Surface And Guided Waves On Structured Surfaces

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SURFACE AND GUIDED WAVES ON STRUCTURED SURFACES AND INHOMOGENEOUS MEDIA

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to my

MOTHER and FATHER

with love
SURFACE AND GUIDED WAVES ON STRUCTURED SURFACES AND INHOMOGENEOUS MEDIA

by

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Abstract

Surface and guided waves on structured surfaces and inhomogeneous media studies the propagation of waves in systems with spatially varying parameters. In the rainbow case (chapter 1), the dielectric constant changes with coordinates. In the cylinder case: boundary and the metal (chapter 2), it is a curved surface. Finally, in the last case (chapter 3), the dielectric constant changes in z-direction.

In chapter 1, we study theoretically the propagation of a wave packet that is a superposition of three s-polarized guided waves with different frequencies in a planar waveguide consisting of a dielectric medium with a graded index of refraction, sandwiched between perfectly conducting walls. The electric field at each point within the waveguide is calculated, and it is shown that each of the constituent modes ceases to propagate at a specific distance along the waveguide that depends on its frequency and on the geometrical and material parameters defining the waveguide. This simple model displays the phenomenon of rainbow trapping of guided waves in an explicit fashion, without the use of a negative index metamaterial.

In chapter 2, the dispersion relation is derived and solved for s-polarized surface polaritons propagating circumferentially around a portion of a cylindrical interface between vacuum and an isotropic dielectric. In the case that the dielectric is convex toward the vacuum these modes are found to be radiative, and consequently are attenuated as they propagate on the cylindrical surface. When the dielectric is concave toward the vacuum the resulting surface polaritons are nonradiative and propagate unattenuated on the cylinder. Such modes do not exist in the case of a planar interface between a homogeneous isotropic dielectric and vacuum.

In chapter 3, analytic expressions for the dispersion relation and electric field profiles of guided waves supported by an asymmetric graded-index dielectric waveguide are derived. The system studied consists of vacuum in the region $z < 0$, and a dielectric medium in the
region \( z > 0 \) whose dielectric permittivity decreases continuously with increasing distance into it from the interface \( z = 0 \) according to \( \epsilon(z) = n_0^2\{1 - (1/g)[1 - (1 + (z/L)^2)]\} \). Here \( n_0 \) is the (real) index of refraction of the medium at \( z = 0 \), and it is assumed that \( g > 1 \). It is found that for frequencies below a certain critical value given by \( \Omega_c = (\sqrt{g}c/2L) \), where \( c \) is the speed of light in vacuum, the dispersion curve consists of a single branch that exists in a narrow spectral range. Its electric field decays exponentially with increasing distance into each medium. For frequencies above this critical value the dispersion curve possesses several branches. The corresponding electric field decays exponentially with increasing distance into the vacuum, and decays in an oscillatory fashion with increasing distance into the graded-index dielectric medium. The number of nodes in the latter field equals the branch number, starting with zero for the lowest frequency branch.
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Chapter 1

Rainbow Trapping of Guided Waves

There is currently a great deal of interest in slow light [1]. Experiments carried out involving ultracold atomic gases showed that light could be slowed down in traversing them [2] and even stopped [3]. The ability to slow light is of fundamental interest, but it also has practical uses in classical and quantum optical communication [4].

It is therefore perhaps not surprising that with an increase in the use of surface plasmon polaritons in nanoscale devices attention has also begun to be directed at the possibility of slowing down these surface electromagnetic waves with the expectation that this will increase the scope of photonic devices based on them.

It has been known for some time [5] that the dispersion curve of a surface plasmon polariton propagating normal to the grooves and ridges of a classical metallic grating approaches the boundary of the one-dimensional first Brillouin zone defined by the period of the grating with zero slope, due to periodicity and time reversal symmetry. Thus the group velocity of the surface plasmon polariton slows down as the zone boundary is approached, and vanishes at the boundary itself. The slowing down of a surface plasmon polariton wave packet by this mechanism was experimentally observed recently [6].

This work was followed by several papers in which the slowing down of surface plasmon polaritons and guided waves was studied on the basis of a different approach. In a planar waveguide consisting of a dielectric layer sandwiched between two metals the number of modes than can be supported by this structure depends on the dielectric constant of the layer and its thickness. As the thickness of the dielectric layer decreases the number of modes supported by the waveguide decreases. Thus, if the waveguide does not have parallel walls, but instead has a thickness that decreases with increasing distance along it, at critical
distances along the wave-guide the number of modes decreases successively by one. These critical distances depend on the wavelengths of the modes. The group velocity of the mode that stops propagating at each of these distances is zero. When the thickness of the dielectric layer reaches the value where the waveguide no longer supports a mode, the group velocity of the only remaining waveguide mode at the corresponding distance along the waveguide is zero. If the waveguide is illuminated by polychromatic light consisting of several discrete wavelengths (a light rainbow), different light colors stop propagating at different positions along the waveguide. Thus, the light rainbow has been stopped and trapped.

This concept was used as the basis of an investigation by Tsakmakidis et al. [7] of the trapping of a guided wave packet in which a semi-infinite portion of the dielectric layer had a positive refractive index and parallel walls, while the remaining semi-infinite portion of the layer had a negative refractive index and a slowly decreasing thickness. The substrate and superstrate consisted of positive index dielectric materials. Trapping of the different frequency components of the guided wave packet at different points of the wave-guide was predicted by these authors. This effect was observed experimentally by Zhao et al. [8].

A trapped rainbow was predicted and observed experimentally by Smolyaninova et al. [9] in a system of which a spherical metal structure was placed on a metal substrate. The air between the two metallic surfaces had a circularly symmetric structure whose thickness increased with increasing distance along the substrate from the point of its contact with the sphere. When illuminated from the side by polychromatic light, this structure displayed the trapping of its different wavelength components at specified distances along the substrate from the point of contact.

A variant of the tapered waveguide approach was used by Gan et al. [10] in a theoretical study of the trapping of surface plasmon polaritons. We have noted above that the dispersion curve of the lowest frequency branch of the dispersion relation for surface plasmon polaritons propagating normally to the grooves and ridges of a classical grating approaches the boundary of the first Brillouin zone defined by the period of the grating with a zero
slope and a value (the cutoff frequency) that depends on the depth of the grooves. The deeper the grooves, the lower the cutoff frequency. Gan et al. considered a metallic lamellar grating the depth of whose grooves increased linearly and slowly along the grating. When a surface plasmon polariton of a given frequency propagates along such a grating, it can reach a point along it at which the groove depth has a value such that an infinite grating with that groove depth would have a cut off frequency below that of the surface plasmon polariton. At that point the surface wave ceases to propagate: its frequency falls in the region of the gap in the dispersion relation of an infinite grating with that groove depth. The surface plasmon is trapped at this point. If the incident surface plasmon polariton is a superposition of surface waves with different wavelengths, the different wavelength components will be trapped at different points along the grating.

What characterizes these studies of rainbow trapping is the absence of a quantitative theory underlying them. They are all based on the result that the narrower the thickness of a waveguide the fewer propagating modes it can support. While this result may be sufficient to estimate the points along the waveguide where these modes disappear one by one, it tells us nothing about the amplitudes of the propagating modes, it neglects the backscattering of the waves at the points where the group velocity vanishes, and it does not show how sharp the trapping phenomenon is, in view of the finite lengths of the structures studied.

In this chapter we study the propagation of electromagnetic waves through a waveguide with a linearly graded dielectric constant, with a view to addressing the points raised in the preceding paragraph.

Taking the point of view that the use of a structure incorporating a negative index metamaterial is an unnecessary complication, the structure we study consists of a dielectric medium that occupies the region $-d < x_3 < d$, and has a graded dielectric constant given
by

\[ \epsilon(x_1) = \begin{cases} 
\epsilon_\infty & x_1 < -L \\
\frac{1}{2}(\epsilon_\infty + 1) - \frac{1}{2L}(\epsilon_\infty - 1)x_1 & -L < x_1 < L \\
1 & x_1 > L.
\end{cases} \] (1.1)

For simplicity we assume that the region \( x_3 > d \) and \( x_3 < -d \) are filled with a perfect conductor. The electromagnetic field incident in this graded index waveguide from the region \( x_3 = -\infty \) is a superposition of \( N \) s-polarized modes, each with a different frequency, supported by an infinitely long waveguide of constant thickness \( 2d \) and filled with a uniform dielectric medium whose dielectric constant is \( \epsilon_\infty \). The intensity of the electric field in this graded index waveguide will be calculated as a function of \( x_1 \) and \( x_3 \), from which the trapping of the incident rainbow can be observed.

The single nonzero component of the electric field in the waveguide, \( E_2(x_1, x_3|\omega) \), satisfies the Helmholtz equation

\[ \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2} + \epsilon(x_1)\frac{\omega^2}{c^2} \right) E_2(x_1, x_3|\omega) = 0 \] (1.2)

in the domain \(-\infty < x_1 < \infty, -d < x_3 < d\), subject to vanishing boundary conditions on the planes \( x_3 = \pm d \), and the continuity of \( E_2(x_1, x_3|\omega) \) and \( \partial E_2(x_1, x_3|\omega)/\partial x_1 \) at \( x_1 = \pm L \).

We solve Eq. (1.2) by the method of the separation of variables. If we express \( E_2(x_1, x_3|\omega) \) in the form

\[ E_2(x_1, x_3|\omega) = f(x_1)g(x_3), \] (1.3)

we find that \( f(x_1) \) and \( g(x_3) \) satisfy the equations

\[ \frac{d^2}{dx_1^2}f(x_1) + [\epsilon(x_1)\frac{\omega^2}{c^2} - \alpha^2]f(x_1) = 0 \] (1.4a)

\[ \frac{d^2}{dx_3^2}g(x_3) + \alpha^2 g(x_3) = 0. \] (1.4b)

where \( \alpha^2 \) is the separation constant.
The solution of Eq. (1.4b) that vanishes at $x_3 = -d$ is

$$g(x_3) = A \sin \alpha(x_3 + d). \quad (1.5)$$

The requirement that $g(d) = 0$ yields the result that

$$\alpha = \frac{n\pi}{2d}, \quad n = 1, 2, 3, \ldots \quad (1.6)$$

Consequently we can write the solution of Eq. (1.4b) in the form

$$g_n(x_3) = A_n \sin \frac{n\pi}{2d}(x_3 + d), \quad n = 1, 2, 3, \ldots \quad (1.7)$$

When Eq. (1.6) is substituted into Eq. (1.4a), we can write the resulting equation as

$$\frac{d^2}{dx_1^2} f_n(x_1) + [\epsilon \omega^2 c^2 - \left(\frac{n\pi}{2d}\right)^2] f_n(x_1) = 0. \quad (1.8)$$

We seek the solution of this equation in each of the three regions $(-\infty, -L), (-L, L)$, and $(L, \infty)$.

$x_1 < -L$: In this region Eq. (1.8) becomes

$$\frac{d^2}{dx_1^2} f_n(x_1) + [\epsilon_\infty \omega^2 c^2 - \left(\frac{n\pi}{2d}\right)^2] f_n(x_1) = 0. \quad (1.9)$$

The solution of this equation is

$$f_n(x_1) = a^{(1)}_n \exp\{i[\epsilon_\infty \omega^2 c^2 - \left(\frac{n\pi}{2d}\right)^2]^\frac{1}{2} x_1\}$$

$$+ a^{(2)}_n \exp\{-i[\epsilon_\infty \omega^2 c^2 - \left(\frac{n\pi}{2d}\right)^2]^\frac{1}{2} x_1\}. \quad (1.10)$$

The first term can be regarded as an incident field, while the second term can be regarded as a reflected wave.

$-L < x_1 < L$: In this region $f_n(x_1)$ satisfies the equation

$$\frac{d^2}{dx_3^2} f_n(x_1) + \left\{\omega^2 \left[\frac{1}{2} (\epsilon_\infty + 1) - \frac{1}{2L}(\epsilon_\infty - 1)x_1\right] \right.\]$$

$$\left. - \left(\frac{n\pi}{2d}\right)^2\} f_n(x_1) = 0. \quad (1.11)$$
The solution of this equation is

\[ f_n(x_1) = b_n^{(1)}Ai\left(\left(\frac{a\omega^2}{c^2}\right)^{\frac{1}{3}}(x_1 - \beta_n)\right) + b_n^{(2)}Bi\left(\left(\frac{a\omega^2}{c^2}\right)^{\frac{1}{3}}(x_1 - \beta_n)\right), \]  \hspace{1cm} (1.12)

where

\[ \beta_n = b - \frac{1}{a}\frac{(n\pi)^2}{2d\omega}, \]  \hspace{1cm} (1.13)

with

\[ a = \frac{1}{2L}(\epsilon_{\infty} - 1), \quad b = \frac{1}{2}(\epsilon_{\infty} + 1). \]  \hspace{1cm} (1.14)

In Eq. (1.12) \( Ai(z) \) and \( Bi(z) \) are two linearly independent Airy functions [11].

\( x_1 > L \): In this region the equation satisfied by \( f_n(x_1) \) is

\[ \frac{d^2}{dx_1^2}f_n(x_1) + \left[\frac{\omega^2}{c^2} - \left(\frac{n\pi}{2d}\right)^2\right]f_n(x_1) = 0 \]  \hspace{1cm} (1.15)

The solution of this equation is

\[ f_n(x_1) = c_n\exp\left\{i\left[\frac{\omega^2}{c^2} - \left(\frac{n\pi}{2d}\right)^2\right]^{\frac{1}{2}}x_1\right\}, \]  \hspace{1cm} (1.16)

which corresponds to a transmitted wave in this region. There is no wave incident from \( x_1 = \infty \).

The boundary conditions satisfied by \( f_n(x_1) \) are the continuity of \( f_n(x_1) \) and of \( df_n(x_1)/dx_1 \).
at $x_1 = -L$ and at $x_1 = L$. At $x_1 = -L$ we obtain the pair of equations

$$
\begin{align}
& a_n^{(1)} \exp\{-i[\epsilon_\infty \omega^2 / c^2 - (n\pi / 2d)^2]^{1/2} L\} \\
& + a_n^{(2)} \exp\{i[\epsilon_\infty \omega^2 / c^2 - (n\pi / 2d)^2]^{1/2} L\} \\
& = b_n^{(1)} \text{Ai}\left(\left(\frac{a\omega^2}{c^2}\right)^{1/3} (-L - \beta_n)\right) \\
& + b_n^{(2)} \text{Bi}\left(\left(\frac{a\omega^2}{c^2}\right)^{1/3} (-L - \beta_n)\right) \\
& i[\epsilon_\infty \omega^2 / c^2 - (n\pi / 2d)^2]^{1/2} \\
& \times (a_n^{(1)} \exp\{-i[\epsilon_\infty \omega^2 / c^2 - (n\pi / 2d)^2]^{1/2} L\} \\
& - a_n^{(2)} \exp\{i[\epsilon_\infty \omega^2 / c^2 - (n\pi / 2d)^2]^{1/2} L\}) \\
& = \left(\frac{a\omega^2}{c^2}\right)^{1/4} \left[b_n^{(1)} \text{Ai}'\left(\left(\frac{a\omega^2}{c^2}\right)^{1/3} (-L - \beta_n)\right)\right] \\
& + b_n^{(2)} \text{Bi}'\left(\left(\frac{a\omega^2}{c^2}\right)^{1/3} (-L - \beta_n)\right),
\end{align}
\tag{1.17a}$$

where the prime denotes differentiation with respect to argument. At $x_1 = L$ we obtain a
second pair of equations:

\[
\begin{align*}
&b_n^{(1)} Ai \left( \left( \frac{\omega^2 c^2}{a^2} \right)^{\frac{1}{3}} (L - \beta_n) \right) \\
&+ b_n^{(2)} Bi \left( \left( \frac{\omega^2 c^2}{a^2} \right)^{\frac{1}{3}} (L - \beta_n) \right) \\
&= c_n \exp \left\{ i \left[ \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right] \frac{1}{2} L \right\} \\
&= i \left[ \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right] \frac{1}{2} c_n \\
&\times \exp \left\{ i \left[ \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right] \frac{1}{2} L \right\}.
\end{align*}
\]  

\[
(1.18a)
\]

Equations (1.17) and (1.18) allow us to obtain the coefficients \(a^{(2)}_n\), \(b^{(1)}_n\), \(b^{(2)}_n\), and \(c_n\) in terms of \(a^{(1)}_n\). Thus, we rewrite Eqs. (1.17) and (1.18) in the matrix form

\[
M^{(n)} \begin{pmatrix} a^{(2)}_n \\ b^{(1)}_n \\ b^{(2)}_n \\ c_n \end{pmatrix} = \begin{pmatrix} a^{(1)}_n \\ a^{(1)}_n \\ 0 \\ 0 \end{pmatrix} 
\]  

\[
(1.19)
\]

where the nonzero elements of the matrix \(M^{(n)}\) are presented in Appendix A. We can then write

\[
\begin{pmatrix} a^{(2)}_n \\ b^{(1)}_n \\ b^{(2)}_n \\ c_n \end{pmatrix} = N^{(n)} \begin{pmatrix} a^{(1)}_n \\ a^{(1)}_n \\ 0 \\ 0 \end{pmatrix} 
\]  

\[
(1.20)
\]
where $N^{(n)}$ is the matrix inverse to $M^{(n)}$. Therefore we have the results

\begin{align}
    a_n^{(2)} &= \left( N_{11}^{(n)} + N_{12}^{(n)} \right) a_n^{(1)} \quad (1.21a) \\
    b_n^{(1)} &= \left( N_{21}^{(n)} + N_{22}^{(n)} \right) a_n^{(1)} \quad (1.21b) \\
    b_n^{(2)} &= \left( N_{31}^{(n)} + N_{32}^{(n)} \right) a_n^{(1)} \quad (1.21c) \\
    c_n &= \left( N_{41}^{(n)} + N_{42}^{(n)} \right) a_n^{(1)}. \quad (1.21d)
\end{align}

We now assume that the portion of the waveguide in the region $x_1 < L$ is a single mode waveguide, i.e. we assume that $n = 1$. From Eq. (1.10) we see that we must have

\[ \sqrt{\epsilon_\infty} \frac{\omega}{c} > \frac{\pi}{2d} \quad (1.22) \]

and

\[ \sqrt{\epsilon_\infty} \frac{\omega}{c} < \frac{n\pi}{2d} \quad n \geq 2. \quad (1.23) \]

These inequalities restrict $\omega/c$ to the interval

\[ \frac{1}{\sqrt{\epsilon_\infty}} \frac{\pi}{2d} < \frac{\omega}{c} < \frac{2}{\sqrt{\epsilon_\infty}} \frac{\pi}{2d}. \quad (1.24) \]

At the same time we wish to have no propagating modes in the region $x_1 > L$. From Eq. (1.16) we see that for this to be the case we must satisfy the inequality

\[ \frac{\omega}{c} < \frac{\pi}{2d}. \quad (1.25) \]

We will assume that $\epsilon_\infty$ is sufficiently greater than unity that $2/\sqrt{\epsilon_\infty}$ is smaller than unity. Therefore the right-hand inequality in (1.24) is more restrictive than the inequality (1.25). If we introduce the dimensionless frequency $\Omega$ by

\[ \frac{\omega}{c} = \Omega \frac{\pi}{2d}, \quad (1.26) \]

the inequalities (1.24) become

\[ \frac{1}{\sqrt{\epsilon_\infty}} < \Omega < \frac{2}{\sqrt{\epsilon_\infty}}. \quad (1.27) \]
We will choose for $\epsilon_\infty$ the value that corresponds to silicon, namely $\epsilon_\infty = 12$. With this value of $\epsilon_\infty$, the inequalities (1.27) become

$$0.2887 < \Omega < 0.5774.$$  (1.28)

With these results in hand we will assume that the incident electric field is a wave packet formed by the superposition of $N = 3$ modes whose frequencies $\Omega_j (j = 1, 2, 3)$ satisfy the inequalities (1.28), namely

$$\Omega_1 = 0.35, \quad \Omega_2 = 0.45, \quad \Omega_3 = 0.55.$$  (1.29)

It now remains to determine the half-width $d$ of the waveguide and the length $2L$ of the portion filled with the graded index dielectric medium. We do this by first noting that the wavelength $\lambda$ of the mode of the frequency $\omega$ in the region $x_1 < -L$ is obtained from the relation

$$\sqrt{\epsilon_\infty} \frac{\omega}{c} = \frac{2\pi}{\lambda},$$  (1.30)

so that

$$\lambda = \frac{4d}{\sqrt{\epsilon_\infty} \Omega},$$  (1.31)

where we have used the relation (1.26). The wavelengths corresponding to the frequencies $\Omega_1, \Omega_2, \Omega_3$ are therefore given by

$$\lambda_1 = 3.2991d$$  (1.32a)

$$\lambda_2 = 2.5660d$$  (1.32b)

$$\lambda_3 = 2.0995d.$$  (1.32c)

We will choose for $\lambda_1$ the value $\lambda_1 = 0.6328\mu m$. It follows from Eqs. (1.32) that $\lambda_2 = 0.4922\mu m$, and $\lambda_3 = 0.4027\mu m$. From the same equations we find that $d = 0.1918\mu m$.

In choosing a value for $L$ we wish to make the ratio $d/L$ sufficiently small that the dielectric constant within the region $-L < x_1 < L$ of the waveguide is changing slowly with $x_1$. We have chosen the value $L = 6\mu m$, which yields the ratio $d/L = 0.032$. 

10
The incident electric field in the region $x_1 < -L$ can be written in the form

$$E_2(x_1, x_3)_{\text{inc}} = \sum_{j=1}^{3} a_1^{(1j)} \sin \frac{\pi}{2d} (x_3 + d)$$

$$\times \exp \left\{ \frac{i \pi}{2d} [\epsilon_\infty \Omega_j^2 - 1]^\frac{1}{2} x_1 \right\}, \quad (1.33)$$

where $\omega_j/c = \Omega_j(\pi/2d)$. Since the scattering problem is a linear one, the reflected field in the region $x_1 < -L$ is given by

$$E_2(x_1, x_3)_{\text{ref}} = \sum_{j=1}^{3} \left( N_{11}^{(1j)} + N_{12}^{(1j)} \right) a_1^{(1j)}$$

$$\sin \frac{\pi}{2d} (x_3 + d) \exp \left\{ -i \frac{\pi}{2d} [\epsilon_\infty \Omega_j^2 - 1]^\frac{1}{2} x_1 \right\}. \quad (1.34)$$

Similarly, the transmitted field in the region $x_1 > L$ is

$$E_2(x_1, x_3)_{\text{tr}} = \sum_{j=1}^{3} \left( N_{41}^{(1j)} + N_{42}^{(1j)} \right) a_1^{(1j)}$$

$$\sin \frac{\pi}{2d} (x_3 + d) \exp \left\{ i \frac{\pi}{2d} [\Omega_j^2 - 1]^\frac{1}{2} x_1 \right\}. \quad (1.35)$$

The field inside the region $-L < x_1 < L$ containing the graded index material is

$$E_2(x_1, x_3)_{\text{gr}} = \sum_{j=1}^{3} a_1^{(1j)} \sin \frac{\pi}{2d} (x_3 + d)$$

$$\times \left\{ \left( N_{21}^{(1j)} + N_{22}^{(1j)} \right) Ai \left[ \left( a \frac{\omega_j^2}{c^2} \right)^\frac{1}{2} \left( x_1 - \beta_1^{(1j)} \right) \right] \right.$$ \n
$$+ \left( N_{31}^{(1j)} + N_{32}^{(1j)} \right) Bi \left[ \left( a \frac{\omega_j^2}{c^2} \right)^\frac{1}{2} \left( x_1 - \beta_1^{(1j)} \right) \right] \right\}. \quad (1.36)$$

The matrix elements $N_{mn}^{(1j)}$ are the elements of the matrix inverse to the matrix $M^{(n)}$ whose elements are given in Appendix A, when $n = 1$ and $\omega$ is replaced by $\omega_j = \Omega_j(\pi c/2d)$.

The Airy functions $Ai(z)$ and $Bi(z)$ are both oscillatory functions of $z$ for negative values of $z$, and have a descending or ascending exponential behavior, respectively, for positive values of $z$. Therefore, we can expect that the mode whose frequency is $\omega_j$ in the incident field will stop propagating at the position $x_1^{(j)}$ given by

$$x_1^{(j)} = \beta_1^{(j)} = \frac{1}{a} \left[ b - \left( \frac{\pi c}{2d \omega_j} \right)^2 \right] = \frac{1}{a} \left( b - \frac{1}{\Omega_j^2} \right). \quad (1.37)$$
With our assumption that $\epsilon_{\infty} = 12$, we find from Eqs. (1.14) that

$$a = \frac{5.5}{L}, \quad b = 6.5.$$  \hspace{1cm} (1.38)

With these values of $a$ and $b$, and the values of $\Omega_j$ given by Eqs. (1.29), we find that the stopping points $x_1^{(j)}$ are

$$x_1^{(1)} = -0.3024L = -1.8144\mu m$$  \hspace{1cm} (1.39a)
$$x_1^{(2)} = 0.2480L = 1.7037\mu m$$  \hspace{1cm} (1.39b)
$$x_1^{(3)} = 0.5805L = 3.4845\mu m,$$  \hspace{1cm} (1.39c)

where the second equality in each case follows from our assumption that $L = 6\mu m$.

We illustrate the preceding results by numerical calculations of the electric field and its intensity in each of the regions $x_1 < -L$, $-L < x_1 < L$, and $x_1 > L$. In these calculations, to simplify the resulting figures, we have set $x_3 = 0$ and have assumed that each amplitude $a_1^{(j)} (j = 1, 2, 3)$ is equal to unity.

In Fig. 1.1 we plot $|E_2(x_1, 0)_{inc}|$ (a), $|E_2(x_1, 0)_{ref}|$ (b), and the magnitude of the total field $|E_2(x_1, 0)_{inc} + E_2(x, 0)_{ref}|$ (c) in the region $x_1 < -L$. The shorter period oscillations
Figure 1.1: Plots of $|E_2(x_1,0)_{inc}|$ (a), $|E_2(x_1,0)_{ref}|$ (b), and $|E_2(x_1,0)_{inc} + E_2(x_1,0)_{ref}|$ (c) in the region $x_1 < -L$ of the waveguide.
observed in (c) compared with those in (a) and (b) arise from the interference of the incident and reflected waves in this region of the waveguide. It is seen from these results that the intensity of the reflected field is comparable to the intensity of the incident field.

More interesting is the behavior of the electric field in the region $-L < x_1 < L$ of the waveguide occupied by the graded refractive index medium. In Figs. 1.2 (a)-(c) we plot the real and imaginary parts of $E_2^{(1)}(x_1, 0)_{gr}$, $E_2^{(2)}(x_1, 0)_{gr}$, and $E_2^{(3)}(x_1, 0)_{gr}$, respectively, as functions of $x_1$. It is seen that each of these fields decreases exponentially to zero rapidly as $x_1$ increases past the distances $x_1^{(1)}$, $x_1^{(2)}$, and $x_1^{(3)}$, respectively. This is the rainbow trapping effect. However, a stepwise decrease in the real and imaginary parts of the total electric field at each of these distances is less clearly present in Fig. 1.2 (d), although a trend to smaller values of $ReE_2(x_1, 0)_{gr}$ as $x_1$ crosses $x_1^{(j)}$ ($j = 1, 2, 3$) is seen. The same can be said of the plot of $|E_2(x_1, 0)_{gr}|$ presented in Fig. 1.2 (e).

Propagation of the electric field stops at $x_1 \approx x_1^{(3)}$, and the magnitude of the field decreases exponentially for $x_1$ greater than $x_1^{(3)}$.

The magnitude of the electric field in the region $x_1 > L$, $|E_2(x_1, 0)_{tr}|$, also decreases exponentially with increasing $x_1$, as can be seen from the result plotted in Fig. 1.3.
Figure 1.2: (a)-(c) Plots of the real and imaginary parts of $E_{2}^{(1)}(x_{1}, 0)_{gr}$, $E_{2}^{(2)}(x_{1}, 0)_{gr}$, and $E_{2}^{(3)}(x_{1}, 0)_{gr}$, respectively. (d) Plots of the real and imaginary parts of $|E_{2}(x_{1}, 0)_{gr}|$. (e) A plot of $|E_{2}(x_{1}, 0)_{gr}|$. The dashed vertical lines indicate the values of $x_{1}^{(j)} (j = 1, 2, 3)$. 
Figure 1.3: A plot of $|E_2(x_1,0)_tr|$ in the region $x_1 > L$ of the waveguide.
Chapter 2

Propagation of S-Polarized Surface Polaritons Circumferentially Around a Locally Cylindrical Surface

Several years ago, as part of a general study of the propagation of electromagnetic surface waves over gently bent surfaces, Berry[12] investigated in detail the propagation of p-polarized surface polaritons circumferentially around a portion of a cylindrical surface. To our knowledge a corresponding analysis of the propagation of s-polarized surface polaritons circumferentially around a portion of a cylindrical surface has not been carried out. This case is of interest because surface polaritons of this polarization do not exist on a planar surface. Their occurrence on a cylindrical surface would therefore be a consequence of the curvature of the surface.

In this note we obtain the electromagnetic field and dispersion relation for an s-polarized surface polariton propagating circumferentially around a portion of a cylindrical surface of radius $R$ separating a dielectric from vacuum. The dielectric is characterized by an isotropic, frequency-dependent, dielectric function $\epsilon(\omega)$.

It is convenient to work in cylindrical coordinates $(r, \theta, z)$, where the $z$-axis is parallel to the axis of the cylinder. An s-polarized surface polariton propagating around the cylinder is described by an electric field of the form

$$\mathbf{E}(r, \theta, z) = \hat{z}E_z(r, \theta), \quad (2.1)$$
where $\mathbf{i}_z$ is a unit vector in the $z$ direction. The associated magnetic field has the form

$$
\mathbf{H}(r, \theta, z) = \mathbf{i}_r \frac{c}{i\omega} \frac{1}{r} \frac{\partial E_z}{\partial \theta} - \mathbf{i}_\theta \frac{c}{i\omega} \frac{\partial E_z}{\partial r},
$$

(2.2)

where $\mathbf{i}_r$ and $\mathbf{i}_\theta$ are unit vectors in the $r$- and $\theta$-directions, respectively. In writing Eqs. (2.1) and (2.2) we have omitted writing explicitly a factor $\exp(-i\omega t)$ describing the time-dependence of the wave, where $\omega$ is its frequency.

The Maxwell equation satisfied by $E_z(r, \theta)$ is

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + \epsilon(\omega) \frac{\omega^2}{c^2} E_z = 0
$$

(2.3a)

inside the dielectric, and

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + \frac{\omega^2}{c^2} E_z = 0
$$

(2.3b)

in the vacuum in contact with it. The boundary conditions satisfied by $E_z(r, \theta)$ at the surface $r = R$ are

$$
E_z(r, \theta) \bigg|_{r=R^-} = E_z(r, \theta) \bigg|_{r=R^+},
$$

(2.4a)

$$
\frac{\partial E_z(r, \theta)}{\partial r} \bigg|_{r=R^-} = \frac{\partial E_z(r, \theta)}{\partial r} \bigg|_{r=R^+},
$$

(2.4b)

which express the continuity of the tangential components of the electric and magnetic fields, respectively, across this surface. There are now two cases to consider: (i) the dielectric is convex toward the vacuum; and (ii) the dielectric is concave toward the vacuum. We take them up in turn.
2.1 The Dielectric is Convex Toward the Vacuum

In this case the dielectric occupies the region \( r < R \), while the vacuum occupies the region \( r > R \). The solutions of Eqs. (2.3) in each region are

\[
E_z(r, \theta) = A e^{i \nu \theta} I_\nu\left(|\epsilon(\omega)|^{1/2} \frac{\omega}{c} r\right) \quad 0 \leq r \leq R \quad (2.5a)
\]

\[
= B e^{i \nu \theta} H_\nu^{(1)}\left(\frac{\omega}{c} r\right) \quad r \geq R, \quad (2.5b)
\]

where \( \nu \) is the separation constant, and \( H_\nu^{(1)} \) and \( I_\nu \) are Hankel and modified Bessel functions of the first kind and order \( \nu \), respectively[13]. In obtaining Eq. (2.5a) we have assumed that we are in a frequency range where \( \epsilon(\omega) \) is negative. Consequently, the solution given by Eq. (2.5a) decays exponentially with increasing distance from the surface \( r = R \) toward the origin \( r = 0 \), as is required for a surface wave. In obtaining the solution given by Eq. (2.5b) we have used the fact that all solutions of Eq. (2.3b) are oscillatory functions of \( r \), and the one given by Eq. (2.5b) is the only one that describes an outgoing wave as \( r \to \infty \). Thus, as in the case of p-polarized surface electromagnetic waves[12], it is found that s-polarized surface polaritons that decay with increasing \( r \) cannot exist in the case under consideration: they must radiate. The separation constant \( \nu \) in Eqs. (2.5) is not required to be an integer because we are not considering a complete cylinder, but only a locally cylindrical surface. If we write \( \exp(i \nu \theta) \) as \( \exp[i(\nu/R)(R\theta)] \), and note that \( R\theta = s \) measures distance along the cylindrical surface, we see that \( (\nu/R) = k \) has the physical significance of a surface wave number.

Substitution of Eqs. (2.5) into the boundary conditions (2.4) yields as the solvability condition

\[
\frac{H_\nu^{(1)}(\frac{\omega}{c} R)}{H_\nu^{(1)}(\frac{\omega}{c} R)} = |\epsilon(\omega)|^{1/2} \frac{I_\nu(|\epsilon(\omega)|^{1/2} \frac{\omega}{c} R)}{I_\nu(|\epsilon(\omega)|^{1/2} \frac{\omega}{c} R)}, \quad (2.6)
\]

where primes denote differentiation with respect to argument. Equation (2.6) is the dispersion relation for s-polarized surface polaritons in the present case. If a solution of Eq.
(2.6) exists, it gives \( \nu = kR \) as a function of \( y \). It follows from general considerations that Eq. (2.6) has no real solutions, and the surface wave number is a complex quantity, \( k(\omega) = k_R(\omega) + i k_I(\omega) \), with \( k_I(\omega) > 0 \). This result expresses the fact that the surface polariton is attenuated as it propagates around the cylinder because it radiates as it travels along the surface, and energy conservation requires that the energy radiated must be extracted from the wave itself.

### 2.2 The Dielectric is Concave Toward the Vacuum

In this case the dielectric occupies the region \( r > R \), while the vacuum occupies the region \( r < R \). The solutions of Eqs. (2.3) in each region in this case are

\[
E_z(r, \theta) = \begin{cases} 
A e^{i \nu \theta} J_{\nu}(\frac{\omega}{c} r) & 0 \leq r \leq R \\
B e^{i \nu \theta} K_{\nu}(\sqrt{\frac{\epsilon(\omega)}{c^2}} r) & r \geq R, 
\end{cases}
\]

(2.7a)

where \( J_{\nu} \) is a Bessel function of the first kind and order \( \nu \), while \( K_{\nu} \) is a modified Bessel function of the second kind of order \( \nu \). The solution given by Eq. (2.7b) decays Bessel exponentially as \( r \to \infty \), as is required of a surface wave. In order to obtain such a localized solution we have had to assume that \( \epsilon(\omega) < 0 \). The choice of the solution given by Eq. (2.7a) was dictated by the fact that \( J_{\nu}(x) \) is regular at \( x = 0 \) and increases exponentially with increasing \( x \) until a value \( x \sim \nu \) is reached, at which it acquires an oscillatory dependence on \( x \) that continues for \( x > \nu \). Since we seek a solution for \( r \) in the range \( 0 < r < R \) that is localized for \( r \) in the vicinity of \( R \), i.e. that it tends to zero as \( r \to 0 \), \( J_{\nu}(\omega/c r) \) has this behavior provided that \( \nu \) is of the order of \( (\omega/c)R \). The oscillatory behavior of \( J_{\nu}(\omega/c r) \) when this condition on \( \nu \) is satisfied makes it possible to satisfy the boundary condition (2.4b) when the field in the dielectric is an exponentially decreasing function of \( r \). Use of Eqs. (2.7) in the boundary conditions (2.4) yields the condition that

\[
|\epsilon(\omega)|^{\frac{1}{2}} \frac{K_{\nu}(\sqrt{\frac{\epsilon(\omega)}{c^2}} R)}{K_{\nu}(\sqrt{\frac{\epsilon(\omega)}{c^2}} R)} = \frac{J_{\nu}(\frac{\omega}{c} R)}{J_{\nu}(\frac{\omega}{c} R)}.
\]

(2.8)
Equation (2.8) is the dispersion relation for s-polarized surface polaritons in the present case. In contrast with the result obtained when the dielectric is convex toward the vacuum, Eq. (2.8) admits real solutions for the surface wave number \(k = k(\omega)\), so that surface waves that are nonradiative exist in this case.

We now turn to the solution of the dispersion relations given by Eqs. (2.6) and (2.8).

In solving Eqs. (2.6) and (2.8) we assume for the dielectric function of the dielectric the simple free electron form

\[
\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2},
\]

where \(\omega_p\) is the plasma frequency of the charge carriers in the metal. Thus, the surface polaritons we study are surface plasmon polaritons. In our calculations we have used the value \(\omega_p = 12.708 \times 10^{15} \text{ s}^{-1}\). This value was obtained by fitting the value \(\epsilon(\omega) = -17.2\) for the real part of the dielectric function of silver at a wavelength \(\lambda = 632.8 \text{ nm}\) [14] by the expression (2.9). In addition, we introduce the dimensionless variables

\[
\frac{\omega}{\omega_p} = y, \quad \frac{\omega_p R}{c} = a.
\]

The dimensionless frequency \(y\) is restricted to the range \(0 < y < 1\) within which \(\epsilon(\omega)\) is negative.

Equation (2.6) can now be rewritten as

\[
\frac{H^{(1)'}_{\nu}(ya)}{H^{(1)}_{\nu}(ya)} = \frac{(1 - y^2)^{1/2}}{y} \frac{I'_{\nu}(1 - y^2)^{1/2} a}{I_{\nu}(1 - y^2)^{1/2} a} 0 < y < 1.
\]

We have sought solutions of this equation in the form \(\nu(y) = \nu_R(y) + i\nu_I(y)\), with \(\nu_I(y) > 0\), by numerical and analytical approaches, and have not found any such solutions. We believe that no such solution of Eq. (2.11) exists, just as no solution of the dispersion relation for s-polarized surface plasmon polaritons exists at a planar vacuum-metal interface [15].

Equation (2.8) can be rewritten as

\[
\frac{J^{(1)'}_{\nu}(ya)}{J_{\nu}(ya)} = \frac{(1 - y^2)^{1/2}}{y} \frac{K'_{\nu}(1 - y^2)^{1/2} a}{K_{\nu}(1 - y^2)^{1/2} a} = 0
\]

\[
0 < y < 1.
\]
For a fixed real positive value of $\nu$ the variable $y$ is increased in a stepwise fashion with a constant step size in the interval $0 < y < 1$. A change of sign of the left-hand side of this equation occurs when $y = y(\nu)$. When this calculation is repeated for a series of values of $\nu$, the function $y(\nu)$ can be constructed.

In Fig. 2.1 the results of such calculations are presented for two values of the parameter $a$: $a = 21.18$ (a) and $a = 42.36$ (b). These values correspond to values of $R$ given by $R = 0.5 \mu m$ and $R = 1 \mu m$, respectively. It is seen from these results that the dispersion relation consists of many branches for each value of $a$, and that the number of branches increases, and their separations decrease, as $a$ is increased.

The electric field $E_z(r, \theta)$ can now be written as

$$E_z(r, \theta) = A e^{i\nu \theta} J_\nu((\omega/c)r) \quad 0 \leq r \leq R \quad (2.13a)$$

$$= A e^{i\nu \theta} \frac{J_\nu(\omega/c)R}{K_\nu(|\epsilon(\omega)|^{\frac{1}{2}}(\omega/c)R)}$$

$$\times K_\nu(|\epsilon(\omega)|^{\frac{1}{2}}((\omega/c)r) \quad r \geq R. \quad (2.13b)$$
Figure 2.1: The dispersion curves $y = y(\nu)$ for s-polarized surface plasmon polaritons propagating circumferentially around a portion of a circularly cylindrical interface between vacuum and a metal, when the metal is concave toward the vacuum. These curves represent the solutions of Eq. (2.12). (a) $a = 21.18$; (b) $a = 42.36$. 
Figure 2.2: The radial dependence of the electric field $E_z(r, \theta)$ is plotted as a function of $(r/R)$ for the values of $y$ and $\nu$ indicated by the circle on the lowest frequency branch of the dispersion curve presented in Fig. 2.1(a).
The radial dependence of the field

\[ R(r) = J_\nu((\omega/c)r) \quad 0 \leq r \leq R \]  
\[ = \frac{J_\nu((\omega/c)R)}{K_\nu(|\epsilon(\omega)|^{\frac{1}{2}}(\omega/c)R)}K_\nu(|\epsilon(\omega)|^{\frac{1}{2}}(\omega/c)r) \quad r \geq R, \]  

or equivalently

\[ R(r) = J_\nu(ay(r/R)) \quad 0 \leq r \leq R \]  
\[ = \frac{J_\nu(ay)}{K_\nu(a(1 - y^2)^{\frac{1}{2}})}K_\nu(a(1 - y^2)^{\frac{1}{2}}(r/R)) \quad r \geq R, \]  

corresponding to the point \((\nu = 10, y = 0.6514)\) marked by an open circle on the lowest frequency branch of the dispersion curves depicted in Fig. 2.1(a), is plotted as a function of \(r/R\) in Fig. 2.2. The localization of this field to the vacuum-metal interface is clearly seen in this figure.
Chapter 3

S-Polarized Guided Electromagnetic Waves at a Planar Interface Between Vacuum and a Graded-index Dielectric

3.1 Introduction

Electromagnetic waves whose fields are localized to the vicinity of a planar interface between two media are generally of two types. One type consists of surface polaritons [16]. Their existence requires that the dielectric functions of the two media have opposite signs in some frequency range, and that their electromagnetic field is p polarized. The field in such a mode decays purely exponentially into each medium with increasing distance from the interface. Examples of such surface electromagnetic waves are surface plasmon polaritons, which exist at a dielectric-metal interface in the frequency range below the plasma frequency of the metal, and surface phonon polaritons, which exist at the interface between a dielectric and, e.g. a diatomic cubic polar crystal with two inequivalent ions in a primitive unit cell, in the frequency range between the frequencies of the infinite wavelength transverse and longitudinal vibration modes.

The second type of surface-localized waves are guided waves [17]. These waves can exist in purely dielectric structures, and can be of either p or s polarization. The electromagnetic field of such a wave decays exponentially at large distances from the interface in each
medium. However, in the vicinity of the interface the field has an oscillatory dependence on the distance from the interface, which enables the satisfaction of the electromagnetic boundary conditions at the interface.

In the case of a purely dielectric structure, the case we consider in this chapter, a way of establishing an electromagnetic field with these properties is to work with a film whose dielectric constant is $\epsilon_2$ that is sandwiched between semi-infinite dielectric media whose dielectric constants are $\epsilon_1$ and $\epsilon_3$ [18]. All three dielectric constants are real and positive. Solutions of Maxwell’s equations for p- and s-polarized fields that decay exponentially at infinity in each of the media surrounding the film exist. If $\epsilon_2$ is larger than both $\epsilon_1$ and $\epsilon_3$, standing wave solutions can be obtained for the field within the film. Their oscillatory nature enables the satisfaction of the boundary conditions at both surfaces of the film. This gives rise to a series of discrete, dispersive, eigenmodes that propagate in a wavelike manner in directions parallel to the surfaces of the film, but whose fields are localized to the vicinity of the film. One of the earliest experimental studies of wave propagation in such slab waveguides was carried out by Osterberg and Smith in 1964 [19]. These guided waves form the basis for much of today’s integrated optics technology [20].

A second way of establishing guided waves in an all dielectric structure is to use a system that consists of a semi-infinite homogeneous dielectric medium in contact across a planar interface with a semi-infinite graded-index dielectric medium. Once methods for fabricating such structures [21, 22] and for exciting the guided waves they support [19] were developed, the dispersion relations and the corresponding electromagnetic field profiles of these waves began to be calculated for various forms of the index profile. Many of these calculations were carried out by means of the Wentzel-Kramers-Brillouin (WKB) method [23], which yields approximate results for the dispersion relation and the corresponding field profiles of these waves [24, 25, 26, 27, 28, 29, 30]. Exact, analytic, dispersion relations were obtained for waves supported by a continuous index profile consisting of a small number of straight lines [24], and for an index profile represented by a descending exponential [31]. Exact results for the dispersion curves and the associated electromagnetic field profiles
were obtained for other index profiles by purely numerical methods [32, 33, 34, 35, 36]. The majority of these theoretical studies were devoted to guided waves of s polarization.

In more recent work [37, 38] it was shown that exact analytic dispersion relations and electric field profiles for s-polarized guided waves can be obtained by the use of a dielectric function that is a smooth continuously decreasing function of the coordinate normal to the interface in one of the two dielectric media in contact. In Ref. [37] the dielectric function in the medium in the region $z > 0$ had the simple free electron form with a $z$-dependent plasma frequency

$$\epsilon(z|\omega) = \epsilon_L \left[ 1 - \frac{\omega_p^2(z)}{\omega^2} \right]$$

(3.1.1)

where $\omega_p^2(z) = (4\pi e^2/\epsilon_L m_{\text{eff}})N_0\{1 - (1/b)[1 - (1 + za^{-1})^{-2}]\}$. In Ref. [38] the dielectric function of a slab occupying the region $0 \leq z \leq D$, and surrounded by air, had the Drude form with a $z$-dependent plasma frequency,

$$\epsilon(z|\omega) = \epsilon_L \left[ 1 - \frac{\omega_p^2(z)}{\omega(\omega + i\gamma)} \right],$$

(3.1.2)

where $\omega_p^2(z) = (4\pi e^2/\epsilon_L m_{\text{eff}})N_0\{1 - (1/b)[1 - (1+(D-z)a^{-1})^{-2a}]\}$. Thus, the graded-index dielectric medium studied in these papers can be considered to be an n-type semiconductor with a $z$-dependent conduction electron number density.

Because so few index profiles are known that produce exact analytical results for the dispersion relation and the electromagnetic field profiles of the guided waves in asymmetric graded-index dielectric waveguides, it seems worthwhile to present a new index profile for which such results can be obtained for guided waves of s polarization. It characterizes a purely dielectric medium that contains no conduction electrons.

Thus, we assume that the medium in the region $z > 0$ is characterized by a real, positive, dielectric constant that is frequency independent and decreases with increasing $z$ until it saturates at a bulk value. It is given by

$$\epsilon(z) = n_0^2 \left[ 1 - \frac{1}{g} + \frac{1}{g(1 + \frac{z}{L})^2} \right].$$

(3.1.3)
Here \( n_0 \) is the real index of refraction at \( z = 0 \). We will be interested in the case that \( g > 1 \). In this case, as \( z \to \infty \) the dielectric constant saturates at the volume value \( n_v^2[1 - (1/g)] = n_v^2 \), which satisfies the inequality \( n_v^2 < n_0^2 \). We also assume that the volume value of \( \epsilon(z) \) is larger than that of vacuum, namely that \( n_v > 1 \). The region \( z < 0 \) is vacuum. We note that this system is invariant in the \( y \) direction. We study the guided waves of \( s \) polarization that propagates in the \( x \) direction along the interface \( z = 0 \).

The result that the graded-index waveguide defined by Eq. (3.1.3) yields an exact dispersion relation and electric field profiles of \( s \)-polarized guided waves it supports in analytic forms was noted briefly in the review article [39], but no results were presented there. In the present chapter we expand on the work reported in Ref. [39] and present plots of the dispersion curves and the associated electric field profiles, both in the regime of the low frequencies and in the regime of high frequencies. The regimes of low and high frequencies are defined with respect to a characteristic frequency that arises in the analysis carried out here. The natures of the solutions in these two frequency regimes are markedly different.

### 3.2 The Guided Wave Dispersion Relation

If we write the single nonzero component of the electric field in the region \( z > 0 \) in the form

\[
E_y^>(x, z; t) = E_y^>(z|\omega) \exp(ikx - i\omega t),
\]

the amplitude \( E_y^>(z|\omega) \) satisfies the equation

\[
\left\{ \frac{d^2}{dz^2} - \left[ k^2 - \epsilon(z)\left(\frac{\omega}{c}\right)^2 \right] \right\} E_y(z|\omega) = 0.
\]

To obtain a solution of this equation that decays to zero as \( z \to \infty \) we make the \textit{Ansatz}

\[
E_y^>(z|\omega) = u^2 f(u),
\]

30
where

\[ u = 1 + \frac{z}{L}. \]  

(3.2.4)

On substituting Eq. (3.2.3) into Eq. (3.2.2), and making use of Eqs. (3.1.3) and (3.2.4), we find that the function \( f(u) \) satisfies the equation

\[
\frac{d^2 f(u)}{du^2} + \frac{1}{u} \frac{df(u)}{du} - \left[ L^2 p^2 + \frac{s^2}{u^2} \right] f(u) = 0, 
\]  

(3.2.5)

where

\[ p^2 = \frac{\omega^2}{c^2} (b^2 - n_v^2), \quad s^2 = \frac{1}{4} \left( 1 - \frac{\omega^2}{\Omega_c^2} \right), \]  

(3.2.6)

with

\[ b = \frac{c}{\omega} k, \quad \Omega_c = \frac{c}{2L} \sqrt{\frac{1}{n_v^2 - n_v^2}}. \]  

(3.2.7)

Equation (3.2.5) is the equation satisfied by the modified Bessel functions. To obtain the solution of it that decays exponentially as \( u \) (and hence \( z \)) tends to infinity, the coefficient \( p^2 \) must be real and positive. We can assume that \( p \) is also real and positive. From Eqs. (3.2.6) and (3.2.7) we see that this condition restricts the wavenumber \( k \) to be in the region of the \((\omega, k)\) plane where \( k > n_v(\omega/c) \). The solution of Eq. (3.2.5) is then given by the modified Bessel function of the second kind and order \( s \),

\[ f(u) = AK_s(Lpu), \]  

(3.2.8)

where \( A \) is an arbitrary amplitude. The order \( s \), Eq. (3.2.6), however, can be real or imaginary. Either choice produces a solution \( f(u) \) that decreases to zero exponentially as \( u \to \infty \) \[40\].

When \( s \) is real and positive, so that \( 0 < \omega < \Omega_c \), and \( 0 < s < (1/2) \), the function \( K_s(z) \) has the integral representation \[41\]

\[ K_s(z) = \int_0^\infty dx e^{-x} \cosh x \cosh sx, \]  

(3.2.9)
is real for \( z \) real and positive, and is an even function of \( s \). It is shown in appendix B that for fixed \( s \) \((0 < s < 1/2)\) and \( z \to 0^+ \),
\[
K_s(z) \sim \Gamma(s) \left(\frac{z}{2}\right)^{-s} - \frac{\Gamma(1-s)}{2s} \left(\frac{z}{2}\right)^s + O(z^{2-s}), \tag{3.2.10}
\]

where \( \Gamma(z) \) is the gamma function. For fixed \( s \) and large \( z \) it has the asymptotic form [42]
\[
K_s(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left[ 1 + \frac{4s^2 - 1}{8z} + \frac{(4s^2 - 1)(4s^2 - 9)}{2!(8z)^2} + \cdots \right]. \tag{3.2.11}
\]

When \( s \) is pure imaginary \( s = is_1 \), where \( s_1 = \frac{1}{2}[(\omega/\Omega_c)^2 - 1]\frac{1}{2} \), so that \( \omega > \Omega_c \), the solution \( f(u) \) becomes
\[
f(u) = A K_{is_1}(Lpu). \tag{3.2.12}
\]

The function \( K_{is_1}(z) \) has the integral representation [43]
\[
K_{is_1}(z) = \int_0^{\infty} dx e^{-z \cosh x} \cos s_1 x, \tag{3.2.13}
\]
is real for \( s_1 \) real and \( z \) real and positive, and is an even function of \( s_1 \). For fixed \( s_1 \) and \( z \to 0^+ \), \( K_{is_1}(z) \) is an oscillatory function of \( z \) [40],
\[
K_{is_1}(z) \sim -\left(\frac{\pi}{s_1 \sinh(s_1 \pi)}\right)^{\frac{1}{2}} \left[ \sin(s_1 \ln z) - \phi_{s_1,0} \right] + O(z^2), \tag{3.2.14}
\]
where
\[
\phi_{s_1,0} = \text{arg } \Gamma(1 + is_1). \tag{3.2.15}
\]

For fixed \( s_1 \) and large \( z \) it has the asymptotic form
\[
K_{is_1}(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left[ 1 - \frac{4s_1^2 + 1}{8z} + \frac{(4s_1^2 + 1)(4s_1^2 + 9)}{2!(8z)^2} + \cdots \right]. \tag{3.2.16}
\]

We now turn to the single nonzero component of the electric field in the vacuum region \( z < 0 \). We write it in the form
\[
E_y^<(x, z; t) = E_y^<(z|\omega) \exp(ikx - i\omega t) \tag{3.2.17}
\]
and find that the amplitude $E_y^<(z|\omega)$ is a solution of the equation
\[ \left\{ \frac{d^2}{dz^2} - \left[ k^2 - \left( \frac{\omega}{c} \right)^2 \right] \right\} E_y^<(z|\omega) = 0. \] (3.2.18)
The solution of this equation that decays to zero as $z \to -\infty$ is
\[ E_y^<(z|\omega) = B \exp[\beta_0(k,\omega)z], \] (3.2.19)
where $B$ is an arbitrary constant, while
\[ \beta_0(k,\omega) = \left[ k^2 - \left( \frac{\omega}{c} \right)^2 \right]^{\frac{1}{2}}, \quad Re\beta_0(k,\omega) > 0, \quad Im\beta_0(k,\omega) < 0. \] (3.2.20)
The function $\beta_0(k,\omega)$ must be real and positive in order that the field component given by Eq. (3.2.19) decay to zero as $z \to -\infty$. This requires that $k > (\omega/c)$. However, as we are interested in structures for which $n_v > 1$, the condition on $k$ obtained earlier, namely $k > n_v(\omega/c)$, is the more restrictive one, and it is in this range of values of $k$ that the dispersion relation for waves localized to the surface will be sought.

This dispersion relation is obtained from the satisfaction of the boundary conditions at the interface $z = 0$. These require the continuity of the tangential components of the electric and magnetic fields across this interface. They can be written as
\[ E_y^<(z|\omega) = E_y^>(z|\omega) \] (3.2.21a)
\[ \frac{dE_y^<(z|\omega)}{dz} = \frac{dE_y^>(z|\omega)}{dz}. \] (3.2.21b)

With the results given by Eqs. (3.2.3) and (3.2.19), and recalling Eq. (3.2.4) these equations take the forms
\[ B = u_1^\frac{1}{2} f(u) \bigg|_{u=1} \] (3.2.22a)
\[ \beta_0(k,\omega)B = \frac{1}{L} \left[ \frac{1}{2u_1^\frac{1}{2}} f(u) + u_1^\frac{1}{2} \frac{df(u)}{du} \right] \bigg|_{u=1}. \] (3.2.22b)

There are now two cases to consider.
When the frequency $\omega$ lies in the interval $0 < \omega < \Omega_c$, the function $f(u)$ is given by Eq. (3.2.8). The dispersion relation for s-polarized guided electromagnetic waves in the region $0 < \omega < \Omega_c$, $k > n_v(\omega/c)$ of the $(\omega,k)$ plane therefore becomes

$$L\beta_0(k,\omega) = \frac{1}{2} + Lp \frac{K''(Lp)}{K_s(Lp)},$$

(3.2.23)

where the prime denotes differentiation with respect to argument. This equation has to be solved numerically. The resulting dispersion curve consists of a single branch. It is plotted in Fig. 3.1. The values of the material parameters assumed in obtaining the results presented in Fig. 3.1 are $L = 30$ nm, $n_0 = 2.0$, and (a) $n_v = 1.05$, (b) $n_v = 1.2$, and (c) $n_v = 1.42$. The corresponding values of the characteristic frequency $\Omega_c$ are (a) $\Omega_c = 2.9353 \times 10^{15}$ s$^{-1}$, $\lambda_c = 641.72$ nm; (b) $\Omega_c = 3.1228 \times 10^{15}$ s$^{-1}$, $\lambda_c = 603.19$ nm; and (c) $\Omega_c = 3.5477 \times 10^{15}$ s$^{-1}$, $\lambda_c = 530.96$ nm. The lower edge of the frequency range within which the wave exists for each of these cases is (a) $\omega/\Omega_c = 0.3634$, $\omega = 1.0667 \times 10^{15}$ s$^{-1}$; (b) $\omega/\Omega_c = 0.7078$, $\omega = 2.2103 \times 10^{15}$ s$^{-1}$; and (c) $\omega/\Omega_c = 0.9532$, $\omega = 3.3817 \times 10^{15}$ s$^{-1}$. The intervals of the dimensionless wave number $kL$ within which the wave exists in these three cases are (a) $(0.1121, 0.3189)$; (b) $(0.2654, 0.3766)$; and (c) $(0.4805, 0.5041)$, respectively. Thus, we see that as $n_v$ increases, with the values of the other parameters defining the dielectric constant (3.1.3) kept fixed, the parameter $g$ increases, so that the initial slope of $\epsilon(z)$ decreases in magnitude. As a result the frequency range within which this surface-localized wave exists shifts to higher values and becomes narrower, while the range of wave numbers within which the wave exists shifts to larger values and becomes narrower.

An approximate analytic dispersion relation can be obtained from Eq. (3.2.22) in the case that $(Lp)^2 \ll 1$. In this limit $K_s(z)$ can be approximated by Eq. (3.2.10). With this result it is found in appendix B that

$$z \frac{K'_s(z)}{K_s(z)} = -s - 2s \frac{\Gamma(1-s)}{\Gamma(1+s)} \left( \frac{z}{2} \right)^{2s}.$$

(3.2.24)
Figure 3.1: The dispersion curve for an s-polarized guided electromagnetic wave in the frequency range $0 < \omega < \Omega_c$ propagating along the planar interface between vacuum and a graded index dielectric medium, characterized by the values $L = 30 \text{ nm}$, $n_0 = 2.0$, and (a) $n_v = 1.05$, (b) $n_v = 1.2$, (c) $n_v = 1.42$.

The dispersion relation (3.2.23) consequently takes the form

$$L \beta_0(k, \omega) = \frac{1}{2} - s - 2s \frac{\Gamma(1-s)}{\Gamma(1+s)} \left( \frac{Lp}{2} \right)^{2s}. \quad (3.2.25)$$

This equation has to be solved numerically. In Fig. 3.2 we plot the solution as a dashed curve for comparison with the numerical solution of Eq. (3.2.23) plotted in Fig. 3.1 for the case where $L = 30 \text{ nm}$, $n_0 = 2.0$, and (a) $n_v = 1.05$, (b) $n_v = 1.2$, and (c) $n_v = 1.42$. It is seen that this approximate dispersion curve is in very good agreement with the exact dispersion curve over nearly the entire frequency range within which this guided wave exists, for each value of $n_v$. This agreement breaks down only for $\omega/\Omega_c$ very close to unity. The difference between the approximate and exact dispersion curves decreases as $n_v$ increases.

We have seen that this guided electromagnetic wave exists in a narrow spectral range. A simple analytic estimate of this range is obtained if we use only the first term on the right-hand side of Eq. (3.2.24) in the dispersion relation (3.2.23), which becomes

$$2L \beta_0(k, \omega) = 1 - 2s. \quad (3.2.26)$$

If we introduce the dimensionless frequency $\nu = (\omega/\Omega_c)$, so that $0 < \nu < 1$, the substitution of Eqs. (3.2.6) and (3.2.20) into Eq. (3.2.26) yields an explicit expression for the dependence
Figure 3.2: The dispersion curve for an s-polarized guided electromagnetic wave in the frequency range $0 < \omega < \Omega_c$ propagating along the planar interface between vacuum and a graded index dielectric medium characterized by the values $L = 30$ nm, $n_0 = 2.0$, and (a) $n_v = 1.05$, (b) $n_v = 1.2$, (c) $n_v = 1.42$. The numerical solution of Eq. (3.2.25) is depicted by a dashed curve.

of the wave number $k$ on the frequency $\nu$,

$$k^2 = \frac{\omega^2}{c^2} + \frac{1}{4L^2} [1 - \sqrt{1 - \nu^2}]^2. \quad (3.2.27)$$

We now recall the inequality $n_v^2 < b^2$ that follows from the necessity of $p^2$ being positive. With the expression for $b$ given by Eq. (3.2.7), the expression for $k^2$ given by Eq. (3.2.27), and the definition of $\Omega_c$ given by Eq. (3.2.7), this inequality can be written as

$$\frac{1}{\nu} [1 - \sqrt{1 - \nu^2}] > \left( \frac{n_v^2 - 1}{n_0^2 - n_v^2} \right)^{\frac{1}{2}}. \quad (3.2.28)$$

This inequality is satisfied for

$$\frac{2(n_0^2 - n_v^2)^{\frac{1}{2}}(n_v^2 - 1)^{\frac{1}{2}}}{n_0^2 - 1} = \nu_c < \nu < 1. \quad (3.2.29)$$

The values of $\nu_c$ obtained from Eq. (3.2.29) when $n_0 = 2.0$ and $n_v = 1.05$, 1.2, and 1.42, are 0.3633, 0.7075, and 0.9466, respectively. These values are quite close to the values obtained from the numerical solution of Eq. (3.2.23), namely 0.3634, 0.7078, and 0.9532, respectively.

The validity of Eq. (3.2.20) and more especially of Eq. (3.2.26) rests on the assumption that $(Lp)^2 \ll 1$. If we combine the definition of $p^2$ given by Eq. (3.2.6), the expression for
\( k^2 \) given by Eq. (3.2.27), and the definition of \( \Omega_c \) given by Eq. (3.2.7), we obtain

\[
(Lp)^2 = \frac{1}{4} \left[ (1 - \sqrt{1 - \nu^2})^2 - \frac{1}{\nu^2} \frac{n_v^2 - 1}{n_0^2 - n_v^2} \right].
\] (3.2.30)

At \( \nu = 1 \) this function equals

\[
(Lp)^2 \bigg|_{\nu=1} = \frac{1}{4} \left[ 1 - \frac{n_v^2 - 1}{n_0^2 - n_v^2} \right],
\] (3.2.31)

and decreases monotonically to zero from this value as \( \nu \) decreases to \( \nu = \nu_c \). Thus \((Lp)^2\) will be small for \( \nu \gtrsim \nu_c \), which implies that the expression for \( \nu_c \) given by Eq. (3.2.29) is a good approximation to the lower limit of the spectral range within which the guided electromagnetic wave exists. This is confirmed by our numerical results quoted above.

The electric field amplitude \( E_y(z|\omega) \) obtained from Eqs. (3.2.3), (3.2.4), (3.2.8), and (3.2.19) can be written as \((A=1)\)

\[
E_y(z|\omega) = K_\nu L p \exp[\beta_\nu(k, \omega) z] \quad z < 0 \quad (3.2.32a)
\]

\[
= (1 + \frac{z}{L})^{\frac{1}{2}} K_\nu (L p (1 + \frac{z}{L})) \quad z > 0. \quad (3.2.32b)
\]

This function is plotted in Fig. 3.3 as a function of \((z/L)\) for a typical point on the three dispersion curves depicted in Figs. 3.1(a)-(c). These points are given by (a) \( kL = 0.18, \omega/\Omega_c = 0.5798 \), (b) \( kL = 0.32, \omega/\Omega_c = 0.8524 \); and (c) \( kL = 0.504, \omega/\Omega_c = 0.99977 \). The field in the vacuum \((z < 0)\) is well localized to the interface in each case. The field in the dielectric penetrates more deeply into it. However, to place the spatial decay of the latter field in perspective, we note that for the value of \( L = 30 \text{ nm} \) assumed here, a \( 1/e \) decay length of \( 50L \) as in the result plotted in Fig. 3.3(a), corresponds to \( 1.5 \mu\text{m} \), which is approximately three times the wavelength of the guided wave. This is significantly larger than the decay length of the field into a metallic medium. For example, for a surface plasmon polariton of wavelength \( \lambda = 457.9 \text{ nm} \) at a vacuum-silver interface its \( 1/e \) decay length in the silver is approximately \( \lambda/20 \). The decay length of the field in the graded-index dielectric medium can be decreased somewhat by varying the parameters \( n_0, g, \) and \( L \) in Eq. (3.1.3), as well as the frequency \( \omega \), but it does not seem possible to decrease it to the subwavelength level.
Figure 3.3: The electric field amplitude $E_y(z|\omega)$ as a function of $(z/L)$ for the guided wave corresponding to the points on the dispersion curves plotted in Fig. 3.1 (a)-(c) defined by (a) $kL = 0.18, \omega/\Omega_c = 0.5798$, (b) $kL = 0.32, \omega/\Omega_c = 0.8524$, (c) $kL = 0.504, \omega/\Omega_c = 0.99977$.

Nevertheless, localization of the field in the graded dielectric to within about a wavelength from the interface $z=0$ is possible.

3.2.2 $\omega > \Omega_c$

When the frequency $\omega$ is greater than the characteristic frequency $\Omega_c$, the function $f(u)$ is given by Eq. (3.2.12). The dispersion relation for s-polarized guided electromagnetic waves in the region $\omega > \Omega_c, k > n_v(\omega/c)$, of the $(\omega, k)$ plane is then

$$L\beta_0(k, \omega) = \frac{1}{2} + Lp \frac{K'_{is1}(Lp)}{K_{is1}(Lp)},$$

(3.2.33)

where $s_1 = (1/2)[(\omega/\Omega_c)^2 - 1]^{1/2}$. Equation (3.2.33) also has to be solved numerically. The calculations of $K_{is1}(z)$ and its derivative were carried out by the use of the algorithms due to Gil et al. [44, 45]. Due to the oscillatory nature of $K_{is1}(z)$ as $z$ approaches zero through real positive values, a multiplicity of solutions can be expected. This expectation is indeed satisfied. The three lowest frequency branches of the dispersion curves obtained from the solution of Eq. (3.2.33) are plotted in Fig. 3.4 for the cases where the graded index dielectric medium is characterized by the values $L = 30$ nm, $n_0 = 2.0$, and (a) $n_v = 1.05$, (b) $n_v = 1.2$, (c) $n_v = 1.42$. It is seen that for a fixed value of $kL$ the difference between
Figure 3.4: The three lowest frequency branches of the dispersion curve for an s-polarized guided electromagnetic wave in the frequency range $\omega > \Omega_c$ propagating along the planar interface between vacuum and a graded index dielectric medium characterized by the values $L = 30$ nm, $n_0 = 2.0$, and (a) $n_v = 1.05$, (b) $n_v = 1.2$, (c) $n_v = 1.42$.

The values of $\omega/\Omega_c$ on consecutive branches decreases as $n_v$ increases.

The electric field amplitude $E_y(z|\omega)$ obtained from Eqs. (3.2.3), (3.2.4), (3.2.12), and (3.2.19) can be written as ($A = 1$)

$$E_y(z|\omega) = K_{is_1}(Lp) \exp[\beta_0(k, \omega)z] \quad z < 0 \quad (3.2.34a)$$

$$= (1 + \frac{z}{L})^{\frac{1}{2}} K_{is_1}(Lp(1 + \frac{z}{L})) \quad z > 0. \quad (3.2.34b)$$

This function is plotted as a function of $(z/L)$ in Fig. 3.5 for the points on the branches of the dispersion curves plotted in Fig. 3.4(a) corresponding to $kL = 4.5$, for which $\omega/\Omega_c$ equals (a) 10.3049, (b) 12.4182, (c) 13.5108, respectively. It decays to zero in an exponential fashion as $z \to -\infty$. In the region $z > 0$ it decays to zero in an oscillatory fashion as $z \to \infty$. The number of nodes in this function equals the branch number, if the lowest frequency branch is denoted the zero branch. This behavior is reminiscent of the $z$ dependence of the electric field of the waveguide modes supported by a dielectric film, whose dielectric constant is $\epsilon_2$, sandwiched between two semi-infinite dielectric media whose dielectric constants are $\epsilon_1$ and $\epsilon_3$, when $\epsilon_2 > \epsilon_1$ and $\epsilon_2 > \epsilon_3$. This is not surprising, because the dielectric constant of the system studied in this chapter is characterized by the inequalities $\epsilon(z) > n_v^2 > 1$, which are needed for the existence of guided waves.
Figure 3.5: The electric field amplitudes $E_y(z|\omega)$ as functions of $z$ for the points on the branches of the dispersion curves plotted in Fig. 3.4(a) corresponding to $kL = 4.5$, for which (a) $\omega/\Omega_c = 10.3049$, (b) $\omega/\Omega_c = 12.4182$, and (c) $\omega/\Omega_c = 13.5108$.

Figure 3.6: The same as Fig. 3.5 but for the points on the branches of the dispersion curves plotted in Fig. 3.4(b) corresponding to $kL = 4.5$, for which (a) $\omega/\Omega_c = 9.428$, (b) $\omega/\Omega_c = 10.9421$, and (c) $\omega/\Omega_c = 11.5807$.  

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Figure 3.7: The same as Fig. 3.5, but for the points on the branches of the dispersion curves plotted in Fig. 3.4(c) corresponding to $kL = 4.5$, for which (a) $\omega/\Omega_c = 7.896$, (b) $\omega/\Omega_c = 8.6468$, and (c) $\omega/\Omega_c = 8.8542$.

The electric field amplitudes $E_y(z|\omega)$ \eqref{eq:3.2.34} for the points on the branches of the dispersion curves plotted in Figs. 3.4(b) and 3.4(c) corresponding to $kL = 4.5$ are plotted in Figs. 3.6 and 3.7, respectively. The corresponding values of $\omega/\Omega_c$ are (a) 9.428, (b) 10.9421, and (c) 11.5807 in the case of Fig. 3.4(b), and (a) 7.896, (b) 8.6468, and (c) 8.8542, in the case of Fig. 3.4(c).
Chapter 4

Conclusions

In chapter 1, we have studied the propagation of a wave packet consisting of a superposition of three s-polarized guided waves with different frequencies in a planar waveguide consisting of a dielectric medium with a graded index of refraction sandwiched between perfectly conducting walls. This simple model system displays features observed in earlier studies of the rainbow trapping of guided waves, in particular that each frequency component of the incident wave packet stops propagating at a specific distance along the waveguide that depends on its frequency (its color) and on the material and geometrical parameters defining the waveguide. It also shows some features not discussed in these earlier studies. These include the strong reflection of the incident field from the waveguide, which appears to be due to the cessation of transmission of the waves comprising that field at specific distances along the waveguide, and the fact that the trapping phenomenon is not sharp but displays an exponential decay of the electric field strength on the transmission side at each of these distances. An attractive feature of the model system studied is that its properties can be studied analytically rather than purely numerically [48].

In chapter 2, we have shown that s-polarized surface plasmon polaritons, which do not exist at a planar vacuum-metal interface, exist as true surface plasmon polaritons when the planar interface becomes a portion of a circularly cylindrical boundary with the metal concave toward the vacuum [49].

Finally, we can draw several conclusions from the results obtained in the preceding pages of chapter 3.

We have presented a new continuous index profile for an asymmetric graded-index dielectric waveguide that allows the dispersion relation for the guided waves it supports
and the electric field profiles for these waves, to be obtained in analytic form.

The guided waves supported by the dielectric structure considered here have qualitatively different natures in the two frequency regimes $0 < \omega < \Omega_c$ and $\omega > \Omega_c$.

The existence of the critical frequency $\Omega_c$ and the different natures of the modes with frequencies below and above it, does not occur in the slab optical waveguides or in the continuously graded-index optical waveguides studied until now.

In the low frequency region ($0 < \omega < \Omega_c$) only a single guided wave exists, and the spectral range in which it exists is a narrow one that is controlled by the technically controlled heterogeneity scale $L$. The electric field profile of this mode as a function of $z$ has a single maximum and no nodes, and decays to zero exponentially as $z \to \pm\infty$. This mode thus resembles a surface plasmon polariton in its localization to the interface, even though the maximum of its electric field occurs inside the graded-index medium and not at the interface $z = 0$. It differs from a surface plasmon polariton in that it exists in a narrow spectral range, while the surface plasmon polariton exists in the wide spectral range $0 < \omega < \omega_p/\sqrt{2}$, where $\omega_p$ is the plasma frequency of the conduction electrons in the metal supporting it. Since it occurs in a dielectric structure, which has small ohmic losses, its energy mean free path can be longer than that of a surface plasmon polariton. Also, because it occurs in a dielectric structure, this mode can be used in applications in situations, such as in an oxidizing atmosphere, where a metallic surface cannot be used.

In the high frequency range ($\omega > \Omega_c$) the dispersion curve has a multiplicity of branches, of which we have considered only the three lowest frequency modes. The electric fields of these modes possess nodes for $z$ in the vicinity of the interface $z = 0$ whose number equals the mode number, if the lowest frequency branch in this region is denoted the zero branch. They also decay exponentially as $z \to \pm\infty$. These features are characteristic of waveguide modes.

Large values of $\Omega_c$ caused by the negative gradient of the dielectric permittivity offer the possibility of extending the domain of existence of the s-polarized guided electromagnetic waves to the near infrared, and even to the visible region of the optical spectrum [50].
References


[42] Ref. [41], p. 378, entry 9.7.2.


Appendix A

The nonzero elements of the matrix $M^{(n)}$ entering Eq. (1.19)

$$M_{11}^{(n)} = -\exp \left\{ i \left[ \epsilon_\infty \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\}$$  \hspace{1cm} (A.1)

$$M_{12}^{(n)} = \exp \left\{ i \left[ \epsilon_\infty \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\} \times \text{Ai} \left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (-L - \beta_n) \right)$$  \hspace{1cm} (A.2)

$$M_{13}^{(n)} = \exp \left\{ i \left[ \epsilon_\infty \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\} \times \text{Bi} \left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (-L - \beta_n) \right)$$  \hspace{1cm} (A.3)

$$M_{21}^{(n)} = \exp \left\{ i \left[ \epsilon_\infty \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\}$$  \hspace{1cm} (A.4)

$$M_{22}^{(n)} = \exp \left\{ i \left[ \epsilon_\infty \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\} \times \frac{\left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}}}{i \left[ \epsilon_\infty \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}}} \times \text{Ai}' \left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (-L - \beta_n) \right)$$  \hspace{1cm} (A.5)
\[ M_{23}^{(n)} = \exp \left\{ i \left[ \epsilon_{\infty} \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\} \]

\[ \times \frac{(a\omega^2/c^2)^{\frac{1}{3}}}{i[\epsilon_{\infty} \frac{\omega^2}{c^2} - (n\pi/2d)^2]^{\frac{1}{2}}} \times Bi'\left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (-L - \beta_n) \right) \] (A.6)

\[ M_{32}^{(n)} = Ai\left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (L - \beta_n) \right) \] (A.7)

\[ M_{33}^{(n)} = Bi\left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (L - \beta_n) \right) \] (A.8)

\[ M_{34}^{(n)} = -\exp \left\{ i \left[ \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\} \] (A.9)

\[ M_{42}^{(n)} = \frac{(a\omega^2/c^2)^{\frac{1}{3}}}{i[\epsilon_{\infty} \frac{\omega^2}{c^2} - (n\pi/2d)^2]^{\frac{1}{2}}} \times Ai'\left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (L - \beta_n) \right) \] (A.10)

\[ M_{43}^{(n)} = \frac{(a\omega^2/c^2)^{\frac{1}{3}}}{i[\epsilon_{\infty} \frac{\omega^2}{c^2} - (n\pi/2d)^2]^{\frac{1}{2}}} \times Bi'\left( \left( \frac{a\omega^2}{c^2} \right)^{\frac{1}{3}} (L - \beta_n) \right) \] (A.11)

\[ M_{44}^{(n)} = -\exp \left\{ i \left[ \frac{\omega^2}{c^2} - \left( \frac{n\pi}{2d} \right)^2 \right]^{\frac{1}{2}} L \right\}. \] (A.12)
Appendix B

The small-argument expansion of \( K_{\nu}(z) \)

The modified Bessel function of the second kind, \( K_{\nu}(z) \), is defined by \[46\]

\[
K_{\nu}(z) = \frac{\pi}{2} \left[ \frac{I_{-\nu}(z)}{\sin(\nu \pi)} - \frac{I_{\nu}(z)}{\sin(\nu \pi)} \right].
\]  (B.1)

where \( I_{\nu}(z) \) is the modified Bessel function of the first kind and order \( \nu \). For small values of \( z \) \( I_{-\nu}(z) \) and \( I_{\nu}(z) \) possess the following expansions \[47\]:

\[
I_{-\nu}(z) = \left( \frac{z}{2} \right)^{-\nu} \left[ \frac{1}{\Gamma(1 - \nu)} + \frac{1}{\Gamma(2 - \nu)} \left( \frac{z}{2} \right)^2 + O(z^4) \right],
\]  (B.2a)

\[
I_{\nu}(z) = \left( \frac{z}{2} \right)^{\nu} \left[ \frac{1}{\Gamma(1 + \nu)} + \frac{1}{\Gamma(2 + \nu)} \left( \frac{z}{2} \right)^2 + O(z^4) \right].
\]  (B.2b)

Therefore, for small \( z \)

\[
K_{\nu}(z) = \frac{\pi}{2 \sin \nu \pi} \left[ \frac{1}{\Gamma(1 - \nu)} \left( \frac{z}{2} \right)^{-\nu} + \frac{1}{\Gamma(2 - \nu)} \left( \frac{z}{2} \right)^{2-\nu} + \cdots \right.
\]

\[
- \frac{1}{\Gamma(1 + \nu)} \left( \frac{z}{2} \right)^{\nu} - \frac{1}{\Gamma(2 + \nu)} \left( \frac{z}{2} \right)^{2+\nu} - \cdots \right].
\]  (B.3)

In the case of interest in Section 3.2 the order \( \nu \) obeys the inequalities \( 0 < \nu < 1/2 \). Consequently, the first term on the second line of this equation dominates the second term on the first line. As a result the first two terms in the small argument of \( K_{\nu}(z) \) are given by

\[
K_{\nu}(z) = \frac{\pi}{2 \sin \nu \pi} \left[ \frac{1}{\Gamma(1 - \nu)} \left( \frac{z}{2} \right)^{-\nu} - \frac{1}{\Gamma(1 + \nu)} \left( \frac{z}{2} \right)^{\nu} + O(z^{2-\nu}) \right].
\]  (B.4)

With the use of the relation \( \Gamma(\nu)\Gamma(1 - \nu) = (\pi/ \sin \nu \pi) \), this expansion can be rewritten as

\[
K_{\nu}(z) = \frac{\Gamma(\nu)}{2} \left( \frac{z}{2} \right)^{-\nu} - \frac{\Gamma(1 - \nu)}{2\nu} \left( \frac{z}{2} \right)^{\nu} + O(z^{2-\nu}).
\]  (B.5)
It follows that

\[ K'_\nu(z) = -\frac{\nu \Gamma(\nu)}{2^{1-\nu}} z^{-\nu-1} - \frac{\Gamma(1-\nu)}{2^{1+\nu}} z^{\nu-1} + O(z^{1-\nu}). \]  

(B.6)

On combining eqs. (B.5) and (B.6) we obtain

\[
\frac{K'_\nu(z)}{K_\nu(z)} = -\frac{\nu}{z} \left[ 1 + \frac{(\Gamma(1-\nu)/\Gamma(1+\nu))(z/2)^{2\nu}}{1 - (\Gamma(1-\nu)/\Gamma(1+\nu))(z/2)^{2\nu}} \right] \\
\cong -\frac{1}{z} \left[ \nu + 2\nu \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left( \frac{z}{2} \right)^{2\nu} \right],
\]

(B.7)

so that the two leading terms in the small argument expansion of \( z K'_\nu(z)/K_\nu(z) \) are

\[
z \frac{K'_\nu(z)}{K_\nu(z)} = -\nu - 2\nu \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left( \frac{z}{2} \right)^{2\nu}.
\]

(B.8)

This is Eq. (3.2.24).
Curriculum Vitae

Javier Polanco was born on April 2, 1978 in Cd. Juarez, Chihuahua, Mexico. The only son of Jesus Polanco and Dora Luz Gonzalez, he graduated from Instituto Tecnologico de Cd. Juarez in the spring of 2001 and entered The University of Texas at El Paso in the fall to study Master in Physics. From an early age, he showed interest for recreative activities like playing piano and chess. He also developed special interest for mathematics, to the point that before finishing High School, he was selected as part of the team that represented Mexico at the 37th International Mathematical Olimpiad, held at Mumbai, India in 1996, at the age of 18. After that, when studying Bachelor of Science in Computer Science, he kept participating in science-related contests, on the disciplines of Chemistry, Physics, and Mathematics. At the end of his first Masters, he obtained the Cook Scholarship, due to an outstanding GPA. A second Masters followed, this time in Mathematics. After graduating, he worked over a period of one year as Research Associate with Dr. Fitzgerald, at University of Texas at El Paso. Later on, he pursued a PhD degree in Mathematics at Texas A&M University. He is currently enrolled in the PhD Computational Science Program at the University of Texas at El Paso.

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