The Topology Of Statistical Convergence

Khdiga Kalifa Tabib
University of Texas at El Paso, kktabib@miners.utep.edu
THE TOPOLOGY OF STATISTICAL CONVERGENCE

KHDIGA K. TABIB

Department of Mathematical Sciences

APPROVED:

________________________________________
Mohamed A. Khamsi, Chair, Ph.D.

________________________________________
Emil D. Schwab, Ph.D.

________________________________________
Juan C. Noveron, Ph.D.

________________________________________
Benjamin Flores, Ph.D.
Dean of the Graduate School
Dedication

I owe thanks to god, my parents and siblings, who encouraged me so much in life and helped me to become the person who I am today. They taught me the values of self-responsibility and hard work, and were always supportive of me in pursuing my dreams - whether in writing, in poetry or in the study of philosophy of mathematics. They showered me with love, kindness and care, and I am lucky to have been blessed with the family I have. I also owe thanks to my friends - the informal family of people I wasn’t born with, but who nevertheless has shown me so much love and caring.
THE TOPOLOGY OF STATISTICAL CONVERGENCE

by

KHDIGA K. TABIB

THESIS
Presented to the Faculty of the Graduate School of
The University of Texas at El Paso
in Partial Fulfillment
of the Requirements
for the Degree of

MASTER OF SCIENCE

Department of Mathematical Sciences
THE UNIVERSITY OF TEXAS AT EL PASO
August 2012
Acknowledgements

I am deeply indebted to Dr. Mohamed A. Khamisi, my advisor. With great patience, support and trust that he contributed to this thesis with great suggestions, ideas, and corrections where needed. During my Masters, his integrity, exactness with great fidelity in his explains and teaches method was an inspiring example for me. This thesis could not have come into being without him.

I would also like to thank Dr. Emil Schwab and Dr. Juan C. Noveron for being part of my thesis committee. I greatly appreciate their time to listen and participate while reviewing this work. Also, I am very grateful to my brother Yunis for his support, help and patience throughout these two years.
Abstract

A sequence $\{x_n\}$ is said to be statistically convergent to $\ell$ provided that "almost all" of the values of $\{x_n\}$ are arbitrarily close to $\ell$. One can also define what is meant by statistical limit point, statistical limit superior, statistical limit inferior of a sequence and so forth and thus create a theory of convergence that includes ordinary convergence. In this work we investigate all these concepts and prove some new results. We also introduce a topology defined by this new convergence which we call statistical topology. Then we prove that both the statistical topology and the regular topology are identical.
# Table of Contents

<table>
<thead>
<tr>
<th>Acknowledgements</th>
<th>iv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>v</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
</tbody>
</table>

## Chapter

1. Introduction and Background ........................................... 1
2. Statistical Convergence .................................................. 3
   2.1 Preliminaries and Basic Results ................................. 4
3. Statistical Convergence Topology ...................................... 15
4. Statistical limit-superior and limit-inferior ...................... 19
5. Future research related problems ..................................... 23
   5.1 Statistical convergence in metric spaces ....................... 23
   5.2 Extension of Banach contraction principle ..................... 24

## References ........................................................................ 25

## Appendix

Curriculum Vitae .............................................................. 29
Chapter 1

Introduction and Background

The idea of statistical convergence goes back to the first edition (published in Warsaw in 1935) of the monograph of Zygmund. Formally the concept of statistical convergence was introduced by Steinhaus and Fast and later reintroduced by Schoenberg. Statistical convergence, while introduced over nearly fifty years ago, has only recently become an area of active research. Different mathematicians studied properties of statistical convergence and applied this concept in various areas such as measure theory, trigonometric series, approximation theory, locally convex spaces, finitely additive set functions, in the study of subsets of the Stone-Cech compactification of the set of natural numbers, and Banach spaces. However, in a general case, neither limits nor statistical limits can be calculated or measured with absolute precision. To reflect this imprecision and to model it by mathematical structures, several approaches in mathematics have been developed: fuzzy set theory, fuzzy logic, interval analysis, set valued analysis, etc. One of these approaches is the neoclassical analysis. In it, ordinary structures of analysis, that is, functions, sequences, series, and operators, are studied by means of fuzzy concepts: fuzzy limits, fuzzy continuity, and fuzzy derivatives. For example, continuous functions, which are studied in the classical analysis, become a part of the set of the fuzzy continuous functions studied in neoclassical analysis. Neoclassical analysis extends methods of classical calculus to reflect uncertainties that arise in computations and measurements.
In particular if $A \subseteq \mathbb{N}$ then the asymptotic density of $A$, denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{k \to \infty} \frac{1}{n} |\{k \leq n/k \in A\}|$$

where $|B|$ is the cardinality of the $B$. A real valued sequence $x = (x_k)$ is statistically convergent to $\ell$ if

$$\delta(\{k : |x_k - \ell| > \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $\lim_{k \to \infty} x_k = \ell$ and we call $\ell$ the statistical limit of $x$. 
Chapter 2

Statistical Convergence

Statistics is concerned with the collection and analysis of data and with making estimations and predictions from the data. Typically two branches of statistics are discerned: descriptive and inferential. Inferential statistics is usually used for two tasks: to estimate properties of a population given sample characteristics and to predict properties of a system given its past and current properties. To do this, specific statistical constructions were invented. The most popular and useful of them are the average or mean (or more exactly, arithmetic mean) $\mu$ and standard deviation $\sigma$ (variance $\sigma^2$). To make predictions for future, statistics accumulates data for some period of time. To know about the whole population, samples are used. Normally such inferences (for future or for population) are based on some assumptions on limit processes and their convergence. Iterative processes are used widely in statistics. For instance the empirical approach to probability is based on the law (or better to say, conjecture) of big numbers, states that a procedure repeated again and again, the relative frequency probability tends to approach the actual probability. The foundation for estimating population parameters and hypothesis testing is formed by the central limit theorem, which tells us how sample means change when the sample size grows. In experiments, scientists measure how statistical characteristics (e.g., means or standard deviations) converge. Convergence of means/averages and standard deviations have been studied by many authors and applied to different problems. Convergence of statistical characteristics such as the average/mean and standard deviation are related to statistical convergence as we show in Section 4.
2.1 Preliminaries and Basic Results

In this section we are going to recall some basic definitions and background of statistical convergence of sequences in \( \mathbb{R} \). Consider a subset \( K \) of the set \( \mathbb{N} \) of all natural numbers. Set \( K_n = \{ k \in K; k \leq n \} \).

**Definition 2.1.1.** Let \( K \) be a subset of \( \mathbb{N} \) the set of natural numbers. Then the asymptotic density \( \delta(K) \) of the set \( K \) is equal to \( \lim_{n \to \infty} \frac{1}{n} |K_n| \) whenever the limit exists; here \( |B| \) denotes the cardinality of the set \( B \).

It is easy to check that that \( \delta(\mathbb{N}/K) = 1 - \delta(K) \), for any subset \( K \) of \( \mathbb{N} \). Let \( \{ x_n \} \) be a sequence of real numbers, and \( \ell \in \mathbb{R} \). Set

\[
L_\varepsilon(\{x_n\}) = \{ i \in \mathbb{N}; |x_i - \ell| \geq \varepsilon \}.
\]

**Definition 2.1.2.** The asymptotic density, or simply, density \( \delta(\{x_n\}) \) of the sequence \( \{x_n\} \) with respect to \( \ell \) and \( \varepsilon \) is equal to \( \delta(L_\varepsilon(\{x_n\})) \).

Asymptotic density allows us to define statistical convergence of sequences.

**Definition 2.1.3.** A sequence \( (x_n) \) is statistically convergent to \( \ell \) if and only if for each \( \varepsilon > 0 \), we have

\[
\delta(\{ n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon \}) = 0.
\]

We will write \( \text{stat} \lim_{n \to \infty} x_n = \ell \).

One may then wonder if the statistical limit is unique. This is done in the following result.

**Theorem 2.1.4.** If a sequence \( \{x_n\} \) is statistically convergent, then its \( \text{stat} \text{- limit is unique.} \)
Proof. Suppose that \( \text{stat} \lim_{n \to \infty} x_n = \ell_1 \) and \( \text{stat} \lim_{n \to \infty} x_n = \ell_2 \). Given \( \varepsilon > 0 \), define the following sets:

\[
K_1(\varepsilon) = \left\{ n \in \mathbb{N} : |x_n - \ell_1| \geq \frac{\varepsilon}{2} \right\} \quad \text{and} \quad K_2(\varepsilon) = \left\{ n \in \mathbb{N} : |x_n - \ell_2| \geq \frac{\varepsilon}{2} \right\}.
\]

Since \( \text{stat} \lim_{n \to \infty} x_n = \ell_1 \), we have \( \delta(K_1(\varepsilon)) = 0 \). Similarly since \( \text{stat} \lim_{n \to \infty} x_n = \ell_2 \), we have \( \delta(K_2(\varepsilon)) = 0 \). Now let \( K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon) \). Then \( \delta(K(\varepsilon)) = 0 \) which implies \( \mathbb{N} \setminus \delta(K(\varepsilon)) = 1 \). Now if \( k \in \mathbb{N} \setminus K(\varepsilon) \), then we have

\[
|\ell_1 - \ell_2| \leq |\ell_1 - x_n| + |x_n - \ell_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, we get \( |\ell_1 - \ell_2| = 0 \), i.e. \( \ell_1 = \ell_2 \).

In the next result, we compare the statistical convergence to the regular convergence.

**Theorem 2.1.5.** If \( \lim_{n \to \infty} x_n = \ell \), then \( \text{stat} \lim_{n \to \infty} x_n = \ell \). The converse need not be true in general.

**Proof.** Assume that \( \lim_{n \to \infty} x_n = \ell \). Then for every \( \varepsilon > 0 \), there exists a positive integer \( N \geq 1 \), such that \( |x_n - \ell| < \varepsilon \), for all \( n \geq N \). Since \( \delta(A(\varepsilon)) = 0 \), where \( A(\varepsilon) = \{ k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon \} \), we get \( \text{stat} \lim_{n \to \infty} x_n = \ell \).

The following example shows that the converse does not always hold.

**Example 2.1.6.** let \( \{x_k\} \) be defined as

\[
x_k = \begin{cases} 
1 & \text{if } k \text{ is a square} \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly \( \{x_k\} \) is not convergent. In fact, the only limit points of \( \{x_k\} \) are 0 and 1. But \( \{x_n\} \) is statistically convergent and its statistical limit is 0. Indeed let \( \varepsilon > 0 \), we have

\[
\delta(\{ k \in \mathbb{N} : |x_k - 0| \geq \varepsilon \}) = \lim_{n \to \infty} \frac{1}{n}|\{ k \in \mathbb{N} : |x_k - 0| \geq \varepsilon \}|.
\]

Fix \( n \in \mathbb{N} \) and set \( A = \{ k^2 : k^2 \leq n \} \), then \( |A| \leq \sqrt{n} \). So,

\[
\delta(\{ k \in \mathbb{N} : |x_k - 0| \geq \varepsilon \}) = \lim_{n \to \infty} \frac{1}{n}|\{ k \leq n : |x_k - 0| \geq \varepsilon \}| \leq \lim_{n \to \infty} \frac{\sqrt{n}}{n}.
\]

So \( \delta(\{ k \in \mathbb{N} : |x_k - 0| \geq \varepsilon \}) = 0 \). Hence \( \text{stat} \lim_{n \to \infty} x_n = 0 \).
This example suggests the following definition.

**Definition 2.1.7.** A real number \( \ell \) is called a statistical limit point of \( \{x_k\} \) if there is a set \( \{k_1 < k_2 < k_3 < \cdots < k_n < \cdots \} \subseteq \mathbb{N} \), the asymptotic density of which is not zero (i.e., it is greater than zero or does not exist) such that \( \text{stat} \lim_{n \to \infty} x_{k_n} = \ell \). Let \( \Lambda_{\{x_k\}} \) denote the set of statistical limit points of \( \{x_k\} \). \( \ell \) is an ordinary limit point of a sequence \( \{x_k\} \) if there is a subsequence of \( \{x_{n_k}\} \) that converges to \( \ell \). Denote by \( L_{\{x_k\}} \) the set of ordinary limit points of \( \{x_k\} \).

For the sequence considered in Example 2.1, we have \( L_{\{x_k\}} = \{0, 1\} \) while \( \Lambda_{\{x_k\}} = \{0\} \). We will see later on that in general we have \( \Lambda_{\{x_k\}} \subset L_{\{x_k\}} \), for any sequence \( \{x_n\} \). One more difference between regular convergence and statistical convergence is given in the following example.

**Example 2.1.8.** Let us consider the sequence \( \{x_k\} \) of real numbers whose terms are

\[
x_k = \begin{cases} 
  k & \text{when } k = n^2 \\
  0 & \text{otherwise}.
\end{cases}
\]

Then, it is easy to see that the sequence \( \{x_n\} \) is not bounded and statistically converges to 0 since \( \delta(K) = 0 \), where \( K = \{n^2; n = 1, 2, 3, \ldots \} \).

Therefore a statistically convergent sequence may not be bounded while regular convergent sequences are bounded. Another difference between the two types of convergence deal with subsequences. Indeed it is known that a subsequence of a convergent sequence is convergent. This result is not true for statistical convergence. For example, the sequence \( \{a_k\} \), where \( a_k = k \) for any \( k \geq 1 \), is a subsequence of the sequence \( \{x_k\} \) defined in Example 2.2. Clearly \( \{a_k\} \) is not statistically convergent to 0 while \( \{x_k\} \) is statistically convergent to 0. Indeed we have

\[
\delta\left(\{n \in \mathbb{N}; |a_n - 0| \geq 1\}\right) = \delta(\mathbb{N}) = 1.
\]

However, if we consider dense subsequences of statistically convergent sequences, it is possible to prove the corresponding result. Recall that a subset \( K \) of the set \( \mathbb{N} \) is
called statistically dense if \( \delta(K) = 1 \). For example, the set \( \{ n \in \mathbb{N}; \ n \text{ is not a square} \} \) is statistically dense, while the set \( \{3i; i = 1, 2, 3, \ldots \} \) is not statistically dense (in fact we have \( \delta(\{3i; i = 1, 2, 3, \ldots \}) = 1/3 \)).

The following technical lemma will be helpful to prove some algebraic properties of statistically convergent sequences.

**Lemma 2.1.9.** The following statements are true:

(a) A statistically dense subset of a statistically dense set is a statistically dense set.

(b) The intersection and union of two statistically dense sets are statistically dense sets.

**Proof.** The proof of (a) is easy as well as the proof of (b) for the union of two statistically dense subsets. Let us prove (b) for the intersection of two statistically dense subsets. Indeed let \( A \) and \( B \) be two nonempty subsets of \( \mathbb{N} \). In order to prove that \( \delta(A \cap B) = 1 \), we will prove that \( \delta(\mathbb{N} \setminus (A \cap B)) = 0 \). Since

\[
\mathbb{N} \setminus (A \cap B) = (\mathbb{N} \setminus A) \cup (\mathbb{N} \setminus B),
\]

we get

\[
\delta(\mathbb{N} \setminus (A \cap B)) \leq \delta(\mathbb{N} \setminus A) + \delta(\mathbb{N} \setminus B) = 0,
\]

which completes the proof of our claim. \( \square \)

Another similarity with regular convergence lies in the following result:

**Theorem 2.1.10.** Let \( \{x_k\} \) and \( \{y_k\} \) be two statistically convergent sequences. If \( \text{stat} - \lim_{k \to \infty} x_k = a \), \( \text{stat} - \lim_{k \to \infty} y_k = b \), and \( c \) is any real number, then

(i) \( \text{stat} - \lim_{k \to \infty} (x_k + y_k) = a + b \),

(ii) \( \text{stat} - \lim_{k \to \infty} (c \, x_k) = c \, a \).
Proof. Let us prove (i). Let \( \{x_k\} \) and \( \{y_k\} \) be two statistically convergent sequences. Assume \( \text{stat} - \lim_{k \to \infty} x_k = a \), \( \text{stat} - \lim_{k \to \infty} y_k = b \). Then for every \( \varepsilon > 0 \), we have
\[
\delta\left( \{ k \in \mathbb{N}; |x_k - a| < \varepsilon \} \right) = 1 \quad \text{and} \quad \delta\left( \{ k \in \mathbb{N}; |y_k - b| < \varepsilon \} \right) = 1.
\]
Using the Lemma 2.1.9, we get
\[
\delta\left( \{ k \in \mathbb{N}; |x_k - a| < \varepsilon \} \cap \{ k \in \mathbb{N}; |y_k - b| < \varepsilon \} \right) = 1.
\]
Set \( K = \{ k \in \mathbb{N}; |x_k - a| < \varepsilon \} \cap \{ k \in \mathbb{N}; |y_k - b| < \varepsilon \} \). Then, for any \( k \in K \), we have
\[
|x_k - a| < \varepsilon \quad \text{and} \quad |y_k - b| < \varepsilon,
\]
which implies
\[
|x_k + y_k - (a + b)| < \varepsilon + \varepsilon = 2\varepsilon.
\]
Then
\[
\delta(K) = 1 \leq \delta\left( \{ k \in \mathbb{N}; |x_k + y_k - (a + b)| < 2\varepsilon \} \right) \leq 1.
\]
Since \( \varepsilon \) was arbitrary, we get \( \text{stat} - \lim_{k \to \infty} (x_k + y_k) = a + b \).

The proof of (ii) is similar to the one used for the proof of (i). \qed

Definition 2.1.11. A subsequence \( \{x_{k_n}\} \) of the sequence \( \{x_n\} \) is called statistically dense in \( \{x_n\} \) if the set of all indices \( K = \{k_n; n \geq 1\} \) is statistically dense.

Using Lemma 2.1.9, we get the following result:

Corollary 2.1.12. A statistically dense subsequence of a statistically dense subsequence of \( \{x_n\} \) is a statistically dense subsequence of \( \{x_n\} \).

Now we are ready to prove an analogue to the known characterization of convergent sequence using subsequences.

Theorem 2.1.13. A sequence \( \{x_n\} \) is statistically convergent if and only if any statistically dense subsequence of \( \{x_n\} \) is statistically convergent.
Proof. First Assume that \( \{x_n\} \) is statistically convergent and \( \text{stat} - \lim_{n \to \infty} x_n = \ell \). Let us prove that any statistically dense subsequence \( \{y_k\} \) of \( \{x_n\} \) is also statistically convergent and \( \text{stat} - \lim_{k \to \infty} y_k = \ell \). Assume not, i.e., \( \{y_k\} \) does not statistically converge to \( \ell \). Then there is some \( \varepsilon > 0 \) such that
\[
\lim_{n \to \infty} \frac{1}{n} |B_{n, \varepsilon}| = d
\]
for some \( d \in (0, 1) \), where \( B_{n, \varepsilon} = \{k \leq n; |y_k - \ell| \geq \varepsilon\} \). As \( \{y_k\} \) is a subsequence of \( \{x_i\} \), we have \( A_{n, \varepsilon} \supseteq B_{n, \varepsilon} \), where \( A_{n, \varepsilon} = \{i \leq n; |x_i - \ell| \geq \varepsilon\} \). Consequently, \( \lim_{n \to \infty} \frac{1}{n} |A_{n, \varepsilon}| \geq d > 0 \), which yields that \( \delta(\{n \in \mathbb{N}; |x_n - \ell| \geq \varepsilon\}) \neq 0 \). Therefore \( \{x_n\} \) does not statistically converge to \( \ell \). Contradiction.

As for the converse, we use the fact that any sequence is a statistically dense subsequence of itself.

In particular this result implies that a statistically dense subsequence of a statistically convergent sequence is statistically convergent. The next result is may be the most important and useful when dealing with statistical convergence.

**Theorem 2.1.14.** A sequence \( \{x_n\} \) is statistically convergent, and \( \text{stat} - \lim_{n \to \infty} x_n = \ell \), if and only if there exists a set \( K = \{k_1 < k_2 < k_3 < \ldots < k_n < \ldots\} \subset \mathbb{N} \) such that \( \delta(K) = 1 \) and \( \lim_{n \to \infty} x_{k_n} = \ell \).

Proof. Assume first that a set \( K = \{k_1 < k_2 < k_3 < \ldots < k_n < \ldots\} \) exists such that \( \delta(K) = 1 \) and \( \lim_{n \to \infty} x_{k_n} = \ell \). Let \( \varepsilon > 0 \). Then there exists \( n_0 \in \mathbb{N} \) such that
\[
|x_{k_n} - \ell| < \varepsilon, \text{ for any } n \geq n_0.
\]
Put \( A_\varepsilon = \{n \in \mathbb{N}; |x_n - \ell| < \varepsilon\} \). Then \( \{k_n; \text{ for } n \geq n_0\} \subset A_\varepsilon \). Since
\[
\delta(\{k_n; \text{ for } n \geq n_0\}) = \delta(K \setminus \{k_1, \ldots, k_{n_0-1}\}) = 1,
\]
we get \( \delta(A_\varepsilon) = 1 \) or \( \delta(\{n \in \mathbb{N}; |x_n - \ell| \geq \varepsilon\}) = 0 \). Hence the sequence \( \{x_n\} \) is statistically convergent, and \( \text{stat} - \lim_{n \to \infty} x_n = \ell \).
Conversely assume that \( \{x_n\} \) is statistically convergent, and \( \text{stat} - \lim_{n \to \infty} x_n = \ell \). Put
\[
K_j = \left\{ n \in \mathbb{N}; |x_n - \ell| < \frac{1}{j} \right\}, \quad j = 1, 2, \ldots.
\]
Then since \( \{x_n\} \) is statistically convergent to \( \ell \), we have \( \delta(K_j) = 1, \quad j = 1, 2, \ldots. \) It is evident from the definition of \( K_j \), that
\[
(3) \quad K_1 \supset K_2 \supset \ldots \supset K_j \supset K_{j+1} \supset \ldots,
\]
\[
(3') \quad \delta(K_j) = 1, \quad j = 1, 2, \ldots.
\]
Let us choose an arbitrary number \( v_1 \in K_1 \). According to \( (3') \) there exists a \( v_2 > v_1 \), \( v_2 \in K_2 \) such that for each \( n \geq v_2 \), we have \( \frac{K_2(n)}{n} > \frac{1}{2} \) where \( K_j(n) = \{ k \in K_j; k \leq n \} \) and \( \lim_{n \to \infty} \frac{K_j(n)}{n} = \delta(K_j) = 1 \). Further, according to \( (3') \), there exists a \( v_3 > v_2 \), \( v_3 \in K_3 \) such that for each \( n \geq v_3 \), we have \( \frac{K_3(n)}{n} > \frac{2}{3} \). Thus by induction, we can construct a sequence \( v_1 < v_2 < \ldots < v_j < \ldots \) of positive integers such that \( v_j \in K_j \), and
\[
\frac{K_j(n)}{n} > \frac{j-1}{j},
\]
for each \( n \geq v_j \), \( j = 1, 2, \ldots \). Set
\[
K = [1, v_1) \cup \left( \bigcup_{j \geq 1} [v_j, v_{j+1}) \cap K_j \right).
\]
In other words, each natural number of the interval \( [1, v_1) \) belongs to \( K \), and any natural number of the interval \( [v_j, v_{j+1}) \) belongs to \( K \) if and only if it belongs to \( K_j, \quad j = 1, 2, \ldots \). Then for each \( n \in [v_j, v_{j+1}) \), we get
\[
\frac{K(n)}{n} = \frac{K_j(n)}{n} > \frac{j-1}{j}.
\]
Hence \( \delta(K) = 1 \). Let \( \varepsilon > 0 \). Choose a \( j \geq 1 \) such that \( \frac{1}{j} < \varepsilon \). Let \( n \geq v_j, \quad n \in K \). Then there exists a number \( i \geq j \) such that \( n \in [v_i, v_{i+1}) \cap K_i \). Hence
\[
|x_n - \ell| < \frac{1}{i} \leq \frac{1}{j} < \varepsilon.
\]
Thus \( |x_n - \ell| < \varepsilon \), for each \( n \in K, \quad n \geq v_j, \) i.e., \( \lim_{k \to \infty, k \in K} \ x_k = \ell. \)
The next result is a beautiful one that links statistical convergence to the regular convergence. First recall the Cesaro averages or the partial averages of any sequence \( \{x_n\} \) as

\[
\mu(\{x_n\}) = \{\mu_n\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} x_k; n = 1, 2, 3, \ldots \right\}.
\]

**Theorem 2.1.15.** Assume \( \{x_n\} \) is bounded and \( \text{stat} - \lim_{n \to \infty} x_n = \ell \), then \( \lim_{n \to \infty} \mu(x_n) = \ell \).

**Proof.** Assume \( \text{stat} - \lim_{n \to \infty} x_n = \ell \). Let \( \varepsilon > 0 \). We have

\[
\lim_{n \to \infty} \frac{1}{n} |\{i \leq n; |x_i - \ell| \geq \varepsilon\}| = 0.
\]

Set \( L_{\varepsilon} = \{i \in \mathbb{N}; |x_i - \ell| \geq \varepsilon\} \). Then we have \( \delta(L_{\varepsilon}) = 0 \). If \( i \notin L_{\varepsilon} \), then we have \( |x_i - \ell| < \varepsilon \).

Since \( \{x_n\} \) is bounded, there exists \( M \in (0, \infty) \) such that \( |x_n| \leq M \). So for any \( n \in \mathbb{N} \), we have \( |x_n - \ell| \leq M + |\ell| = M* \). Let us consider the set

\[
L_{n,\varepsilon} = \{i \leq n, i \in \mathbb{N}; |x_i - \ell| \geq \varepsilon\},
\]

and denote \( |L_{n,\varepsilon}| \) by \( u_n \). Then \( \lim_{n \to \infty} \frac{u_n}{n} = 0 = \delta(L_{\varepsilon}) \). Since

\[
|\mu_n - \ell| = \left| \frac{1}{n} \sum_{i=1}^{n} x_i - \ell \right| \leq \frac{1}{n} \sum_{i=1}^{n} |x_i - \ell| \]

\[
\leq \frac{1}{n} \left( M^*u_n + (n - u_n)\varepsilon \right) \]

\[
\leq \frac{1}{n} (M^*u_n + n\varepsilon) = \varepsilon + M^* \left( \frac{u_n}{n} \right),
\]

and \( \lim_{n \to \infty} \frac{u_n}{n} = 0 \), we get

\[
|\mu_n - \ell| < \varepsilon + M^* \varepsilon,
\]

for sufficiently large \( n \). Thus, \( \lim_{n \to \infty} \mu(x_n) = \ell \).

However, convergence of the partial averages/means of a sequence does not imply statistical convergence of this sequence as the following example demonstrates.
Example 2.1.16. Let us consider the sequence \((-1)^n \sqrt{n}\). This sequence is statistically divergent although \(\lim_{n \to \infty} \mu(\{x_n\}) = 0\). Indeed, one can show that
\[
\sum_{k=1}^{2n} (-1)^k \sqrt{k} \geq 0 \quad \text{and} \quad \sum_{k=1}^{2n+1} (-1)^k \sqrt{k} \leq 0,
\]
for any \(n \geq 1\). Using these inequalities, we obtain
\[
\left| \sum_{k=1}^{n} (-1)^k \sqrt{k} \right| \leq \sqrt{n}, \ n \geq 1.
\]
Hence
\[
\lim_{n \to \infty} \mu(\{x_n\}) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k \sqrt{k} = 0.
\]
Next let us prove that the sequence \((-1)^n \sqrt{n}\) is statistically divergent. Assume not. Hence there exists \(\ell \in \mathbb{R}\) such that \(\text{stat} - \lim_{n \to \infty} (-1)^n \sqrt{n} = \ell\). In particular, we have
\[
\delta\left(\{n \in \mathbb{N}; |(-1)^n \sqrt{n} - \ell| < 1\}\right) = 1.
\]
This will imply that there infinitely many \(n \in \mathbb{N}\) such that \(|(-1)^n \sqrt{n} - \ell| < 1\). Note that if \(|(-1)^n \sqrt{n} - \ell| < 1\), we get
\[
|(-1)^n \sqrt{n}| \leq |(-1)^n \sqrt{n} - \ell| + |\ell| < 1 + |\ell|,
\]
which leads to a contradiction since \(\{\sqrt{n}\}\) goes to infinity when \(n\) goes to infinity.

Taking a sequence \(\{x_n\}\) of real numbers, it is possible to construct the sequences \(\sigma(\{x_n\}) = \{\sigma_n\}\) and \(\sigma^2(\{x_n\}) = \{\sigma_n^2\}\), where
\[
\sigma_n = \left(\frac{1}{n} \sum_{k=1}^{n} (x_k - \mu_n)^2\right)^{1/2}
\]
of its partial standard deviations and
\[
\sigma_n^2 = \frac{1}{n} \sum_{k=1}^{n} (x_k - \mu_n)^2,
\]
of its partial variances, \(n = 1, 2, 3, \ldots\). We have the following beautiful result:
Theorem 2.1.17. Assume \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} x_n = \ell \), then

\[
\lim_{n \to \infty} \sigma(\{x_k\}) = 0 \quad \text{and} \quad \lim_{n \to \infty} \sigma^2(\{x_k\}) = 0.
\]

Proof. Without loss of generality, we will only show that \( \lim_{k \to \infty} \sigma^2(x_k) = 0 \). By definition, we have

\[
\sigma^2_n(x_k) = \frac{1}{n} \sum_{k=1}^{n} (x_k - \mu_n)^2 = \frac{1}{n} \sum_{k=1}^{n} (x_k)^2 - \mu_n^2.
\]

Thus,

\[
\lim_{n \to \infty} \sigma^2_n(x_k) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (x_k)^2 - \lim_{n \to \infty} \mu_n^2.
\]

Since \( \{x_n\} \) is bounded, there exists \( M \in (0, \infty) \) such that \( |x_k| \leq M \), for all \( k \in \mathbb{N} \). Then

\[
|x_k^2 - \ell^2| = (|x_k - \ell|)(|x_k + \ell|) \leq (M + |\ell|)|x_k - \ell|,
\]

for any \( n \geq 1 \). Next we take the set

\[
L_{n,\varepsilon} = \{k \leq n, k \in \mathbb{N}; |x_k - \ell| \geq \varepsilon\},
\]

and denote \( |L_{n,\varepsilon}| \) by \( u_n \). Hence

\[
|\sigma^2_n| = \left| \frac{1}{n} \sum_{k=1}^{n} (x_k)^2 - \mu_n^2 \right| = \left| \frac{1}{n} \sum_{k=1}^{n} (x_k^2 - \ell^2) - (\mu_n^2 - \ell^2) \right|
\]

\[
\leq \frac{(M + |\ell|)}{n} \sum_{k=1}^{n} |x_k - \ell| + |\mu_n^2 - \ell^2|
\]

\[
\leq \frac{(M + |\ell|)}{n} (u_n + (n - u_n)\varepsilon) + |\mu_n^2 - \ell^2|
\]

\[
\leq \frac{(M + |\ell|)}{n} (u_n + n\varepsilon) + |\mu_n^2 - \ell^2|
\]

\[
= (M + |\ell|) \left( \frac{u_n}{n} \right) + \varepsilon(M + |\ell|) + |\mu_n^2 - \ell^2|,
\]
Theorem 2.1.15 implies \( \lim_{k \to \infty} \mu(x_k) = \ell \), which guarantees that \( \lim_{n \to \infty} \mu_n^2 = \ell^2 \). Also since \( \{x_n\} \) statistically converges to \( \ell \), we get \( \lim_{n \to \infty} u_n/n = 0 \). Since \( \varepsilon > 0 \) was arbitrary, the right hand side of the above inequality tends to 0 when \( n \to \infty \). Therefore, we have

\[
\lim_{k \to \infty} \sigma^2(\{x_k\}) = 0.
\]
Chapter 3

Statistical Convergence Topology

In this section we will study the topology generated by the statistical convergence. This new type of convergence will allow us to define closed subsets mirroring what one knows from the regular convergence.

Definition 3.0.18. Let $F \subset \mathbb{R}$ be nonempty and $\ell \in \mathbb{R}$. Then $\ell$ is in the statistical closure of $F$ if there is a sequence $(x_n)$ of points in $F$ such that $\text{stat}\lim_{n \to \infty} x_n = \ell$. We denote the statistical closure of a set $F$ by $\overline{F}^{st}$. We say that a set is statistically closed if it contains all of the points in its statistical closure, i.e., $\overline{F}^{st} \subset F$.

Let us denote by $\mathcal{T}_f$ the family of all statistically closed sets. Next we investigated whether $\mathcal{T}_f$ defines a topology. Note that the empty set and $\mathbb{R}$ are statistically closed. Also it is obvious that $\mathcal{T}_f$ is stable by intersection. The union was not clear and immediate to us.

Theorem 3.0.19. If $A$ and $B$ are two statistically closed subsets of $\mathbb{R}$, then $A \cup B$ is statistically closed.

Proof. Let $\{x_n\}$ be a sequence in $A \cup B$ such that $\text{stat}\lim_{n \to \infty} x_n = x$. Let us prove that $x \in A \cup B$. The Theorem 2.1.14 implies the existence $K = \{k_1 < k_2 < k_3 < \ldots < k_n < \ldots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and $\lim_{n \to \infty} x_{k_n} = x$. Set

$$A(\{x_n\}) = \{k_n; x_{k_n} \in A\} \quad \text{and} \quad B(\{x_n\}) = \{k_n; x_{k_n} \in B \setminus A\}.$$ 

Clearly we have $A(\{x_n\}) \cup B(\{x_n\}) = K$. Since $\delta(A(\{x_n\})) + \delta(B(\{x_n\})) = \delta(K) = 1$, one of the two sets $A(\{x_n\})$ or $B(\{x_n\})$ is infinite. Assume without loss of generality that
$A(\{x_n\})$ is infinite. Set $A(\{x_n\}) = \{\varphi(n); n \geq 1\}$. Since $\{x_{\varphi(n)}\}$ is a subsequence of $\{x_{k_n}\}$, we get $\lim_{n \to \infty} x_{\varphi(n)} = x$, which implies $\text{stat-} \lim_{n \to \infty} x_{\varphi(n)} = x$, using Theorem 2.1.5. Since $A$ is statistically closed, we get $x \in A$.

The following result shed some light on our investigation of the statistical topology.

**Theorem 3.0.20.** Let $F$ be a nonempty subset of $\mathbb{R}$. $F$ is statistically closed if and only if $F$ is closed, i.e. the statistical topology $T_J$ is identical to the classical topology.

**Proof.** First assume $F$ is a statistically closed subset in $X$, we need to prove that $F$ contains its accumulation points. Let $p$ be an accumulation point of $F$. Let us prove that $p$ is in $F$. Since $p$ is an accumulation point of $F$, there exists a sequence $\{x_n\}$ in $F$ such that $\lim_{n \to \infty} x_n = p$. Using Theorem 2.1.5, we know that $\text{stat-} \lim_{n \to \infty} x_n = p$. Finally since $F$ is statistically closed, then we must have $p \in F$.

Conversely assume that $F$ is a closed subset of $X$. Let us prove that $F$ is statistically closed. Let $\{x_n\}$ be in $F$ such that $\text{stat-} \lim_{n \to \infty} x_n = \ell$. We need to prove that $\ell \in F$. By Theorem 2.1.14, there exists a set $K = \{k_1 < k_2 < k_3 < \ldots < k_n < \ldots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and $\lim_{n \to \infty} x_{k_n} = \ell$. Since $F$ is closed and $\{x_{k_n}\}$ is in $F$, we get $\ell \in F$.

Next we discuss the concept of statistical Cauchy sequences.

**Definition 3.0.21.** A sequence $\{x_k\}$ is said to be a statistically Cauchy sequence if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$
\delta\left(\{n \in \mathbb{N}; |x_n - x_N| \geq \varepsilon\}\right) = 0.
$$

As for the convergent sequences, we have the following result:

**Theorem 3.0.22.** Any sequence $\{x_k\}$ is statistically convergent if and only if it is statistically Cauchy, i.e., $\mathbb{R}$ is statistically complete.

**Proof.** Assume first that $\text{stat-} \lim_{k \to \infty} x_k = \ell$. Then, for any $\varepsilon > 0$, we get $\delta(A(\varepsilon)) = 0$, where

$$
A(\varepsilon) = \left\{n \in \mathbb{N}; |x_n - \ell| \geq \frac{\varepsilon}{2}\right\}.
$$
This implies that
\[ \delta(N \setminus A(\varepsilon)) = \delta \left( \left\{ n \in \mathbb{N}; |x_n - \ell| < \frac{\varepsilon}{2} \right\} \right) = 1. \]

Let \( m, n \not\in A(\varepsilon) \), then \( |x_m - x_n| < \varepsilon \). Fix \( m \not\in A(\varepsilon) \) and let
\[ B(\varepsilon) = \{ n \in \mathbb{N}; |x_m - x_n| < \varepsilon \}. \]

Then \( N \setminus A(\varepsilon) \subset \subset B(\varepsilon) \). Hence
\[ 1 = \delta \left( N \setminus A(\varepsilon) \right) \leq \delta \left( B(\varepsilon) \right) \leq 1. \]

This will imply \( \delta \left( N \setminus B(\varepsilon) \right) = 0 \), where \( N \setminus B(\varepsilon) = \{ n \in \mathbb{N}; |x_m - x_n| \geq \varepsilon \} \). This clearly implies that \( \{ x_n \} \) is statistically Cauchy.

Conversely, assume that \( \{ x_k \} \) is statistically Cauchy. Then, for any \( \varepsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such that \( \delta \left( \{ n \in \mathbb{N}; |x_n - x_{N_0}| < \varepsilon \} \right) = 1 \). In particular, we have
\[ \delta \left( \{ n \in \mathbb{N}; x_n < x_{N_0} + \varepsilon \} \right) = 1 \text{ and } \delta \left( \{ n \in \mathbb{N}; x_{N_0} - \varepsilon < x_n \} \right) = 1. \]

Set
\[ A = \left\{ a \in \mathbb{R}; \delta \left( \{ n \in \mathbb{N}; x_n < a \} \right) = 1 \right\}, \]
and
\[ B = \left\{ b \in \mathbb{R}; \delta \left( \{ n \in \mathbb{N}; x_n > b \} \right) = 1 \right\}. \]

Hence \( x_{N_0} + \varepsilon \in A \) and \( x_{N_0} - \varepsilon \in B \). Moreover let \( a \in A \) and \( b \in B \), then
\[ \delta \left( \{ n \in \mathbb{N}; x_n < a \} \right) = 1 \text{ and } \delta \left( \{ n \in \mathbb{N}; x_n > b \} \right) = 1. \]

Theorem 2.1.9 implies
\[ \delta \left( \{ n \in \mathbb{N}; b < x_n < a \} \right) = 1. \]

This will imply \( b < a \). In particular we have
\[ x_{N_0} - \varepsilon \leq \sup B \leq \inf A \leq x_{N_0} + \varepsilon. \]
Since $\varepsilon$ was arbitrary positive, we get $\sup B = \inf A$. Set $\ell = \sup B = \inf A$. Let $\varepsilon > 0$. Then there exists $a \in A$ and $b \in B$ such that $\ell - \varepsilon < b < a < \ell + \varepsilon$. The definition of $A$ and $B$ imply
\[ \delta \left( \{ n \in \mathbb{N}; \ell - \varepsilon < x_n < \ell + \varepsilon \} \right) = 1. \]
Hence $\delta \left( \{ n \in \mathbb{N}; |x_n - \ell| < \varepsilon \} \right) = 1$, or $\delta \left( \{ n \in \mathbb{N}; |x_n - \ell| \geq \varepsilon \} \right) = 0$. In other words, \( \{x_n\} \) is statistically convergent and its statistical limit is $\ell$. \qed
Chapter 4

Statistical limit-superior and limit-inferior

The purpose of this section is to present natural definitions of the concepts of statistical limit superior and inferior and to develop some statistical analogues of properties of the ordinary limit superior and inferior. The latter results include statistical analogues of Knopp’s Core Theorem and R.C. Buck’s Theorem on Cesaro summability of a sequence to its limit superior.

For a real number sequence \( x = \{x_n\} \), let \( B_x \) denote the set:
\[
B_x = \left\{ b \in \mathbb{R} ; \delta\left( \left\{ k : x_k > b \right\} \right) \neq 0 \right\}.
\]
similarly,
\[
A_x = \left\{ a \in \mathbb{R} ; \delta\left( \left\{ k : x_k < a \right\} \right) \neq 0 \right\}.
\]
Note that, throughout this section, the statement \( \delta(K) \neq 0 \) means that either \( \delta(K) > 0 \) or \( K \) does not have natural density.

**Definition 4.0.23.** If \( \{x_n\} \) is a real number sequence, then the statistical limit superior of \( \{x_n\} \), denoted by \( \text{stat} - \limsup_{n \to \infty} x_n \), is given by
\[
\text{stat} - \limsup_{n \to \infty} x_n = \sup(B_x),
\]
provided that \( B_x \) is not empty. If \( B_x \) is empty, we set \( \text{stat} - \limsup_{n \to \infty} x_n = -\infty \). Also, the statistical limit inferior of \( \{x_n\} \), denoted by \( \text{stat} - \liminf_{n \to \infty} x_n \), is given by
\[
\text{stat} - \liminf_{n \to \infty} x_n = \inf(A_x),
\]
provided that \( A_x \) is not empty. If \( A_x \) is empty, we set \( \liminf_{n \to \infty} x_n = +\infty \).

A simple example will help to illustrate these two concepts just defined.

**Example 4.0.24.** Let the sequence \( \{x_k\} \) be given by

\[
x_k = \begin{cases} 
  k & \text{if } k \text{ is an odd square} \\
  2 & \text{if } k \text{ is an even square} \\
  1 & \text{if } k \text{ is an odd nonsquare} \\
  0 & \text{if } k \text{ is an even nonsquare}
\end{cases}
\]

Note that although \( \{x_k\} \) is unbounded above, it is "statistically bounded" because the set of squares has density zero. Thus \( B_x = (-\infty, 1) \) and \( \limsup_{n \to \infty} x_n = 1 \). Also \( \{x_k\} \) is not statistically convergent since it has two (disjoint) subsequences of positive density that converge to 0 and 1, respectively. Also note that the set of statistical limit points of \( \{x_k\} \) is the set \( \{0, 1\} \), and \( \limsup_{n \to \infty} x_n \) equals the greatest element while \( \liminf_{n \to \infty} x_n \) is the least element of this set, i.e., \( \liminf_{n \to \infty} x_n = 0 \) and \( \limsup_{n \to \infty} x_n = 1 \).

This example suggests the main idea of the following theorem, which can be proved by a straightforward least upper bound argument.

**Theorem 4.0.25.** Let \( \{x_k\} \) be a sequence such that \( \limsup_{n \to \infty} x_n = \beta \) is finite. Then for every \( \varepsilon > 0 \), we have

\[
\delta(\{k \in \mathbb{N}; x_k > \beta - \varepsilon\}) \neq 0 \quad \text{and} \quad \delta(\{k \in \mathbb{N}; x_k > \beta + \varepsilon\}) = 0.
\]

Conversely, if (4.1) holds for every \( \varepsilon > 0 \), then \( \limsup_{n \to \infty} x_n = \beta \).

The dual statement for \( \liminf \) is as follows:

**Theorem 4.0.26.** Let \( \{x_k\} \) be a sequence such that \( \liminf_{n \to \infty} x_n = \alpha \) is finite. Then for every \( \varepsilon > 0 \), we have

\[
\delta(\{k \in \mathbb{N}; x_k < \alpha + \varepsilon\}) \neq 0 \quad \text{and} \quad \delta(\{k \in \mathbb{N}; x_k < \alpha - \varepsilon\}) = 0.
\]

Conversely, if (4.2) holds for every \( \varepsilon > 0 \), then \( \limsup_{n \to \infty} x_n = \alpha \).
From the definition of statistical limit point, we see that Theorem 4.0.25 and Theorem 4.0.26 can be interpreted as saying that \( \text{stat} - \lim \sup \) and \( \text{stat} - \lim \inf \) of a sequence \( \{x_n\} \) are the greatest and the least statistical limit points of \( \{x_n\} \). The next theorem reinforces that observation.

**Theorem 4.0.27.** For any sequence \( x = \{x_k\} \), we have

\[
\text{stat} - \lim \inf_{n \to \infty} x_n \leq \text{stat} - \lim \sup_{n \to \infty} x_n.
\]

**Proof.** Clearly if \( \text{stat} - \lim \inf_{n \to \infty} x_n = -\infty \) or \( \text{stat} - \lim \sup_{n \to \infty} x_n = +\infty \), the conclusion is obvious. Therefore we will assume \( \text{stat} - \lim \inf_{n \to \infty} x_n > -\infty \) and \( \text{stat} - \lim \sup_{n \to \infty} x_n < +\infty \).

Set \( \alpha = \text{stat} - \lim \inf_{n \to \infty} x_n \) and \( \beta = \text{stat} - \lim \sup_{n \to \infty} x_n \). Let \( \varepsilon > 0 \). By Theorem 4.0.25 \( \delta \left( \{k; x_k > \beta + \varepsilon/2\} \right) = 0 \) because \( \beta = \sup B_x \). This implies that

\[
\delta \left( \{k; x_k \leq \beta + \varepsilon/2\} \right) = 1
\]

which, in turn, implies that \( \delta \left( \{k; x_k < \beta + \varepsilon\} \right) = 1 \). Hence, \( \beta + \varepsilon \in A_x \). By definition \( \alpha = \inf A_x \), so we conclude that \( \alpha \leq \beta + \varepsilon \) and since \( \varepsilon \) is arbitrarily positive this gives us \( \alpha \leq \beta \). \( \square \)

From Theorem 4.0.27 and the above definition, it is clear that

\[
\lim \inf_{n \to \infty} x_n \leq \text{stat} - \lim \inf_{n \to \infty} x_n \leq \text{stat} - \lim \sup_{n \to \infty} x_n \leq \lim \sup_{n \to \infty} x_n,
\]

for any sequence \( \{x_n\} \).

The next result is another statistical analogue of a very basic property of convergent sequences. For clarity of presentation we first give a formal definition of another statistical concept.

**Definition 4.0.28.** The real number sequence \( \{x_n\} \) is said to be statistically bounded if there is a number \( M \in \mathbb{R} \) such that

\[
\delta \left( \{n \in \mathbb{N}; |x_k| > M\} \right) = 0.
\]
Note that statistical boundedness implies that $\liminf_{n \to \infty} x_n$ and $\limsup_{n \to \infty} x_n$ are finite. Next we prove an analogue to the known result on regular convergence of sequences.

**Theorem 4.0.29.** The statistically bounded sequence $\{x_n\}$ is statistically convergent if and only if $\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n$.

**Proof.** Assume that the statistically bounded sequence $\{x_n\}$ is statistically convergent, and $\lim_{n \to \infty} x_n = \ell$. Let $\varepsilon > 0$. Then

$$\delta \left( \{n \in \mathbb{N}; |x_n - \ell| \geq \varepsilon \} \right) = 0.$$

So

$$\delta \left( \{n \in \mathbb{N}; x_n \geq \ell + \varepsilon \} \right) = 0,$$

which implies that $\liminf_{n \to \infty} x_n \leq \ell + \varepsilon$. We also have

$$\delta \left( \{n \in \mathbb{N}; x_n \leq \ell - \varepsilon \} \right) = 0,$$

which implies that $\limsup_{n \to \infty} x_n \geq \ell - \varepsilon$. Since $\varepsilon$ was arbitrary, we get $\liminf_{n \to \infty} x_n \leq \ell$ and $\limsup_{n \to \infty} x_n \geq \ell$. Therefore we must have

$$\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = \ell.$$

Conversely assume $\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n$. Set $\ell = \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n$. Let $\varepsilon > 0$. Then Theorems 4.0.25 and 4.0.26 imply

$$\delta \left( \{n \in \mathbb{N}; x_n > \ell + \frac{\varepsilon}{2} \} \right) = 0 \quad \text{and} \quad \delta \left( \{n \in \mathbb{N}; x_n < \ell - \frac{\varepsilon}{2} \} \right) = 0.$$

Hence $\lim_{n \to \infty} x_n = \ell$. \qed
Chapter 5

Future research related problems

5.1 Statistical convergence in metric spaces

Let \((M, d)\) be a metric space. A sequence \(\{x_n\}\) in \(M\) is said to be statistically convergent to \(x \in M\) if for every \(\varepsilon > 0\) we have

\[
\delta \left( \{ n \in \mathbb{N} ; d(x_n, x) \geq \varepsilon \} \right) = 0.
\]

It is easy to check many of the properties showed in the case of \(M = \mathbb{R}\). In particular, we have the following definition:

**Definition 5.1.1.** A sequence \(\{x_k\}\) in \(M\) is said to be a statistically Cauchy sequence if for every \(\varepsilon > 0\) there exists a number \(N = N(\varepsilon) \in \mathbb{N}\) such that

\[
\delta \left( \{ n \in \mathbb{N} ; d(x_n, x_N) \geq \varepsilon \} \right) = 0.
\]

It will be interesting to investigate whether the following theorem is still valid.

**Theorem 5.1.2.** Assume that \((M, d)\) is a complete metric space. Then any sequence \(\{x_k\}\) in \(M\) is statistically convergent if and only if it is statistically Cauchy, i.e., \(M\) is statistically complete.

In the case \(M = \mathbb{R}^m\), any sequence \(\{x_n\}\) will generate \(m\) real sequences (the components of each vector \(x_n\)). It will be interesting to characterize the statistical convergence of a sequence \(\{x_n\}\) in \(\mathbb{R}^m\) in terms of the statistical convergence of their components.
5.2 Extension of Banach contraction principle

Banach’s Contraction Mapping Principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the mapping is simple and easy to test, because it requires only a complete metric space for its setting, and because it finds almost canonical applications in the theory of differential and integral equations.

Definition 5.2.1. Let \((M, d)\) be a metric space. A mapping \(T : M \to M\) is said to be lipschitzian if there is a constant \(k \geq 0\) such that, for all \(x, y \in M\), we have

\[
d(T(x), T(y)) \leq k d(x, y).
\]

The smallest constant \(k\), for which the above inequality holds, is called the Lipschitz constant of \(T\) and is denoted \(\text{Lip}(T)\). The mapping \(T\) is called a contraction whenever \(\text{Lip}(T) < 1\).

A point \(x \in M\) is said to be a fixed point of \(T\) if \(T(x) = x\).

Now we are ready to state Banach’s Contraction Mapping Principle.

Theorem 5.2.2. Let \((M, d)\) be a complete metric space and let \(T : M \to M\) be a contraction mapping. Then \(T\) has a unique fixed point \(x_0\), and moreover, for each \(x \in M\), \(\lim_{n \to \infty} T^n(x) = x_0\).

One of the questions raised when studying the statistical convergence is the validity of Banach’s Contraction Mapping Principle when we assume that the mapping \(T : M \to M\) satisfies the following condition

\[
\text{stat} - \lim_{n \to \infty} \text{Lip}(T^n) < 1.
\]

This question is still open and may require future attention.
References


Curriculum Vitae

Khdiga Kalifa Tabib was born in Libya. She graduated from the mathematic department of Elmerghab University and she was chosen by the University to be an assistant teacher in the same department. When she started to work in the college of science, she also began to teach in the community college for three years. Throughout those years she found herself very interested to finish her degree plan.

When she began to study in a graduate program, she received a scholarship to study abroad, so she had chosen USA to complete her study. After she got a master degree she has a chance from her country to continue studying in PhD program.

Permanent address: 1700 Hawthorn Street El Paso, Texas 79902