was a Yang-Mills-like theory. It shows that quarks must be spin-\(\frac{1}{2}\) fermions, with fractional electric charge and must come in three colors. The carriers of the strong force, the gluons, must be massless spin-1 bosons which, unlike the photon on QED, they carry a color charge and therefore interact among themselves.

The microscopic theory of QCD is formulated with the introduction of a quark fields \(\psi^a_i\) in the fundamental representation of the SU(3)\(_c\) color gauge group, carrying flavor \(i\) and color \(a\). The QCD Lagrangian [39] density reads

\[
\mathcal{L}_{QCD} = \bar{\psi}_i^a \left( i \gamma^\mu \partial_\mu + \gamma^0 \mu - m_i \right) \psi_i^a + g A_\mu^A \bar{\psi}_i^a \gamma^\mu T_{ab}^A \psi_b^a - \frac{1}{4} G_{\mu\nu}^A G^{A,\mu\nu} \quad (2.34)
\]

where \(A_\mu^A\) is the vector gauge field in the adjoint representation of SU(3)\(_c\),

\[
G_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f^{ABC} A_\mu^B A_\nu^C \quad (2.35)
\]

is the field strength, and the generators of color transformation are defined as

\[
T_{ab}^A = \frac{1}{2} (\lambda^A)_{ab} \quad (2.36)
\]

where \(\lambda^A\) are the Gell-Man matrices, \(m_i\) is the mass of the quark \(i\) and \(\mu\) the chemical potential.

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The interaction vertex on (2.34) has a non trivial color structure given by the color generators $T^A_{ab}$

$$\sum_{A=1}^{N_c^2-1} T^A_{ab} T^A_{b'b} = -\frac{N_c + 1}{4N_c} (\delta_{aa'}\delta_{bb'} - \delta_{ab}\delta_{a'b'}) + \frac{N_c - 1}{4N_c} (\delta_{aa'}\delta_{bb'} + \delta_{ab}\delta_{a'b'})$$  \hspace{1cm} (2.37)

The first antisymmetric term corresponds to the attractive antitriplet channel, which form the Cooper pairs. The second to the repulsive sextets channel.

At moderate densities, like may exist in compact stars, the assumption that quark matter is weakly interacting is unlikely. The use of the microscopic theory of strong interaction is not valid, and one has to rely on effective theories like the Nambu-Jona Lasinio (NJL) model.

The NJL model was originally constructed to describes nucleon-nucleon interaction. It does not contain a confinement mechanism, but it does exhibit spontaneous chiral symmetry breaking which generates the large constituent of the nucleon masses. Thus, the NJL model is suitable for situations in which chiral symmetry is the relevant feature of QCD and the lack of confinement is less important like is the case of our study. One of the simplest NJL type model that has the global chiral symmetry of QCD, is given by the Lagrangian density with a local four-fermion interaction. Then,

$$L_{NJL} = \bar{\psi}^a_i \left( i\gamma^\mu \partial_\mu + \gamma^0 \mu - m_i \right) \psi^a_i + G_S \left[ (\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_\sigma \bar{\psi})^2 \right] + G_D \left( i\bar{\psi}_C \epsilon^a \gamma_5 \bar{\sigma} \psi \right) (i\bar{\psi} \epsilon^a \gamma_5 \bar{\sigma} \psi_C),$$  \hspace{1cm} (2.38)

where $\psi_C = C\bar{\psi}^t$ is the charge-conjugate spinor, $C = i\gamma^2\gamma^0$ is the charge conjugation matrix, $\bar{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices in flavor space, while $(\epsilon)^{ik} = \epsilon^{ik}$ and $(\epsilon^a)^{bc} = \epsilon^{abc}$ are the antisymmetric tensors in the flavor and in the color spaces, respectively.

The model (2.38) is very often used to describe CS since it capture the most important features of QCD relevant to CS like the chiral symmetry braking, leaving others aside. NJL models provide the opportunity to tackle complex problems, which are not accessible via other approaches. Nevertheless, the NJL parameters are largely undetermined and must
be adjusted in an appropriate way to reproduce known observable values relevant to the energy scale being explored.

2.8 Hubbard-Stratonovich transformation

In Section 2.6, a free fermionic field at fine temperature and finite density was considered and solved exactly. However, if interacting fields are taking into account in the quantum field model, there are very limited cases where an exact solution could be found, in general the applications of approximations is necessary. The most common techniques are based on perturbation theory, but as it is known, CS is a non-perturbative phenomena. One of the strategies used to approach a non-perturbative quantum field theory within the path integral formalism is the Hubbard-Stratonovich transformation.

The Hubbard-Stratonovich transformation is a method used to rewrite the interaction term on the Lagrangian in term of new bosonic fields introduced via an exact integral transformation. There is an ambiguity in the choice of these fields. However, since the objective is to study the order parameter in CS, it is convenient to relate the expectation value of these fields to the particle-particle expectation value since this parameter describe when two particles become correlated and form Cooper pairs. Therefore, the order parameter for superconductivity is related to the particle-particle expectation value.

The convenient integral Hubbard-Stratonovich transformation used is

\[
\exp \left[ G(\bar{\psi}\psi)^2 \right] = N \int \mathcal{D}\phi \exp \left[ \frac{\phi^2}{4G} + \bar{\psi}\psi\phi \right]
\]

(2.39)

This transformation allows to bring the functional integral over fermionic fields into a quadratic (Gaussian) form so that fermions can be integrated out leaving the generating function \( Z \) in terms of the new introduced auxiliary fields

\[
Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \, e^{-S_{\text{eff}}(\phi,\phi^*)}
\]

(2.40)

where \( S_{\text{eff}} \) is as effective action. This procedure is called Bosonization. Generally, the coupling of the new auxiliary fields is weak such that a perturbative expansions of the
nonlinear effective action could provide useful results already at low orders of this expansion. Then, a mean-field theory is applied to obtain the general form of the gap equation.

### 2.9 The Mean Field Approximation

In the mean-field theory, the fields are expanded in terms of the fluctuations around their expectation value. Now, assuming that there is a non-zero expectation value, or mean field, $\langle \bar{\psi}\psi \rangle$ in the system vacuum, the term $\bar{\psi}\psi$ can be expanded in term of a fluctuation around the this mean field as

$$
\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}\psi \rangle + \bar{\psi}\psi 
$$

Using this approximation, the quark-quark interaction can be expanded as

$$
(\bar{\psi}\psi)(\bar{\psi}\psi) = (\bar{\psi}\psi + \langle \bar{\psi}\psi \rangle)(\bar{\psi}\psi + \langle \bar{\psi}\psi \rangle)
$$

$$
= \bar{\psi}\psi\bar{\psi}\psi + \bar{\psi}\psi\langle \bar{\psi}\psi \rangle + \langle \bar{\psi}\psi \rangle\bar{\psi}\psi + \langle \bar{\psi}\psi \rangle^2
$$

(2.42)

Upon performing the mean field approximation we can neglect the quadratic term $\bar{\psi}\psi\bar{\psi}\psi$, leaving

$$
(\bar{\psi}\psi)(\bar{\psi}\psi) \cong 2\bar{\psi}\psi\langle \bar{\psi}\psi \rangle + \langle \bar{\psi}\psi \rangle^2
$$

(2.43)

Similarly, if we have the diquark-diquark interaction

$$
(\bar{\psi}\psi)(\bar{\psi}\psi)(\bar{\psi}\psi) = (2\bar{\psi}\psi\langle \bar{\psi}\psi \rangle + \langle \bar{\psi}\psi \rangle^2)(2\bar{\psi}\psi\langle \bar{\psi}\psi \rangle + \langle \bar{\psi}\psi \rangle^2)
$$

$$
\cong 4\bar{\psi}\psi\langle \bar{\psi}\psi \rangle^3 + \langle \bar{\psi}\psi \rangle^4
$$

(2.44)

### 2.10 Equation of State

To find the system EoS, it is necessary to find the system energy density and pressure. The energy density and pressure are obtained respectively from the $\langle T_{00} \rangle$ and $\langle T_{ii} \rangle$ components of the quantum-statistical average of the energy momentum tensor. For an
isotropic system, as the one we are considering, the covariant structure of the \(\langle T_{\mu\nu} \rangle\) tensor is given as [40]

\[
\frac{T}{V} \langle T_{\mu\nu} \rangle = (\Omega_0 + B)g_{\mu\nu} + (\mu n_F + TS)u_\mu u_\nu
\]  

(2.45)

where \(V\) is the system volume, \(T\) the absolute temperature, \(S\) the entropy, and \(u_\mu\) the medium 4-velocity with value \(u_\mu = (1, \vec{0})\) in the rest frame. In (2.45) the bag constant \(B\) is introduced to account for the energy difference between the perturbative vacuum and the true one. In that way, Eq. (2.45) is modeling what occurs in the case of quark matter, where the asymptotically-free phase of quarks forms a perturbative regime (inside a bag) which is immersed in the nonperturbative vacuum. This scenario is what is called the MIT bag model [41]. The creation of the bag costs free energy. Then, in the energy density, the energy difference between the perturbative vacuum and the true one should be added. Essentially, that is the bag constant \(B\) characterizing a constant energy per unit volume associated to the region where the quarks live. From the point of view of the pressure, \(B\) can be interpreted as an inward pressure needed to confine the quarks into the bag.

Then, the energy density and pressure of the system in the zero-temperature limit are respectively calculated from

\[
\varepsilon = \Omega_0 + \mu n_F + B, \quad p = -\Omega_0 - B
\]  

(2.46)

where \(\Omega\) is the thermodynamic potential.

To incorporate the contribution of the diquark-diquark repulsion in the EoS, it is necessary to expand the interacting term through the mean field approximation. This expansion will add a term in the thermodynamic potential \(\Omega\), as well as a shift to the gap parameter \(\Delta\). This will be done for each of the studied cases in subsequent chapter.
Chapter 3

The EoS in the BCS-BEC Crossover of a Simple Fermion System

In this section, a simple fermion system is consider. Their interaction is such that it emulates the formation of Cooper pairs through a fermion-fermion interaction with a difermion-difermion repulsion. This simple model serves as a toy model simulating a system of quarks with quark-quark attractive, and diquark-diquark repulsion channels. Therefore, the fermions sometimes will refering as quarks during the rest of the chapter.

The objective of this section is to look for a possible BCS-BEC crossover analyzing, through the mean-field approximation, their EoS. Then, the effects that varying the diquark-diquark repulsion produces on the EoS, are studied.

3.1 Fermion Model

The investigation starts with a simplified pure fermion system represented by the four-fermion interaction Lagrangian density of Ref. [14]

\[
L = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi + \frac{g}{4} (\bar{\psi} i \gamma_5 C \bar{\psi}^T) (\psi^T C i \gamma_5 \psi)
\]  

(3.1)

Here \( \psi \) is a Dirac fermion field, \( C = i \gamma_0 \gamma_2 \) is the charge conjugation matrix, \( m \) the fermion mass, and \( g \) the attractive coupling constant in the \( J^P = 0^+ \) channel that parameterizes the strength of the interaction.

For simplicity, the only internal fermion degrees of freedom considering in this model is the spin. The BCS-BEC crossover is uniquely related to the change in the nature of
the spectrum of the quasiparticles, which is determined by the variation of the diquark-pair binding energy as a function of the coupling constant strength. Therefore, the results obtained should not qualitatively change even when additional internal fermion degrees of freedom are added.

The generating function in imaginary time formalism of model (3.1) is

\[ Z = \int D\bar{\psi} D\psi e^{\int_0^\beta d\tau \int d^3x (\mathcal{L} + \mu \psi^\dagger \psi)} \]  

(3.2)

where \( \beta \) is the inverse of the temperature, \( \beta = 1/T \), and \( \mu \) is the chemical potential defining the Fermi energy. Introducing the Nambu-Gor'kov space in the field basis \( \Psi^r = (\psi, \psi_C) \), with \( \psi_C = C \bar{\psi}^r \) being the charge-conjugate spinors; and performing the Hubbard-Stratonovich transformation with gap parameter \( \Delta = \langle g \bar{\psi}^r C \gamma_5 \psi/2 \rangle \), the partition function (3.2) in imaginary time formalism, and momentum space, is given by:

\[ Z = \int D\bar{\Psi} D\Psi D\Delta D\Delta^* e^{\int_0^\beta d\tau \int d^3p \left( \frac{1}{2} \bar{\Psi} G^{-1} \Psi - \frac{\Delta^2}{\beta} \right) \} \]  

(3.3)

where \( G^{-1} \) is the inverse propagator in terms of the inverse Nambu-Gor'kov propagator

\[ G^{-1} = \gamma^\mu p^\mu - m + \mu \gamma_0 \sigma_3 + i \gamma_5 \Delta \sigma_+ + i \gamma_5 \Delta^* \sigma_- \]  

(3.4)

where \( \sigma_{\pm} = (\sigma_1 \pm \sigma_2)/2 \) are defined in Nambu-Gor'kov space with \( \sigma_i \) denoting the corresponding Pauli matrices.

Integrating out the fermion fields

\[ Z = \int D\Delta D\Delta^* e^{-S_{\text{eff}}} \]  

(3.5)

the bosonic effective action

\[ S_{\text{eff}} = \int_0^\beta d\tau \int d^3p \left( \frac{\left| \Delta \right|^2}{g} - \frac{1}{2\beta} \text{Tr} \ln (\beta G^{-1}) \right) \]  

(3.6)

is obtained. In the mean field approximation, and introducing the fermion Matsubara frequencies \( \omega_n = (2n + 1)\pi/\beta \), the free energy at finite temperature is

\[ \Omega_{\text{FER}} = -\frac{1}{\beta} \sum_{n=0}^\infty \frac{1}{(2\pi)^4} \int d^3p \text{Tr} \ln [\beta G^{-1}(i\omega_n, p)] + \frac{\Delta^2}{g}, \]  

(3.7)
where $G^{-1}(i\omega_n, \mathbf{p})$ is the inverse propagator in terms of the Matsubara frequencies

$$G^{-1}(i\omega_n, \mathbf{k}) = (\omega_n + \mu\sigma_3)\gamma_0 - \gamma \cdot \mathbf{p} - m + i\gamma_5 \Delta \sigma_+ + i\gamma_5 \Delta^* \sigma_- \quad (3.8)$$

The energy dispersion relation, $\epsilon_p^e$, is obtained from the roots of $\det[G^{-1}] = 0$. They are given by

$$\epsilon_p^e = \sqrt{(\epsilon_p - e\mu)^2 + \Delta^2}, \quad \epsilon_p = \sqrt{p^2 + m^2}, \quad e = \pm 1. \quad (3.9)$$

The spectra corresponding to different $e = \pm$ values denote the particle ($e = +1$) and antiparticle ($e = -1$) contributions.

Taking the trace and the sum in Matsubara frequencies in (3.7), and obtaining the zero-temperature limit, (3.7) becomes

$$\Omega_{FER}^0 = -\sum_{e=\pm 1} \int_{\Lambda} \frac{d^3p}{(2\pi)^3} \epsilon_p^e + \frac{\Delta^2}{g} \quad (3.10)$$

where $\Lambda$ is an appropriate momentum cutoff to regularize the momentum integral in the ultraviolet, and $\epsilon_p^e$ the quasiparticle energy spectrum (3.9).

The diquark-diquark interaction is modeled in the context of a $\phi^4$ boson theory [42]. In the case of model (3.1), it can be achieved by introducing a $\lambda \Delta^4$ term in the free energy (3.17). Then

$$\Omega_{FER}^{0,T} = -\sum_{e=\pm 1} \int_{\Lambda} \frac{d^3p}{(2\pi)^3} \epsilon_p^e + \frac{\Delta^2}{g} + \lambda \Delta^4 \quad (3.11)$$

This is basically the model studied in Ref. [43].

### 3.2 Gap Equation at Fixed Particle Number Density

A stable phase, with gap parameter $\Delta$ and chemical potential $\mu$, is obtained solving numerically the gap equation obtained from $\frac{\partial \Omega}{\partial \Delta} = 0$ which from Eq. (3.11) yields

$$\frac{1}{g} + 2\lambda \Delta^2 - \int_{\Lambda} \frac{d^3p}{(2\pi)^3} \left( \frac{1}{2\epsilon_p^+} + \frac{1}{2\epsilon_p^-} \right) = 0 \quad (3.12)$$
Moreover, considering a canonical ensemble, as usual in the study of the BCS-BEC crossover, the particle number density $n_F$ is fixed through the Fermi momentum $P_F$ with the density equation $n_F = \frac{P_F^3}{3\pi^2}$. Here the particle number is defined by the Fermi momenta as $n_F = \frac{P_F^3}{3\pi^2}$. Then, from (3.11) the particle density equation is

$$\frac{P_F^3}{3\pi^2} = -\int_\Lambda \frac{d^3p}{(2\pi)^3} \left( \frac{\xi_p^+}{\epsilon_p^+} - \frac{\xi_p^-}{\epsilon_p^-} \right)$$

(3.13)

with

$$\xi_p^\pm = \epsilon_p \mp \mu \quad \text{(3.14)}$$

There are two parameters that will be adjusted to drive the crossover, the quark-quark coupling $\hat{g}$, and the diquark-diquark repulsion $\lambda$. There are also free parameters on the model, the Fermi momentum $P_F$, the fermion mass $m$, the bag constant $B$, and the cutoff $\Lambda$. Since our model could not be tuned with real observable, the parameter will be chosen as follows

$$P_F = 0.2 \Lambda \quad m = 0.4 \Lambda \quad B^{1/4} = 248.4 \Lambda$$

All the variables are normalized with the cutoff parameter $\Lambda$. Therefore, from now on, the normalized units

$$g \rightarrow g\Lambda^2, \quad m \rightarrow \frac{m}{\Lambda}, \quad P_F \rightarrow \frac{P_F}{\Lambda}, \quad \Delta \rightarrow \frac{\Delta}{\Lambda}, \quad \mu \rightarrow \frac{\mu}{\Lambda}, \quad \text{and} \quad k \rightarrow \frac{k}{\Lambda}.$$  

are used. Therefore the cutoff parameter won’t be explicit on our result. We also define $\hat{g} = \frac{g}{4\pi\Lambda}$.

The problem to be solved is written now on the standard form that will be addressed on Chapter 6. First, it is defined a vector of unknowns variables $\mathbf{x} = (\mu, \Delta)^T$. Then, from the gap equation (3.12) and particle density equation (3.13), taking into account definitions (3.9) and (3.14), and integrating on the solid angle, a system of equation is formulated as $\mathbf{F} = (f_1(\mathbf{x}), f_2(\mathbf{x}))^T$ where

$$f_1(\mathbf{x}) = \frac{1}{\hat{g}} + 2\lambda\Delta^2 - \int_0^1 dp \, p^2 \left( \frac{1}{\epsilon_p^+} + \frac{1}{\epsilon_p^-} \right) \quad \text{(3.15)}$$

$$f_2(\mathbf{x}) = P_F^3 + \frac{3}{2} \int_0^1 dp \, p^2 \left( \frac{\xi_p^+}{\epsilon_p^+} - \frac{\xi_p^-}{\epsilon_p^-} \right)$$  

(3.16)
The problem is to find a solution for $F = 0$ such that the thermodynamic potential (3.17), which is a function of $x$

$$
\Omega_{FER}^{0,T}(x) = -\frac{1}{2\pi^2} \int_0^1 dp p^2 (\epsilon_k^+ + \epsilon_k^-) + \frac{\Delta^2}{4\pi^2 \hat{g}} + \lambda \Delta^4,
$$

becomes a global minimum with respects to the gap parameter $\Delta$.

### 3.3 Numerical Results and EoS

As known, the condition $\mu < m$ is characteristic of a relativistic Bose gas [44]. From Fig. 3.2, we see that for this simple model ignoring the diquark-diquark repulsion, $\lambda = 0$, there exists a critical value for the coupling constant $\hat{g}_{cr} \sim 0.84$ beyond which the condition $\mu < m$ is satisfied. Hence, the quasiparticle spectrum corresponding to coupling constants smaller and larger than $\hat{g}_{cr}$ should correspond to fermion-like and boson-like behaviors, respectively.

The quasiparticle spectra, $\epsilon_p^+$ shown on Fig. 3.1, was obtained from this model where $\lambda = 0$. From their graphical representations, we can see that for the spectra corresponding to $\hat{g} = 0.5$ the minimum of their dispersion relations occurs at $k = \sqrt{\mu^2 - m^2}$, with excitation energy given by the gap $\Delta$, a behavior characteristic of fermionic quasiparticles in the BCS regime. On the other hand, for $\hat{g} = 1$, the minimum of the corresponding spectrum occurs at $k = 0$, with excitation energy $\varepsilon = \sqrt{(\mu - m)^2 + \Delta^2}$, which is typical of bosonic-like quasiparticle. Therefore, it is corroborated that $\hat{g}_{cr}$ is the threshold value for the BCS-BEC crossover in this model. In other words, for $\hat{g} < \hat{g}_{cr}$, we have $\mu > m$ and the quasiparticles exhibit fermionic-like modes, while for $\hat{g} > \hat{g}_{cr}$, we have $\mu < m$ and the quasiparticles are characterized by bosonic-like modes.

Taking into account the diquark-diquark repulsion $\lambda$, the slope of the chemical potential $\mu$ starts to decrease as $\lambda$ increases, and for $\lambda = 1$ the BCS-BEC crossover disappear since $\mu > m$ for the hole region. On the other hand, the gap critical value for the gap parameter is $\Delta_{cr} = 0.051$. As $\lambda$ begins to increase, the slope of $\Delta$ starts decreasing, as it is shown
in Fig. 3.3. For a repulsion \( \lambda = 1 \) the value of \( \Delta \) is not longer enough to create the BEC phase, \( \Delta < \Delta_{cr} \) for the whole range.

The results for \( p \) and \( \varepsilon \), are shown on Fig. 3.4 and Fig. 3.5 respectively. There, it can be seen that the system energy density is decreasing with the coupling strength, as is expected the condensation energy, which is proportional to the gap is increasing. The pressure is also decreasing up to get negative values at coupling constants corresponding to the BEC regime and \( \lambda = 0 \). The appearance of a negative pressure for the diferfion free gas in the BEC region indicates that the free-diferfion system is unstable. It is an expected result since the absence of repulsion between the diferfions makes their BEC inevitable with the corresponding decrease of the matter pressure. Nevertheless, the contribution of the diferfion-diferfion repulsion in the EoS of the strongly interacting system compensates the decreasing tendency due to the BEC and consequently rendering a constant pressure throughout the strongly interacting region. The repulsive interaction between diferfion makes a significant contribution to the energy density and pressure as can be seen in Fig. 3.4. For values of \( \lambda = 10 \), the matter pressure does not decreas significantly. It is apparent that
Figure 3.2: Chemical Potential $\mu$ vs Diquark Coupling $\hat{g}$ at different Diquark-Diquark Repulsion $\lambda$

the instability produced by a negative pressure in the BEC region disappears.
Figure 3.3: Gap Parameter $\Delta$ vs Diquark Coupling $\hat{g}$ at different Diquark-Diquark Repulsion $\lambda$

Figure 3.4: Pressure $p$ vs Diquark Coupling $\hat{g}$ at different Diquark-Diquark Repulsion $\lambda$
Figure 3.5: Energy Density $\varepsilon$ vs Diquark Coupling $\hat{g}$ at different Diquark-Diquark Repulsion $\lambda$
Chapter 4

The EoS in the BCS-BEC Crossover for CFL Matter

The CFL is now a well established phase of QCD at asymptotic densities where the u, d, and s quarks can be treated on equal footing. In the CFL phase, quarks of all three colors and all three flavors form Cooper pair, giving the maximal pairing free energy benefit [45].

The Cooper pairs will have different flavors. To study the pairing structure, the ansatz

\[ \langle \phi^\alpha_C \gamma_5 \phi^\beta \rangle \sim \Delta_1 \epsilon_{\alpha\beta1} \epsilon_{ab1} + \Delta_2 \epsilon_{\alpha\beta2} \epsilon_{ab2} + \Delta_3 \epsilon_{\alpha\beta3} \epsilon_{ab3}, \]  

is considered. It is clear from Eq. (4.1) that the pairing is anti-symmetric in color, and flavor.

In this section, a possible BCS-BEC crossover is investigated on the CFL phase modeled by the three-flavor NJL theory considered in [46]. Similarly than in Chapter 3, the crossover is explored by increasing the diquark coupling strength \( G \), and the diquark-diquark repulsion \( \lambda \).

4.1 CFL Model

The physics described by the CFL phase, at low temperatures and very high baryon densities, is simplified to a model with a four-fermion vertex interacting by a point-like gluon exchange. The NJL effective model Lagrangian density is given in imaginary time formalism as

\[ \mathcal{L} = \bar{\psi}_i^a \left( \gamma^\mu p_\mu + \mu \gamma^4 - M \right) \psi_i^a - \frac{3}{8} G(\bar{\psi}_i^a \Gamma^A \psi_i^a) (\bar{\psi}_i^a \Gamma^A \psi_i^a) \]  

(4.2)
with $\psi^a_i$ is a Dirac fermion field carrying flavor $i = \{1, 2, 3\}$ and color $a = \{g, b, r\}$, and the generators of color transformation are $\Gamma^A_\mu = \frac{1}{2}(\lambda^A)_{ab}$ with $\lambda^A$ the Gell-Man matrices. In this Lagrangian, neutrality conditions are not considered, hence $\mu$ is a scalar density chemical potential. The mass matrix in flavor space is defined by $M = \text{diag} \ (0, 0, m)$ were $m$ is the s quark mass.

The generating function of model (4.2) is

$$Z = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \ e^{\int_0^\beta \ d\tau \ d^3x \ \mathcal{L}}$$  \hspace{1cm} (4.3)$$

where $\beta$ is the inverse of the temperature, $\beta = 1/T$.

Considering the ansatz (4.1), the interaction term in the Lagrangian density (4.2) (the last term) could be written as

$$\mathcal{L}_{int} = \frac{G}{4} \sum_\eta (\bar{\psi} P_\eta \psi^T) \ (\psi^T \bar{P}_\eta \psi)$$  \hspace{1cm} (4.4)$$

with $(P_\eta)^{\alpha\beta}_{ij} = C \gamma^5 \epsilon^{\alpha\beta\eta} \epsilon_{ij}$ where $\eta$ labels the pairing channels $\eta = \{1, 2, 3\} = \{ds, us, ud\}$. Then, introducing the Nambu-Gor’kov space and performing the Hubbard-Stratonovich transformation with the gap parameters $\phi_\eta$ the generating function becomes

$$Z = \int \mathcal{D} \bar{\Psi} \mathcal{D} \Psi \mathcal{D} \phi_\eta \mathcal{D} \phi^*_\eta \ e^{\int_0^\beta \ d\tau \ d^3x \ \left( \frac{1}{2} \bar{\Psi} G^{-1} \Psi - \phi_\eta \phi^*_\eta \right)}$$  \hspace{1cm} (4.5)$$

where the inverse Nambu-Gor’kov propagator is given by

$$G^{-1} (p) = \begin{pmatrix} \gamma^\mu p_\mu + i \gamma_4 & M \\ \bar{P}_\eta \phi^*_\eta & (\gamma^\mu p_\mu - i \gamma_4 + M)^T \end{pmatrix}$$  \hspace{1cm} (4.6)$$

Integrating out the fermion fields, in the mean field approximation the generating function reads

$$Z = \det [\beta \ G^{-1} (i \omega_n, p)]^{1/2} \ e^{-\beta V \Delta_n \Delta_n}$$  \hspace{1cm} (4.7)$$

where $G^{-1} (i \omega_n, p)$ is the inverse propagator, and $\omega_n = (2n + 1) \pi / \beta$ the fermion Matsubara frequencies.
From (4.7), using (2.16), the thermodynamical potential at finite temperature is
\[
\Omega_{CFL} = -\frac{1}{\beta} \sum_{n=0}^{\infty} \int d^3p \frac{1}{(2\pi)^3} \frac{1}{2} \text{Tr} \ln [\beta G^{-1}(i\omega_n, \mathbf{p})] + \frac{\Delta_n^2}{G},
\] (4.8)

Taking the trace and the sum in Matsubara frequencies in (4.9), and in the zero-temperature limit
\[
\Omega_{CFL}^0 = -\sum_j \int_{\Lambda} d^3p \frac{1}{(2\pi)^3} |\epsilon_j| + \frac{1}{G}(\Delta_1^2 + \Delta_2^2 + \Delta_3^2)
\] (4.9)

where \(\Lambda\) is an appropriate momentum cutoff to regularize the momentum integral in the ultraviolet, and the quasiparticle energy spectrum, \(\epsilon_j\), is obtained as the roots of 
\[
\det G^{-1}(i\omega_n, \mathbf{p}) = 0.
\]
There are 36 roots. In the case of the CFL pairing, the gaps are equal \(\Delta \equiv \Delta_1 = \Delta_2 = \Delta_3\), and there exist 4 different degenerate roots. They are
\[
\begin{align*}
\epsilon_1 &= \pm \sqrt{(\epsilon_p + \mu)^2 + \Delta^2} \quad \text{Degeneracy 16} \\
\epsilon_2 &= \pm \sqrt{(\epsilon_p - \mu)^2 + \Delta^2} \quad \text{Degeneracy 16} \\
\epsilon_3 &= \pm \sqrt{(\epsilon_p + \mu)^2 + 4\Delta^2} \quad \text{Degeneracy 2} \\
\epsilon_4 &= \pm \sqrt{(\epsilon_p - \mu)^2 + 4\Delta^2} \quad \text{Degeneracy 2}
\end{align*}
\] (4.10)
where
\[
\epsilon_p = \sqrt{p^2 + m^2}
\] (4.11)

The thermodynamic potential of the system at \(T = 0\) is given by
\[
\Omega_{CFL}^0 = 3 \frac{\Delta^2}{G} - \frac{1}{2} \int_{\Lambda} \frac{d^3p}{(2\pi)^3} (16 |\epsilon_1| + 16 |\epsilon_2|) \\
- \frac{1}{2} \int_{\Lambda} \frac{d^3p}{(2\pi)^3} (2 |\epsilon_3| + 2 |\epsilon_4|)
\] (4.12)
Adding the diquark-diquark repulsion in the mean field theory
\[
\Omega_{CFL}^{0,T} = 3 \frac{\Delta^2}{G} - \frac{1}{2} \int_{\Lambda} \frac{d^3p}{(2\pi)^3} (16 |\bar{\epsilon}_1| + 16 |\bar{\epsilon}_2|) \\
- \frac{1}{2} \int_{\Lambda} \frac{d^3p}{(2\pi)^3} (2 |\bar{\epsilon}_3| + 2 |\bar{\epsilon}_4|) + 48\lambda \frac{\Delta^4}{G^4}
\] (4.13)

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where

\[
\tilde{\epsilon}_1 = \pm \sqrt{(\epsilon_p + \mu)^2 + \tilde{\Delta}^2} \\
\tilde{\epsilon}_2 = \pm \sqrt{(\epsilon_p - \mu)^2 + \tilde{\Delta}^2} \\
\tilde{\epsilon}_3 = \pm \sqrt{(\epsilon_p + \mu)^2 + 4\tilde{\Delta}^2} \\
\tilde{\epsilon}_4 = \pm \sqrt{(\epsilon_p - \mu)^2 + 4\tilde{\Delta}^2}
\]  (4.14)

with

\[
\tilde{\Delta} = \Delta + 96\lambda \frac{\Delta^3}{G}
\]  (4.15)

### 4.2 Gap Equation at Fixed Particle Density

To obtain a stable phase, we must minimize the free energy (4.13) with respect to the variation of the gap parameter thought the gap equation \(\frac{\partial \Omega}{\partial \Delta} = 0\) which yields

\[
\frac{6\Delta}{G} + 192\lambda \frac{\Delta^3}{G^4} - \frac{\tilde{\Delta}}{2} \left( 1 + 288\lambda \frac{\Delta^2}{G^3} \right) \int_A \frac{d^3p}{(2\pi)^3} \left( \frac{16}{|\tilde{\epsilon}_1|} + \frac{16}{|\tilde{\epsilon}_2|} + \frac{2}{|\tilde{\epsilon}_3|} + \frac{2}{|\tilde{\epsilon}_4|} \right) = 0
\]  (4.16)

The particle density equation, in the canonical ensemble, \(n_F = -\frac{\partial \Omega}{\partial \mu}\) is fixed by

\[
\frac{P_F^3}{3\pi^2} = -\frac{1}{2} \int_A \frac{d^3p}{(2\pi)^3} \left( \frac{16\xi^+_p}{|\tilde{\epsilon}_1|} - \frac{16\xi^-_p}{|\tilde{\epsilon}_2|} + \frac{2\xi^+_p}{|\tilde{\epsilon}_3|} - \frac{2\xi^-_p}{|\tilde{\epsilon}_4|} \right)
\]  (4.17)

The particle number density \(n_F\) is fixed through the Fermi momentum \(P_F\) as \(n_F = \frac{P_F^3}{3\pi^2}\), and \(\xi^\pm_p\) is defined as

\[
\xi^\pm_p = \epsilon_p \pm \mu
\]  (4.18)

There are two parameters that will be adjusted, the quark-quark coupling \(G\), and the diquark-diquark repulsion \(\lambda\). In order to present the results in the same scale than in Chapter 5, the dimensional coupling constant \(G\) is taken to be proportional fixed value of \(G_s\). The value of the momentum integration cutoff \(\Lambda = 602.3\,\text{MeV}\) as well as the value of \(G_s = 1.835\Lambda^2\) are adjusted to reproduce the pion condensate [36]. Nevertheless, in the CFL phase, the value of the cutoff should be higher since its energy scale is higher than in the region of the pion condensate. Therefore, the values of \(\Lambda\) is adjusted to reproduce a
gap parameter of approximately $\Delta \sim 10$ MeV like is estimated to be in the core of neutron stars. Thus

$$G_D = \eta G_s$$

$$\Lambda = 1000 \text{ MeV} \quad (4.19)$$

Moreover, all the variables are normalized with the cut off parameter $\Lambda$. Therefore, from now on, we use normalized units

$$g \rightarrow g\Lambda^2, \quad m \rightarrow \frac{m}{\Lambda}, \quad P_F \rightarrow \frac{P_F}{\Lambda}, \quad \Delta \rightarrow \frac{\Delta}{\Lambda}, \quad \mu \rightarrow \frac{\mu}{\Lambda}, \quad \text{and} \quad k \rightarrow \frac{k}{\Lambda},$$

where the cutoff parameter is not explicitly shown in the result.

In the model, there are also free parameters that hast to be adjusted, the Fermi momentum $P_F$, the s quark mass $m$, the bag constant $B$, and the cutoff $\Lambda$. The mass of the s quark is taken to be of the order of the chemical potential of the CFL phase $\sim 0.45$. This value is high considering that the condition to obtain a stable phase is known to be $\mu \geq \frac{m^2}{2\Delta}$. However, the objective is to look for a region where a crossover could occur. The system considered has a very high coupling constant ($\eta = 1$). The QCD diagram of strong interacting systems shown in Fig. 2.2 was obtained considering a value of $\eta$ between 3 and 4. For the values we use, the s quark mass should be higher than the value of $m = 0.3$ on Ref. [36]. Then, the values used are

$$P_F = 0.2 \quad m = 0.4 \quad B = 248.4$$

where the Bag constant $B$ was chosen to decrease the values of the pressure to zero for $\lambda = 0$. The true vacuum pressure is unknown since there is no current experiment under this condition that could provide an appropriate value.

Then, the problem to solve is written in the standard form on Chapter 6. First, it is defined a vector of unknowns variables $\mathbf{x} = (\mu, \Delta)^T$. Then, from the gap equation (4.16) and particle density equation (4.17), taking into account definitions (4.14) and (4.18), and integrating on the solid angle, a system of equation is formulated as $\mathbf{F} = (f_1(\mathbf{x}), f_2(\mathbf{x}))^T$.
where
\[
f_1(x) = 6\frac{\Delta}{G} + 192\lambda \frac{\Delta^3}{G^4} - \frac{\tilde{\Delta}}{4\pi^2} \left( 1 + 288\lambda \frac{\Delta^2}{G^3} \right) \int_0^1 dp p^2 \left( \frac{16}{|\tilde{\epsilon}_1|} - \frac{16}{|\tilde{\epsilon}_2|} + \frac{2}{|\tilde{\epsilon}_3|} - \frac{2}{|\tilde{\epsilon}_1|} \right)
\]
\[
f_2(x) = \frac{P_F^3}{3\pi^2} + \frac{1}{4\pi^2} \int_0^1 dp p^2 \left( \frac{16\xi^+_p}{|\tilde{\epsilon}_1|} - \frac{16\xi^-_p}{|\tilde{\epsilon}_2|} + \frac{2\xi^+_p}{|\tilde{\epsilon}_3|} - \frac{2\xi^-_p}{|\tilde{\epsilon}_1|} \right)
\]
(4.20)
(4.21)

The problem is to find a solution for \( F = 0 \) such that the thermodynamic potential (4.13), now a function of \( x \)

\[\Omega^{0,T}_{CFL} = 3\frac{\Delta^2}{G} - \frac{1}{2} \int_0^1 dp p^2 \left( 16|\tilde{\epsilon}_1| + 16|\tilde{\epsilon}_2| \right) - \frac{1}{4\pi^2} \int_0^1 dp p^2 \left( 2|\tilde{\epsilon}_3| + 2|\tilde{\epsilon}_4| \right) + 48\lambda \frac{\Delta^4}{G^4} \]
(4.22)

be a global minimum with respect to the gap parameter \( \Delta \).

### 4.3 Numerical Results and EoS on Fermionic System

The condition \( \mu = m \) is the sign that characterizes the BCS-BEC crossover. Fig. 4.1 shows that ignoring the diquark-diquark repulsion, \( \lambda = 0 \), there exists a threshold value for the BCS-BEC crossover in this model at \( \eta_{cr} \sim 4.1 \) when the condition \( \mu = m \) is satisfied. Hence, the quasiparticle spectrum corresponding to coupling constants smaller and larger than \( \eta_{cr} \) should correspond to fermion-like and boson-like behaviors, respectively.

Taking into account the diquark-diquark repulsion \( \lambda \), the slope of the chemical potential does not change until values extremely higher. It is shown in Fig. 4.1, the variation for \( \mu \) when \( \lambda = 100 \). Those values are extraordinary high and are outside the model’s predictions. The same pattern is shown on the gap parameter on Fig. 4.2, the pressure on Fig. 4.3, and the energy density on Fig. 4.4.

Then, from the results shown, it is evident that the BCS-BEC crossover exist on the CFL phase model we are considering, and the crossover is fixed for a wide range of the diquark-diquark repulsion \( \lambda \).
Figure 4.1: Chemical Potential $\mu$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$

Figure 4.2: Gap Parameter $\Delta$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$
Figure 4.3: Pressure $p$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$

Figure 4.4: Energy Density $\varepsilon$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$
Chapter 5

No BCS-BEC Crossover for 2CS Matter

Compact stars are formed by superdense matter. The estimated central densities of such stars might be as large as $10\rho_0$, where $\rho_0 = 0.15\text{ fm}^{-3}$ is the nuclear saturation density. The corresponding ground state is characterized by a condensate of Cooper pairs made of quarks. The ground state of QCD, at asymptotic high chemical potential, is the CFL phase. However, experiment at RHIC suggest that if neutron star cores are made of CS quark matter, they may not reach the densities at which CFL predominates [32].

In this chapter, a possible phase down in density, the 2SC, is consider. This is a color superconducting phase with the spin-0 Cooper pairing in quark matter made of only u and d quarks. In this state, quark matter is unlikely to be weakly interacting. The microscopic theory of QCD could not be applied, and one has to rely on an effective theory like NJL model described in Chapter 2, Eq. (2.38). The chromomagnetic instabilities produced in this phase could be avoided if the strong coupling constant becomes sufficiently high. That is, the diquark ($G_D$) and quark-antiquark ($G_S$) coupling constants should become of the order of $\Lambda^2$.

The 2SC is a more realistic model, which could be applied to the EoS of compact stars. Matter in the bulk of compact stars should be neutral with respect to electrical as well as color charges. Also, such matter should remain in $\beta$-equilibrium. Such relations could substantially influence the pairing dynamics between quarks. Therefore, neutrality and $\beta$-equilibrium are considered in the model presented in this chapter.
5.1 2SC Model

The NJL effective model Lagrangian density describing the 2SC model [36] is

\[ \mathcal{L}_{NJL} = -\bar{\psi}^{a}_i (i\gamma^\mu \partial_\mu + \gamma^0 \hat{\mu} - M_i) \psi^a_i + G_s \left[ (\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma_5 \bar{\tau}\psi)^2 \right] + G_D (i\bar{\psi}_C \varepsilon^a \gamma_5 \bar{\tau}\psi)(i\bar{\psi}\varepsilon^a \gamma_5 \bar{\tau}\psi_C) \] (5.1)

where \( \psi_C = C\bar{\psi}^\tau \) is the charge-conjugate spinor and \( C = i\gamma^2\gamma^0 \) is the charge conjugation matrix, \( \bar{\tau} = (\tau^1, \tau^2, \tau^3) \) are the Pauli matrices in flavor space, while \( (\varepsilon)^{ik} = \varepsilon^{ik} \) and \( (\varepsilon^a)^{bc} = \varepsilon^{abc} \) are the antisymmetric tensors in the flavor and in the color spaces, respectively. The matrix \( \hat{\mu} = \text{diag} (\mu, \mu_e, \mu_8) \) where \( \mu, \mu_e, \) and \( \mu_8 \) are the baryonic, electrical and color chemical potentials. The matrix \( M = \text{diag} (m, m, 0) \) where \( m \) is the mass of the u and d quarks.

The color structure of condensate \( (i\bar{\psi}_C \varepsilon^a \gamma_5 \bar{\tau}\psi) \sim \varepsilon_{ik}\varepsilon^{abc} \) is antisymmetric in color as well as in flavor and in Dirac indexes. It has an arbitrary orientation in the color space. By making use of the global color transformations, the orientation is conveniently fixed in the third blue direction. In this case, the Cooper pairs in the 2SC phase are made of the r and g quarks only; therefore, the flavor SU(3) symmetry is reduced to SU(2). The approximate chiral symmetry is not broken, the masses of the quarks remain small compared with the coupling.

In \( \beta \)-equilibrium, the reactions

\[ \text{d} \rightarrow \text{u} + e^{-} + \bar{\nu}_e \quad \text{and} \quad \text{u} + e^{-} \rightarrow \text{d}_b + \bar{\nu}_e \] (5.3)

should occur at the same rate.

The electric neutrality requires

\[ n_d \simeq 2n_u \] (5.4)

where \( n_u \) is the number of u quarks (electric charge +\( \frac{2}{3} \)) and \( n_d \) the number of d quarks (−\( \frac{1}{3} \)), even in the presence of a non vanishing electron density. This condition is implicit with the introduction of the electric chemical potential \( \mu_e \). Similarly, the color neutrality with the color chemical potential \( \mu_8 \).
The matrix $\hat{\mu}$ is determined as

$$
\mu_{ij,\alpha\beta} = (\mu \delta_{ij} - \mu_e Q_{ij}) \delta_{\alpha\beta} + \frac{2}{\sqrt{3}} \mu_8 \delta_{ij} (T_8)_{\alpha\beta}
$$

(5.5)

with $Q$ and $T_8$ are the generators of the electromagnetism and color gauge group. The quark chemical potential explicit expression are

$$
\mu_{ur} = \mu_{ug} = \mu - \frac{2}{3} \mu_e + \frac{1}{3} \mu_8
$$

(5.6)

$$
\mu_{dr} = \mu_{dg} = \mu + \frac{1}{3} \mu_e + \frac{1}{3} \mu_8
$$

(5.7)

$$
\mu_{ub} = \mu - \frac{2}{3} \mu_e - \frac{2}{3} \mu_8
$$

(5.8)

$$
\mu_{db} = \mu + \frac{1}{3} \mu_e - \frac{2}{3} \mu_8
$$

(5.9)

Following the procedure in Chapter 3, and in Chapter 4, the effective potential in the NJL model in the mean field approximation is obtained using the standard Hubbard-Stratonovich transformation, performing the functional trace in Matzubara frequencies and obtaining the Zero Temperature limit,

$$
\Omega^0_{2SC} = \frac{\Delta^2}{4G_D^4} - \frac{\Lambda^4}{2\pi^2} - \frac{\mu_e^4}{12\pi^2} - 4 \int_{\Lambda} \frac{d^3 p}{(2\pi)^3} \left( \sqrt{(\epsilon_p + \tilde{\mu})^2 + \Delta^2} + \sqrt{(\epsilon_p - \tilde{\mu})^2 + \Delta^2} \right)
+ 2 \int_{\Lambda} \frac{d^3 p}{(2\pi)^3} (\mu_{ub} - \epsilon_p) \Theta(\mu_{ub} - \epsilon_p) + 2 \int_{\Lambda} \frac{d^3 p}{(2\pi)^3} (\mu_{db} - \epsilon_p) \Theta(\mu_{db} - \epsilon_p)
$$

(5.10)

where $\epsilon_p = \sqrt{p^2 + m^2}$, $\Theta$ is the step function, and it is introduced the effective chemical potential

$$
\tilde{\mu} = \mu - \frac{1}{6} \mu_e + \frac{1}{3} \mu_8
$$

(5.11)

Taking into account the diquark-diquark repulsion in the mean-field approximation

$$
\Omega^{0,T}_{2SC} = \frac{\Delta^2}{4G_D^4} - \frac{\Lambda^4}{2\pi^2} - \frac{\mu_e^4}{12\pi^2} + 48 \lambda \frac{\Delta^4}{G_D^4}
+ 2 \int_{\Lambda} \frac{d^3 p}{(2\pi)^3} (\mu_{ub} - \epsilon_p) \Theta(\mu_{ub} - \epsilon_p) + 2 \int_{\Lambda} \frac{d^3 p}{(2\pi)^3} (\mu_{db} - \epsilon_p) \Theta(\mu_{db} - \epsilon_p)
- 4 \int_{\Lambda} \frac{d^3 p}{(2\pi)^3} \left( \sqrt{(\epsilon_p + \tilde{\mu})^2 + \Delta^2} + \sqrt{(\epsilon_p - \tilde{\mu})^2 + \Delta^2} \right)
$$

(5.12)

with

$$
\tilde{\Delta} = \Delta + \lambda \frac{\Delta^3}{4G_D^3}
$$

(5.13)
5.2 Gap Equation and Neutrality Conditions at Fixed Particle Density

A stable phase is obtained minimizing the free energy (3.17) with respect to the gap parameter and imposing the neutrality conditions

$$\frac{\partial \Omega_0}{\partial \Delta} = 0, \quad \frac{\partial \Omega_0}{\partial \mu_e} = 0, \quad \frac{\partial \Omega_0}{\partial \mu_8} = 0 \quad (5.14)$$

at a fixed particle density

$$n_F = -\frac{\partial \Omega_0}{\partial \mu}. \quad (5.15)$$

The parameters are normalized with respect to the cutoff $\Lambda$. The dimensional coupling constant $G_S$ and the momentum integration cutoff $\Lambda$ are adjusted to fit observable values. In Ref [36], the pion decay adjust the value of $G_S = 1.835\Lambda^2$ and $\Lambda = 602.3$ MeV. However, the region of interest in our study is the center of neutron stars where the values of $\Delta = 10$ MeV [47]. Consequently, the value of $\Lambda$ is chosen with the objective of meeting this requirement, and the strength of the coupling constant, $G_D$, is taken to be proportional to $G_S$ as follows

$$G_D = \eta G_S, \quad (5.16)$$

$$\Lambda = 1000 \text{ MeV}, \quad (5.17)$$

where $\eta$ is a dimensionless parameter of order 1 and $G_S = 1.835\Lambda^2$ as in Ref. [36].

Then, the problem can be written in the standard form of Chapter 6. To obtain the explicit expression of Eq. (5.14) and Eq. (5.15), we consider the integral

$$I(x) = 2 \int_{\Lambda} \frac{d^3p}{(2\pi)^3} (x - \epsilon_p) \Theta(\mu_{db} - x)$$

$$= \frac{1}{\pi^2} \int_0^{\sqrt{x^2 - m^2}} p p^2 (\epsilon_p - x) \Theta(x - m)$$

$$= -\frac{1}{24 \pi^2} \sqrt{x^2 - m^2} (5m^2 x - 2x^3) - \frac{m^4}{8 \pi^2} \ln \left( \frac{\sqrt{x^2 - m^2} + |x|}{m} \right) \quad (5.18)$$
and its derivative
\[
\frac{\partial I(x)}{\partial y} = -\frac{5m^4 x - 16m^2 x^3 + 8x^5 + 3m^4 |x|}{24 x \pi^2 \sqrt{x^2 - m^2}} \frac{\partial x}{\partial y} \tag{5.19}
\]

The vector of unknown variables is defined as \( \mathbf{x} = (\Delta, \mu, \mu_e, \mu_8)^T \). Then, from the gap equation and neutrality conditions (5.14), and density equation (5.15), the system of equation is formulated as \( \mathbf{F} = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}), f_4(\mathbf{x}))^T \) where

\[
f_1(\mathbf{x}) = \frac{\Delta}{2G_D} + \lambda \frac{\Delta^3}{4G_D^3} \tag{5.20}
\]

\[
f_2(\mathbf{x}) = \frac{P_F^3}{3\pi^2} - \frac{2}{\pi} \int_0^1 dp p^2 \left( \frac{\xi_\mu}{\sqrt{(\epsilon_p + \bar{\mu})^2 + \Delta^2}} + \frac{\xi_\mu}{\sqrt{(\epsilon_p - \bar{\mu})^2 + \Delta^2}} \right) + \frac{\partial I(\mu_{ab})}{\partial \mu_{ab}} \frac{\partial \mu_{ab}}{\partial \mu} + \frac{\partial I(\mu_{db})}{\partial \mu_{db}} \frac{\partial \mu_{db}}{\partial \mu} \tag{5.21}
\]

\[
f_3(\mathbf{x}) = -\frac{\mu_e}{3\pi^2} - \frac{2}{\pi} \int_0^1 dp p^2 \left( \frac{\xi_{\mu_e}}{\sqrt{(\epsilon_p + \bar{\mu})^2 + \Delta^2}} + \frac{\xi_{\mu_e}}{\sqrt{(\epsilon_p - \bar{\mu})^2 + \Delta^2}} \right) + \frac{\partial I(\mu_{ab})}{\partial \mu_{ab}} \frac{\partial \mu_{ab}}{\partial \mu} + \frac{\partial I(\mu_{db})}{\partial \mu_{db}} \frac{\partial \mu_{db}}{\partial \mu} \tag{5.22}
\]

\[
f_4(\mathbf{x}) = -\frac{2}{\pi} \int_0^1 dp p^2 \left( \frac{\xi_{\mu_8}}{\sqrt{(\epsilon_p + \bar{\mu})^2 + \Delta^2}} + \frac{\xi_{\mu_8}}{\sqrt{(\epsilon_p - \bar{\mu})^2 + \Delta^2}} \right) + \frac{\partial I(\mu_{ab})}{\partial \mu_{ab}} \frac{\partial \mu_{ab}}{\partial \mu} + \frac{\partial I(\mu_{db})}{\partial \mu_{db}} \frac{\partial \mu_{db}}{\partial \mu} \tag{5.23}
\]

The problem is to find a solution for \( \mathbf{F} = 0 \) such that the thermodynamic potential (4.13), now a function of \( \mathbf{x} \),

\[
\Omega_{2sc}^{0,T} = \frac{\Delta^2}{4G_D} - \frac{\Lambda^4}{2\pi^2} - \frac{\mu_8^4}{12\pi^2} - 2 \int_0^\Lambda \frac{dp p^2}{\pi^2} \left( \sqrt{(\epsilon_p + \bar{\mu})^2 + \Delta^2} + \sqrt{(\epsilon_p - \bar{\mu})^2 + \Delta^2} \right) + \frac{1}{\pi^2} (I(\mu_{ab}) + I(\mu_{db})) + 48 \lambda \frac{\Delta^4}{G_D^4} \tag{5.24}
\]

be a global minimum with respects to the gap parameter \( \Delta \), and chemical potentials \( \mu_e \) and \( \mu_8 \).
5.3 Numerical Results and EoS on 2SC

The condition that characteristic the BCS-BEC crossover for this model is that the effective chemical potential $\bar{\mu}$ crosses the value of the mass $m$. Fig. 5.1 shows that such condition is never reached, even at zero diquark-diquark repulsion. Thus, the 2SC phase is confined to be on the BCS state. The following can be confirmed in the subsequent figures.

In Fig. 5.2, it is shown a small variation of the gap with respect to the exaggerate value of $\Lambda = 100$. The behavior of the energy density and pressure is also stable: Fig. 5.3 and Fig. 5.4. The baryonic chemical potential in Fig. 5.5 decreases as the coupling increases, however, their effects are nullified by the decrease of the color chemical potential, while the electric chemical potential remain constant.

![Graph showing the effective chemical potential $\bar{\mu}$ vs $\eta$ at different diquark-diquark repulsion $\lambda$.](image)

Figure 5.1: Effective Chemical Potential $\bar{\mu}$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$
Figure 5.2: Gap Parameter $\Delta$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$

Figure 5.3: Energy Density $\varepsilon$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$
Figure 5.4: Pressure Density $p$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$.

Figure 5.5: Chemical Potential $\mu$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$.
Figure 5.6: Chemical Potential $\mu_e$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$

Figure 5.7: Chemical Potential $\mu_s$ vs $\eta$ at different Diquark-Diquark Repulsion $\lambda$
Chapter 6

Numerical Method

In this section, the numerical methods employed during the last chapters are described. As it was mentioned before, the analytical procedure of the studied problems begins defining the Lagrangian density and resulting in the introduction of a grand canonical potential $\Omega$. The numerical problem starts there, and could be formulated as follows: we have a thermodynamic potential, $\Omega$, defined as an integro-differential-equation. $\Omega$ depends on several variables denoted as $x = (x_1, x_2, \ldots, x_n)^T$. Then, it is required to minimize the function $\Omega$ with respect to $n - 1$ of the $n$ variables (gap parameter, and possible neutrality conditions), and besides $\Omega$ must satisfy a density equation of the form $\frac{\partial \Omega}{\partial x_n} + c = 0$. To solve this problem in an efficient way, it is required to have the first and second derivative information from $\Omega$. First derivatives are obtained analytically, and second derivatives (Jacobian) are estimated numerically. With the vector of variables $x$ at hand, the physical quantities of interest, like the pressure and energy densities of the system, are calculate.

The main computations needed are:

- Numerical integration
- Numerical derivatives.
- The solution of a non-linear system of equations

In the following, each one of these points are discussed. The main objective of the numerical analysis will be to obtain results with at least 4 digits of precision. This accuracy has to be tested at the end of each calculation since the behavior of our equations are not trivial, and an appropriate bound could not be made.
6.1 Numerical Integration

The numerical integration is the heart in the evaluation of the non-linear equations. This computation is the deeper in the sense that the error introduced here is propagated through the others calculations. Thus, this error has to be strongly bounded with the least computational work possible.

The integrals appearing on the evaluations are divergent

\[ I = \int_0^{\infty} dk \ g(k) \sim k^4 \bigg|_0^{\infty} \]  

and a proper regularization has to be applied to make them finite. As is usual in a non-renormalized theory, a cutoff ultraviolet parameter \( \Lambda \) cuts the upper limit of the integral. This cutoff is the measure of the energy scale that is been observed, all the variables \( x \) will be normalized with respect to this parameter. Then (6.1) is finite

\[ I = \int_0^{1} dk \ g(k) \]  

where \( g(k) \) is the dispersion relation of the quasiparticles and is positive defined, and is smooth.

In order to decrease the function evaluation, the integral (6.2) is discretized using a Gaussian quadrature instead of the more common trapezoidal or Simpson method. The \( n \)-points Gaussian quadrature is given by

\[ \int_0^{1} dk \ g(k) = \sum_{i=0}^{n} A_i g(k_i) \]  

The nodal points \( x_i \) and the weights \( A_i \) in the Gaussian quadrature are chosen such that the integral (6.3) is exact for polynomials of degree \( 2n + 1 \), as in Ref. [48]. Then, the error term of using approximation (6.3) is given by

\[ E = \frac{g^{(2n)}(\xi)}{(2n)!} \int_0^{1} dk \ \prod_{i=0}^{n-1} (k - k_i)^2 \]  

where \( g^{(2n)}(\xi) \) is the \( 2n \) derivative of \( g \) evaluated at a point \( \xi \in (0, 1) \).
To have an estimate of the error term using Gaussian quadratures, Table 6.1 shows the integral factor on (6.4) for different numbers $n$.

In principle, a Gaussian quadratures with 10 points will be enough considering that $g(k)$ is usually smooth. However, we employ a dynamical choice of quadrature for each problem. First, 50 and 40 points are compared. If both results do not coincide at 10 digits of precision, 50 points are used. Otherwise, 40, and 30 points are compared, and we repeat the procedure until a quadrature is determined. This dynamical election is done at the first run of the optimization algorithm described on Section 6.3.

### 6.2 Numerical Jacobian

As we mentioned before, it is needed to calculate first (Gradient $\nabla f(x)$) and then the second derivatives of a function $f(x)$ (Jacobian $Jf(x)$). Even though it was decided to obtain explicit expressions for the first derivative, the cost of doing it numerically is for free once we have calculated the second derivatives. Then, the analytical expression for $\nabla f(x)$ is used to adjust parameters of the numerical differentiation.

The method of finite difference is used to obtain the gradient and the Jacobian of a

<table>
<thead>
<tr>
<th>$n$</th>
<th>Error Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$g^{n}(\xi)/(2n)! \cdot 1.3950 \times 10^{12}$</td>
</tr>
<tr>
<td>20</td>
<td>$g^{n}(\xi)/(2n)! \cdot 1.2835 \times 10^{-24}$</td>
</tr>
<tr>
<td>30</td>
<td>$g^{n}(\xi)/(2n)! \cdot 1.1720 \times 10^{-36}$</td>
</tr>
<tr>
<td>40</td>
<td>$g^{n}(\xi)/(2n)! \cdot 1.0681 \times 10^{-48}$</td>
</tr>
<tr>
<td>50</td>
<td>$g^{n}(\xi)/(2n)! \cdot 9.7268 \times 10^{-61}$</td>
</tr>
</tbody>
</table>
function $f(x)$. The method will be described here for two variables, however, it can be generalized for any number of variables by repeating the steps for each pair of variables. First, we create a mesh of points around the evaluation point $(x, y)$ as is shown in Fig. 6.1. The space $h$, between adjacent points, is a parameter that are adjusted dynamically.

![Figure 6.1: Discretization](image)

The schemes used to discretized first and second derivatives of $f$ with respect to a variable $x$ are

$$
\frac{\partial f(x, y)}{\partial x} \approx f_x = f(x + h, y) - f(x - h, y)
$$

and similar expressions are obtained for the derivatives with respect to $y$. The mixed derivatives that appears on the Jacobian are discretized applying scheme in Eq. (6.5)
twice

\[
\frac{\partial^2 f(x, y)}{\partial x \partial y} \approx f_{xy} = \frac{1}{2h} \left( \frac{f(x + h, y + h) - f(x - h, y + h)}{2h} \right. \\
\left. - \frac{f(x + h, y - h) - f(x - h, y - h)}{2h} \right)
\]  

(6.6)

\[
\frac{\partial^2 f(x, y)}{\partial y \partial x} \approx f_{yx} = \frac{1}{2h} \left( \frac{f(x + h, y + h) - f(x + h, y - h)}{2h} \right. \\
\left. - \frac{f(x - h, y + h) - f(x - h, y - h)}{2h} \right)
\]  

(6.7)

Using Taylor expansion, it can be shown that the error bound on each finite difference evaluation is \( Err = O(h^2) \). The parameter \( h \) is usually chosen as \( h = \sqrt{\varepsilon} \), where \( \varepsilon \) is the machine precision. However, since the explicit expressions for the first derivatives are already known, they are used to chose the appropriate value of \( h \) dynamically. The initial value is \( h = 10^{-8} \), then, both analytical and numerical first derivatives are compare in order to adjust the value of \( h \). If \( h = 10^{-8} \) is introducing roundoff errors, \( h \) will be decreased by a factor of 10.

Then, the gradient of \( f(x, y) \)

\[
\nabla f(x, y) = \begin{pmatrix} \frac{\partial f(x, y)}{\partial x} \\ \frac{\partial f(x, y)}{\partial y} \end{pmatrix}
\]  

(6.8)

is discretized using the skins on (6.5), (6.6) and (6.7)

\[
\nabla f(x, y) \approx \begin{pmatrix} f_x \\ f_y \end{pmatrix}
\]  

(6.9)

The Jacobian of \( f(x, y) \)

\[
J f(x, y) = \begin{pmatrix} \frac{\partial^2 f(x, y)}{\partial x^2} & \frac{\partial^2 f(x, y)}{\partial x \partial y} \\ \frac{\partial^2 f(x, y)}{\partial y \partial x} & \frac{\partial^2 f(x, y)}{\partial y^2} \end{pmatrix}
\]  

(6.10)

is discretized as

\[
J f(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}
\]  

(6.11)

Notice that the Function, Gradient, and Jacobian at pint \((x, y)\) are evaluated using just the 9 points shown in the mesh of Fig. 6.1.
6.3 Minimization Method

The problem described at the beginning of the chapter is strictly formulated as a minimization problem with a constrain

\[
\min_{x_1, \ldots, x_{n-1}} \Omega(x) \quad \text{subject to} \quad \frac{\partial \Omega}{\partial x_n} + c = 0 \quad (6.12)
\]

There are different methods to approach (6.12), sequential quadratic programming, interior point methods, quadratic penalty methods, etc. However, the number of variables \( n \) in our case is small, from two to four, and the function evaluation is not too expensive; such that we can reformulate problem (6.12) in a little less efficient way that will save some complication inherent in the minimization problem.

The problem, in (6.12), could be redefined as

\[
r(x) = 0 \quad \text{subject to} \quad C > 0 \quad \text{and} \quad |x^*|_{\infty} < 1 \quad (6.13)
\]

where \( r(x) \) is the residual function

\[
r(x) = \begin{pmatrix}
r_1(x) \\
r_2(x) \\
\vdots \\
r_n(x)
\end{pmatrix}
\]

whit \( r_1(x) = \frac{\partial \Omega}{\partial x_1}, \ r_2(x) = \frac{\partial \Omega}{\partial x_2}, \ldots, r_{n-1}(x) = \frac{\partial \Omega}{\partial x_{n-1}}, \) and \( r_n(x) = \frac{\partial \Omega}{\partial x_n} + c. \) The last equation represents the density condition, and \( C \) is the curvature of \( \Omega \) with respect to the variables \( x_1, x_2, \ldots, x_{n-1}. \) In other words, the solution \( x^* \) of (6.13) also has to be a minimum of \( \Omega, \) but not any minimum, it has to be a global minimum where \( |x^*|_{\infty} < 1. \) The last restriction is imposed because of the physics that is being studied. All the variables are normalized with respect to a momentum cutoff \( \Lambda. \) This value is our energy scale and non observable could be even close to its value.

To solve (6.13), a Newton-like method with a merit function approach is used. Taking into account the characteristics of the method needed, the name of the method is a merit
function approach to solve non-linear equations using a globalized Newton-like method with a Line Search and curvature constrain. The name of the method summarizes the algorithms it uses (for more information about the different parts of the algorithm see [49]).

6.3.1 Merit Function

The merit function is designed to measure the progress made by each Newton step. In our case the function is proportional to the residual function r(x)

\[ f(x) = \frac{1}{2}|r(x)|^2 \]  \hspace{1cm} (6.15)

Equation (6.15) is the same merit function minimized in the Least Square Problem. However, in this case, the number of equations is equal to the number of variables; then, the minimum \( x^* \), if it exists, it will make \( f(x^*) = 0 \).

6.3.2 Newton Method

Newton method is an iterative method, which forms a linear model \( M_k(p_k) \) of the function \( f(p_k) \) at each iteration \( k \). The linear model is obtained from the first two terms of the Taylor polynomial of the residual function \( r(x) \). The root of the model \( M_k(p_k) = 0 \), gives the step \( p_k \) to the next approximation

\[ x_{k+1} = x_k + p_k \]  \hspace{1cm} (6.16)

The model is

\[ M_k(p_k) \equiv r_k + J_k p_k \]  \hspace{1cm} (6.17)

where \( r_k \) is a short notation for \( r(x_k) \), \( J_k \) is the Jacobian of \( r_k \) and \( p_k \) is the Newton step

\[ p_k = -J_k^{-1} r_k \]  \hspace{1cm} (6.18)

If the first approximation \( x_0 \) is close to the solution \( x^* \), Newton method is guaranteed to converge q-quadratically. Its convergence is its main advantage when the approximation is close to the solution.
Algorithm 1 Newton-Method Root Finder

Require: $x_0$ and $Tol$

repeat
    Obtain Newton Step
    $$p_k = -J_k^{-1} r_k$$
    Update Solution
    $$x_{k+1} = x_k + p_k$$
until $f(x_{k+1}) < Tol$

After a Newton iteration, if the new approximation $x_{k+1}$ is close enough to the solution of (6.15), the method has succeeded; otherwise, the procedure is repeated. To measure how close the approximation $x_{k+1}$ is to the solution, the variable $Tol$ is introduced. Then, the criteria to stop the method is $f(x_{k+1}) < Tol$. The variable $Tol$ is chosen dynamically. The initial value is $Tol = 10^{-8}$, and will be adjusted, by a factor of 10, if needed at the end of the algorithm if the four digits of precision are not met.

6.3.3 Line Search

The Newton direction, $p_k$, is designed to be a descendent direction

$$p_k^T \nabla f_k = -p_k^T J_k^T r_k = -|r_k|^2 < 0$$

(6.19)

if a sufficient small step is taken, it will decrease the merit function and a progress to the solution is guaranteed. However, if big quantities of small steps are taken, the q-quadratic convergency of a full Newton step is lost. There are two common strategies to approach this problem a backtracking line search and a trust region method. Both strategies have similar efficiency, and globalize the Newton method ensuring a progress to the solution at each step. Thus, the strategy used is the line search because is more simple.

The line search needs a condition to know if the considered step is accepted or not. The requirement imposed is that the step reduces the merit function (6.15) in proportion to the
**Algorithm 2 Line Search**

**Require:** $p_k$, $J_{max}$

Set $\alpha = 1$

for $J = 0, 1, \ldots, J_{max}$ do

if $f(x_k + \alpha_k p_k) > f(x_k) + c \alpha_k \nabla f(x_k)$ then

return $\alpha$

end if

$\alpha = \frac{1}{2} \alpha$

end for

return $\alpha$

the step length $\alpha$ and the $\nabla f$.

$$f(x_k + \alpha_k p_k) > f(x_k) + c \alpha_k \nabla f(x_k) \quad (6.20)$$

Notice that close to the solution, $\nabla f \approx 0$. The parameter $c$ should be small; the algorithms takes $c = 10^4$ which has been used with very good results [50].

The line search strategy is then to modify Newton step as

$$x_k + \alpha_k p_k \quad (6.21)$$

first, the full Newton step is tested with $\alpha_k = 1$, if it does not satisfy (6.20) the step is decreased by a factor $\frac{1}{2}$ and tested again, until a proper step is found or a maximum number of iterations $J_{max}$ has been reached. Restricting the number of iterations prevents to get stuck accepting small steps, but could let it increase the merit function. This is a risk that has to be taken.

### 6.3.4 Global Strategy

There are several points that have to be taken into account in the developing of a general strategy. In the search for a global minimum, a common strategy is to create a discrete mesh of points and use the value of each point as initial approximation in Newton
method. The solution is expected to satisfy $|x|_{inf} < 1$ due to physical restriction. Besides, in this region, our function has probably just one minimum. Then, we create a mesh, with no more than three or four points per dimension, inside the unit square like is shown in Fig. 6.2.

![Figure 6.2: Mesh of Initial Points](image)

Each point in the mesh is used as a starting point $x_0$ in Newton method and will converge to its own local minimum. When the method finish exploring each initial point, the solutions found are compared. The one that minimizes the potential $\Omega$ and satisfies the restriction on (6.13) is chosen. Finally, all the parameters of interest like pressure and energy density are calculated, and the accuracy of the results are checked. It is expected that changing the variables by a value proportional to $\sim 10^{-5}$, the parameters of interest does not change by more than a quantity $\sim 10^{-5}$. If the parameter changes, the precision is adjusted, and the calculation continues.

The pseudo-code for the complete method are shown below. It does a very good job on all the cases studied on the last chapters, and in reproducing some other result already published similar to the ones tackled on this thesis.
Algorithm 3 Newton-Method Root Finder

Require: $Mesh = \{M_1, M_2, \ldots, M_N\}, IT_{max}$

Begin Local Search

for $k = 1, k \leq N, k \rightarrow k + 1$ do

\begin{align*}
    x_0 &= M_k \\
    \text{Begin Newton Iteration} \\
    \text{for } IT = 0, IT < IT_{max}, IT \rightarrow IT + 1 \text{ do} \\
    \quad \text{Obtain Newton Step} \quad p_k = -J_k^{-1}r_k \\
    \quad \text{Run Line search Algorithm} \\
    \quad \text{Update Solution} \quad x_{k+1} = x_k + p_k \\
    \quad \text{Check toping Criteria} \quad f(x_{k+1}) < Tol
    \end{align*}

end for

if $IT < IT_{max}$ and $|x|_{inf} < 1$ and $C > 0$ then

    Solution Found.........!!! $SOL(k) = x_{k+1}$

else

    Solution not Found...!!! $SOL(k) = -1$

end if

end for

Find global Solution

return Global Solution
Chapter 7

Discussion and Remarks

During this thesis, we study the evolution of the EoS of different phases of strong-interacting-quark-matter for vanishing temperatures and densities beyond hadronic matter. Under these conditions, the color-superconducting phases, characterized by diquark condensates, emerge. The goal was to explore the possible crossover from the BCS phase to the BEC one and its implications for the EoS of the different systems.

We started by considering a simple fermion system. In this system, we found that, as the strength of the attractive coupling between quarks increases, the chemical potential turns from being larger than the quark mass to being smaller, an indication of the BCS-BEC crossover. This transition was confirmed in the characteristics of its quasiparticle spectrum. A consequence of the crossover to BEC phase, in this system, is that the matter pressure decays to zero, and even reaches negative values. Then, we considered the introduction of a repulsive force between diquarks. In this case, the pressure collapse was prevented by a sufficiently strong diquark-diquark repulsion. But the implications of reaching a stable state in the strong diquark coupling regime is that the system maintains its BCS nature for the whole range of values of the quark-coupling. Thus, this self-interacting diquark system will not form a Bose gas.

The second system considered was quark matter in the CFL phase. In this case, we observed a similar behavior that in the first system, a BCS-BEC crossover for increasing values of the diquark coupling, while is characterized by the decrease of the chemical potential to values smaller than the quark mass. However, when introducing the diquark repulsion, the system does not change their behavior for a wide range of the diquark coupling values. Its nature is extremely stable being only affected by the quark coupling,
but ignores completely their diquark-diquark repulsion.

Finally, we considered a more realistic model for the interior of a compact start, the 2SC phase at strong coupling. The effective chemical potential in this phase was fixed to be around 400 MeV, and their gap parameter around 10 MeV, condition that are estimated to be found in the interior of neutron stars, and in the phase diagram obtained on [36] for the 2SC phase at $\eta = 1$. Moreover, we considered neutrality of electric and color charge. In this system we observed that the effective chemical potential does not change with the increase of both quark and diquark coupling. The baryonic chemical potential decreases with the quark coupling, but its effect is compensated by the decrease of the color chemical potential, while the electric chemical potential remain fixed.

The last result implies that even when in a simple model, and in a ideal CFL phase with a very high coupling, the BCS-BEC crossover is realized, in a more realistic situation for strongly coupled quark matter this phenomena does not appears.

On the other hand, the numerical methods developed during this theses have proved to work efficiently. The dynamical mode in which some of their parameters change allows to face a wide range of different integro-differential equation just introducing the specifications of the problem in the main file without worried about the details of the calculations.

7.1 Future Work

A nontrivial problem that is worth to investigate is the effect of the EoS of the different systems of none general four-fermion interaction that can include for example vector-vector interaction, or other more complex tensor structures.

Another interesting question to be studied in this scenario is the possible effect of an applied magnetic field. As already estimated in Ref. [40], magnetic fields of order $10^{19} - 10^{20}$ G can coexist in the core of neutron stars with quark matter. By increasing the magnetic field strength, $\Delta$ increases [51], and the system is lead to crossover from the BEC region to the BCS one [16]. At very strong magnetic fields, when all the particles
are localized in the lowest Landau level, only BCS diquarks are allowed [16]. On the other hand, the pure magnetic contribution to the pressure is negative [47]. Thus, the magnetic field will have a double effect in the pressure whose consequences should be elucidated in the frame of the strongly interacting system.

In those regions of relatively low densities there is of course the possibility that the repulsion between the diquarks catalyzes a phase transition to other ground state configurations such as a hadronic phase with a well identified fermion nature able to produce the degeneracy pressure needed to compensate for the gravitational pull. Other possibilities to be investigated are the viability, through their EoS, of some inhomogeneous phases, such as those formed by density waves [52], quarkyonic chiral spirals [53], inhomogeneous Fulde-Ferrel state [54] or quark clusters in solid or liquid states [55], that can in principle be realized in quark matter at moderate density.
References


Curriculum Vitae

Israel Portillo Vazquez was born in Hidalgo del Parral, Chihuahua, Mexico. In June, 2008, Israel obtained his bachelor degree in electronic engineering at the Instituto Tecnologico de Chihuahua, in Chihuahua, Chihuahua, Mexico. Form January of 2009 to June 2011, Israel studied a MS in Physics at the University of Texas at El Paso (UTEP). During this time, Israel worked in collaboration with the theoretical-high-energy-physics group at UTEP headed by Dr. Efrain Ferrer and Dr. Vivian Incera. In May 2011, Israel was honored by the Physics Department at UTEP for his outstanding research work. Israel continue his Ph. D. studies at UTEP in the Computational Science Program (CPS), where now, with the defence of this thesis, is becoming a CPS Ph. D. candidate.

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