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A Characterization of Directly Ordered Subspaces of R^n

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to my husband

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with love

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by

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Abstract

In the finite dimensional ordered vector space $\mathbb{R}^n$, we consider the standard positive cone to be the set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \; x \geq 0\}$. Given a subspace V of $\mathbb{R}^n$, we define the *positive cone* of V as $V_+ = V \cap \mathbb{R}^n_+$. The cone V_+ is said to be generating if $V = V_+ - V_+$, that is, if any vector $v \in V$ can be expressed as the difference of two vectors, $v = x - y$ where $x, y \in V_+$. Ordered vector spaces with generating cones are generally referred to as *directly ordered*. Well-known from Order Theory is that all lattices and thus lattice-subspaces are directed. However, not all directly ordered spaces are lattices, and often it is difficult to determine when a space is directed. Since directly ordered spaces enjoy a number of desirable qualities, it is useful to know when one is working in such a space. In this work, we characterize those collections of vectors in $\mathbb{R}^n$ that span directly ordered subspaces. The theory we develop naturally gives rise to a method of determining when a subspace is directed by means of a simple algorithm.

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Chapter 1

Preliminaries

This work provides a characterization of directly ordered subspaces of $\mathbb{R}^n$. All results presented are considered in $\mathbb{R}^n$. In this chapter, we present preliminary results on ordered vector spaces and cones which will be used in the proof of the main theorem in Chapter 4.

1.1 The Ordered Vector Space $\mathbb{R}^n$

Throughout this work we denote by lowercase letters u, v, x, and y vectors in $\mathbb{R}^n$. When enumerating a list of vectors, we will use the notation $x_1, x_2, \ldots, x_m$. By the Greek letters $\alpha, \beta, \ldots,$ we denote scalars in $\mathbb{R}$.

Definition 1.1. Recall that an order relation $\leq$ on a vector space X is called a *vector space ordering* if it satisfies the following five properties:

- (i) **Reflexivity**: $x \leq x$ for all $x \in X$,
- (ii) **Antisymmetry**: $x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in X$,
- (iii) **Transitivity**: $x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in X$.

Additionally, $\leq$ must be compatible with the algebraic structure of X so that for all $x, y \in X$ with $x \leq y$, we have:

- (iv) $x + z \leq y + z$ for all $z \in X$,
- (v) $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

The vector space X together with $\leq$, henceforth denoted by $(X, \leq)$, is called an ordered vector space.

Throughout by $\mathbb{R}^n$ we will indicate the finite dimensional vector space $\bigoplus_{i=1}^{n} \mathbb{R}$. By the *standard order* in $\mathbb{R}^n$, we mean the *coordinate-wise order*: let $x, y \in \mathbb{R}^n$ with

$$x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad y = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Then we say that x is less than or equal to y if $\alpha_i \leq \beta_i$ for each $1 \leq i \leq n$, and we write $x \leq y$. Two examples follow.

Example 1.1. Let $x_1, x_2 \in \mathbb{R}^4$ such that

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}.$$

Then we see that $x_2 \leq x_1$, since each coordinate of x_2 is less than or equal to each coordinate of x_1:

$$\begin{aligned} -1 &\leq 1 \\ 2 &\leq 2 \\ -1 &\leq 0 \\ 1 &\leq 3 \end{aligned}$$

Example 1.2. Consider x_1, x_2 in $\mathbb{R}^3$ with

$$x_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -1 \\ 7 \\ 1 \end{pmatrix}.$$

Then neither $x_1 \leq x_2$ is true, nor is $x_2 \leq x_1$ true, since $-1 \leq 4$ but $-2 \leq 7$.

By $(\mathbb{R}^n, \leq)$, we mean $\mathbb{R}^n$ equipped with the standard order.

Definition 1.2. In the ordered vector space $(\mathbb{R}^n, \leq)$, a vector x is called a *positive vector* if $x \geq 0$. If it is true that each coordinate of x is strictly greater than zero, then we say that x is a *strictly positive* vector.

We make a slight modification of the definition of *strictly positive vector* for reasons that will become clear in the proofs that follow. Let $x_1, \ldots, x_m$ be m linearly independent vectors in $\mathbb{R}^n$, and let $V = \langle x_1, \ldots, x_m \rangle$ be the m-dimensional vector subspace they generate, where $1 \leq m < n$. Suppose that each x_i contains a zero in its jth coordinate, for one or more $j \in \{1, \ldots, n\}$. Then any vector $v \in V$ must also contain a zero in its jth coordinate. In this case, we will still consider v to be strictly positive provided that each coordinate of v for which at least one x_i has a nonzero coordinate is positive.

Example 1.3. Let $x_1, x_2 \in \mathbb{R}^4$ such that

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Both x_1 and x_2 have zero in the third coordinate. Hence, any v in the linear span of $V = \langle x_1, x_2 \rangle$ must also have a zero as its third coordinate. We have that

$$x_1 + x_2 = u = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$

which we consider strictly positive since each coordinate of u for which there exists a nonzero coordinate of at least one of x_1, x_2 is positive.

Hereafter, the term *strictly positive vector* $v \in V$ is understood in this sense. We will not remind the reader of this again.

1.2 Cones and Directly Ordered Spaces

In this section we develop basic results regarding cones in $\mathbb{R}^n$. For a full discussion of these notions, see [2]. Here, we paraphrase the results of that text.

Definition 1.3. A nonempty subset K of a vector space X is called a *cone* whenever it satisfies the following:

- (i) $K + K \subseteq K$
- (ii) $\mathbb{R}_+ K \subseteq K$
- (iii) $K \cap -K = \{0\}$.

Definition 1.4. The set $X_+ = \{x \in X : \ x \geq 0\}$ is called the *positive cone* of X.

The positive cone of $\mathbb{R}^n$ is the set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x \geq 0\}$ and is generally referred to as the *standard cone* of $\mathbb{R}^n$. Though cones may exist which are not positive cones of a given ordered vector space, this work is concerned exclusively with positive cones. For this reason, we may refer to a cone X_+ of X simply as *the* cone of X. Given a subspace V of $\mathbb{R}^n$, we define the positive cone of V by $V_+ = V \cap \mathbb{R}^n_+$. Some cones have the desirable property of being *generating*, which we now define:

Definition 1.5. The cone X_+ of the ordered vector space X is called *generating* if $X = X_+ - X_+$. Equivalently, we have that for each $x \in X$, there exist $u, v \in X_+$ such that $x = u - v$. If X_+ is generating, we may say that X is directly ordered, or that the order is directed.

The standard cone of $\mathbb{R}^n$ is generating. However, not all cones are generating. For example, consider one cone of $\mathbb{R}^3$ defined as

$$K = \left\{ v \in \mathbb{R}^3 : \ \xi_1 \geq 0, \ \xi_2 \geq 0, \ \xi_3 = 0 \right\}.$$

It is clear that K satisfies the definition of a cone. Any element $v \in K$ is of the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}$$

where $\xi_1, \xi_2 \geq 0$. But then a vector such as

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

cannot be written as a difference of vectors from K, since $0 - 0 = 0$ must appear as the third coordinate of u.

The following will be useful in the proof of the main result of this chapter.

Definition 1.6. A vector $0 < e \in V$ is an order unit of V if for each $x \in V$ there exists $\alpha > 0$ such that $x \leq \alpha e$.

The following lemma is adapted from [2].

Lemma 1.1. *If a subspace V of an ordered vector space contains an order unit, then the cone V_+ is generating.*

Proof. Let $e \in V_+$ be an order unit and let $x \in V$. Since e is an order unit, there exists $\alpha > 0$ such that $x \leq \alpha e$. This is equivalent to $0 \leq \alpha e - x$, i.e., $\alpha e - x \in V_+$. Clearly, $\alpha e \in V_+$, and if we let $u = \alpha e$ and $v = \alpha e - x$ we then have that

$$x = u - v \in V_+ - V_+.$$

□

We now give consideration to some basic topological notions regarding generating cones. To this end, we introduce the following:

Definition 1.7. Given a subset S of $\mathbb{R}^n$ and a vector $v \in S$, we say that v is an *interior point* of S if there exists an open set U such that $v \in U \subseteq S$, where U is open under the induced topology on S.

Lemma 1.2. *Let V be a subspace of $\mathbb{R}^n$. A vector $u \in V_+$ is strictly positive if and only if it is an interior point of V_+.*

Proof. First, assume that u is an interior point of V_+, and let $u = (\alpha_1, \ldots, \alpha_n)$. Since $u \in V_+$, by definition, $\alpha_i \geq 0$ for each i. Suppose that for some $i = 1, \ldots, n$, we have that $\alpha_i = 0$. Then for any $\epsilon > 0$, there exists $v \in B(u, \epsilon)$ such that the ith coordinate of v is equal to $0 - \frac{\epsilon}{2} < 0$. Then $v \notin V_+$. Therefore, every coordinate of u must be strictly positive. For the converse, assume that $u \in V_+$ is strictly positive, and again let $u = (\alpha_1, \ldots, \alpha_n)$. Define $\mu = \min_{1 \leq i \leq n} \{\alpha_i\}$ and let $\epsilon = \frac{\mu}{2}$. Then $\epsilon > 0$ and clearly $B(u, \epsilon) \subseteq V_+$, since any $v \in B(u, \epsilon)$ has all coordinates strictly greater than zero. □

The following theorem is stated in [2], and the following Corollary is strongly tied to our main result.

Theorem 1. *Let V be a subspace of $\mathbb{R}^n$. Then the cone V_+ is generating if and only if it contains an interior point of V.*

Proof. Assume first that V_+ contains an interior point u. By Lemma 1.2, this means that u is strictly positive in V. We claim that u is an order unit. To see this, let $u = (\alpha_1, \ldots, \alpha_n)^T$ and $x \in V$, say $x = (\beta_1, \ldots, \beta_n)^T$. According to our previous agreement, we can assume without loss of generality that $\alpha_i > 0$ for each $i = 1, \ldots, n$. Let $\alpha = \max\left\{\frac{\beta_i}{\alpha_i} : \ i = 1, \ldots, n\right\}$. Then $\alpha\alpha_i \geq \beta_i$ for $i = 1, \ldots, n$, so $x \leq \alpha u$. Thus u is an order unit and so by Lemma 1.1, V_+ is generating. For the converse, assume V_+ is generating. We claim that there exists a basis of V in V_+. To see this, let $\{e_1, \ldots, e_k\}$ be a maximal set of linearly independent vectors in V_+. Then any vector in V_+ must be a linear combination of the vectors $e_1, \ldots, e_k$. Since V_+ is generating, for any $x \in V$ there exist $y, z \in V_+$ such that $x = y - z$. But then $x \in V$ can be expressed as a linear combination of the vectors

$e_1, \ldots, e_k$. Therefore $\{e_1, \ldots, e_k\}$ is a basis for V. Now we construct an interior point from this basis. Set $e = \sum_{i=1}^{k} e_i$. It is well-known that under the Euclidean topology, the set $U = \left\{ \sum_{i=1}^{k} \lambda_i e_i : \ \sum_{i=1}^{k} |\lambda_i| < 1 \right\}$ is an open neighborhood of zero. Then any $v \in U$ satisfies $e + v = \sum_{i=1}^{k} (1 + \lambda_i) e_i$, with $1 + \lambda_i > 0$ for each i, so for any $v \in U$, we have $e + v \in V_+$. This is equivalent to $e + U \subseteq V_+$, so e is indeed an interior point of V_+. □

Corollary 1. *A cone V_+ is generating in a subspace V of $\mathbb{R}^n$ if and only if it contains a strictly postive vector.*

Proof. This follows immediately from Lemma 1.2 and Theorem 1. □

Chapter 2

Vector Subspaces and Orthogonal Complements

In this chapter we present basic results from Linear Algebra regarding a subspace V of $\mathbb{R}^n$ and its orthogonal complement, $V^{\perp}$.

2.1 Vector Subspaces

We briefly review conventions from Linear Algebra which will be used in the remainder of this work.

Let $x_1, \ldots, x_m$ be m linearly independent vectors from $\mathbb{R}^n$, where $1 \leq m < n$. We denote by V the m-dimentsional subspace generated by the set of x_i, that is, $V = \langle x_1, x_2, \ldots, x_m \rangle$. We can form a matrix A whose columns are the x_i as follows: For $x \in \mathbb{R}^n$, let $x(i)$ denote the ith coordinate of x. Then the matrix whose columns are formed by the x_i can be written as

$$A = \begin{pmatrix} x_1, x_2, \ldots, x_m \end{pmatrix} = \begin{pmatrix} x_1(1) & x_2(1) & \cdots & x_m(1) \\ x_1(2) & x_2(2) & \cdots & x_m(2) \\ \vdots & \vdots & \vdots & \vdots \\ x_1(n) & x_2(n) & \cdots & x_m(n) \end{pmatrix}.$$

Then we have that

$$A^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} x_1(1) & x_1(2) & \cdots & x_1(n) \\ x_2(1) & x_2(2) & \cdots & x_2(n) \\ \vdots & \vdots & \vdots & \vdots \\ x_m(1) & x_m(2) & \cdots & x_m(n) \end{pmatrix}.$$

2.2 Orthogonal Complements

The *null space* of an $n \times m$ matrix A is the set of all vectors $v \in \mathbb{R}^n$ such that $Av = \mathbf{0}$. The null space of a matrix is denoted by $ker\,(A)$. For two vectors $x, y \in \mathbb{R}^n$, recall that the *inner product* of x by y is denoted by $x \cdot y$ and defined as

$$x \cdot y = \sum_{i=1}^{n} \alpha_i \beta_i$$

where $x = (\alpha_1, \ldots, \alpha_n)^T$ and $y = (\beta_1, \ldots, \beta_n)^T$.

Definition 2.1. The *orthogonal complement* of a vector subspace $V \subseteq \mathbb{R}^n$ is the set of all vectors $x \in \mathbb{R}^n$ such that $v \cdot x = 0$, for each $v \in V$ and is denoted by $V^\perp$.

The orthogonal complement of a vector subspace V is characterized in terms of matrices as follows: Let $x_1, \ldots, x_m$ be m linearly independent vectors in $\mathbb{R}^n$, and let $V = \langle x_1, x_2, \ldots, x_m \rangle$. If we build the matrix $A = \begin{pmatrix} x_1, x_2, \ldots, x_m \end{pmatrix}$ as above, then $V^\perp = ker\left(A^T\right)$. We give a proof of this below.

Theorem 2. *For* $V = \langle x_1, x_2, \ldots, x_m \rangle$ *and* $A = \begin{pmatrix} x_1, x_2, \ldots, x_m \end{pmatrix}$, $V^\perp = ker\left(A^T\right)$.

Proof. First let $x \in V^\perp$, so that $v \cdot x = 0$ for each $v \in V$. Then $x_i \cdot x = 0$ for each $i = 1, \ldots, m$, so by the definition of matrix multiplication, $A^T x = \mathbf{0}$. For the converse, let $A^T x = \mathbf{0}$. This is equivalent to $x_i \cdot x = 0$ for each $i = 1, \ldots, m$. Since the x_i form a basis for V, for any $v \in V$ there exist scalars $\xi_1, \xi_2, \ldots, \xi_m$ such that $v = \sum_{i=1}^{m} \xi_i x_i$. Then

$$\begin{aligned} v \cdot x &= \left(\sum_{i=1}^{m} \xi_i x_i \right) \cdot x \\ &= \sum_{i=1}^{m} \xi_i \left(x_i \cdot x \right) \\ &= \sum_{i=1}^{m} \xi_i \cdot 0 \\ &= 0. \end{aligned}$$

□

There exist many useful characterizations of a vector subspace V in terms of its orthogonal complement, $V^{\perp}$. In the following section, we present one which is crucial to the main result of this work.

2.3 A Subspace and its Orthogonal Complement

We saw in Theorem 1 that the cone of a subspace is generating if and only if it contains a strictly positive vector. In applications, it may be difficult to determine whether or not a subspace contains a positive vector. It may be easier to determine equivalent characteristics of vectors in the orthogonal complement of a subspace. For this reason, we present the following theorem.

Theorem 3. *For a subspace V of $\mathbb{R}^n$, exactly one of the following mutually exclusive alternatives holds:*

- *(i) V contains a strictly positive vector.*
- *(ii) $V^{\perp}$ contains a positive vector.*

Before proving this, we establish the following result from [3], Theorem 2.9. For a full discussion, see Chapter 2 in [3].

Lemma 2.1. *For an $n \times m$ matrix A, exactly one of the following mutually exclusive alternatives holds:*

- *(i) The equation $A^T x = 0$ has a positive solution $x \in \mathbb{R}^n$, or*
- *(ii) The inequality $Ay > 0$ has a solution $y \in \mathbb{R}^m$.*

We are now ready to prove Theorem 3:

Proof. Let $\{x_1, \ldots, x_m\}$ be a basis for V and suppose that V does not contain a strictly positive vector. This is equivalent to saying that the inequality $\sum_{i=1}^{m} \xi_i x_i > 0$ has no solution $(\xi_1, \ldots, \xi_m)$. If A is the matrix $A = \Big(x_1, x_2, \ldots, x_m\Big)$, then this implies that the inequality $Ay > 0$ has no solution. Then by Lemma 2.1, there exists $x \in \mathbb{R}^n$ with $x \geq 0$ and $A^T x = 0$. Thus $x \in V^{\perp}$. □

Recall Corollary 1:

A cone is generating if and only if it contains a positive vector.

Combining this with Theorem 3 above yields the following result:

Corollary 2. *A cone of a vector subspace V is generating if and only if its orthogonal complement $V^{\perp}$ contains no positive vector.*

We will use this result in the proof of our main theorem in Chapter 4.

Chapter 3

Motivation for Main Problem

In this chapter we will give the motivation for the approach taken in this work for determining when a given set of linearly independent vectors span a directly ordered subspace. For a full discussion of the results which follow, see [1]. Below is a summary of the main parts of their paper which are relevant to our problem.

3.1 Lattices

Though our main problem does not involve lattices, we mention them briefly here since their study is the main focus of [1], and we use similar methods to solve our problem.

Definition 3.1. In an ordered vector space X, a vector x is the *supremum* of the set $\{y, z\}$, where $y, z \in X$ if x satisfies the following:

- (i) x is an upper bound of the set $\{y, z\}$; that is, $y \leq x$ and $z \leq x$.
- (ii) x is the least upper bound satisfying (i); that is, if v is any other vector in X such that $y \leq v$ and $z \leq v$, then we must have $x \leq v$.

Analogously, we can define the $infimum$ of two vectors.

Definition 3.2. A vector x is the $infimum$ of the set $\{y, z\}$ if it satisfies:

- (i) x is a lower bound of the set $\{y, z\}$; that is, $x \leq y$ and $x \leq z$.
- (ii) x is the greatest lower bound satisfying (i); that is, if v is any other vector which satisfies $v \leq y$ and $v \leq z$, then we must have $v \leq x$.

With these definitions, we can now define a *lattice*:

Definition 3.3. An ordered vector space X is called a *vector lattice* if for every $y, z \in X$ the set $\{y, z\}$ has a supremum and an infimum in X.

Of particular interest to the authors of [1] are the *lattice-subspaces* of $\mathbb{R}^n$. We consider a subspace V of a vector space X to be ordered under the induced ordering from X.

Definition 3.4. A vector subspace V of an ordered vector space X is called a *lattice-subspace* if for every $y, z \in V$ the set $\{y, z\}$ has a supremum and an infimum, where the supremum and infimum are taken in V.

We are now ready to explore the results presented in [1].

3.2 The AAP Method

For convenience, we refer to the algorithm developed as a consequence of Theorem 2.6 in [1] as the *AAP Algorithm*, after the authors Y. A. Abramovich, C. D. Aliprantis, and I. A. Polyrakis. For a full presentation of this material see [1], for here we give only a synopsis which is relevant our problem.

In [1], Abramovich et al seek to answer the question:

> *When does a collection of linearly independent positive vectors generate a lattice subspace?*

They begin solving the problem as follows: Given a set of m linearly independent positive vectors $x_1, \ldots, x_m$ in $\mathbb{R}^n$, let $V = \langle x_1, \ldots, x_m \rangle$ be the m-dimensional vector subspace they generate, where $1 \leq m < n$.

As above in Chapter 1, for $x \in \mathbb{R}^n$, let $x(i)$ denote the ith component of x. Then the matrix whose columns are formed by the x_i can be written as

$$A = \Big(x_1, x_2, \ldots, x_m\Big) = \begin{pmatrix} x_1(1) & x_2(1) & \cdots & x_m(1) \\ x_1(2) & x_2(2) & \cdots & x_m(2) \\ \vdots & \vdots & \vdots & \vdots \\ x_1(n) & x_2(n) & \cdots & x_m(n) \end{pmatrix}.$$

As before, we then transpose the matrix A to form

$$A^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} x_1(1) & x_1(2) & \cdots & x_1(n) \\ x_2(1) & x_2(2) & \cdots & x_2(n) \\ \vdots & \vdots & \vdots & \vdots \\ x_m(1) & x_m(2) & \cdots & x_m(n) \end{pmatrix}.$$

From the columns of A we form the following n vectors of $\mathbb{R}^m$:

$$y_1 = \begin{pmatrix} x_1(1) \\ x_2(1) \\ \vdots \\ x_m(1) \end{pmatrix}, \quad y_2 = \begin{pmatrix} x_1(2) \\ x_2(2) \\ \vdots \\ x_m(2) \end{pmatrix}, \ldots, y_n = \begin{pmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_m(n) \end{pmatrix}.$$

Notice that since A has rank m, A^T also has rank m. Therefore, among the y_k, $k = 1, \ldots, n$, there exist m linearly independent vectors. Now, we consider the indices of these vectors. We recall terminology in [1] to present the following definition.

Definition 3.5. A set of m indices $\{n_1, \ldots, n_m\} = I$ is called a *fundamental set of indices* for the vectors $x_1, \ldots, x_m$ whenever I satisfies

- (i) the m vectors $y_{n_1}, \ldots, y_{n_m}$ are linearly independent,
- (ii) for each $j \notin I$, y_j is a non-negative combination of the y_{n_r} where $n_r \in I$. That is, for each $j \notin I$,

  $$y_j = \sum_{r=1}^{n} \alpha_{j,r} y_{n_r}$$

  with each $\alpha_{j,r} \geq 0$ for all $1 \leq r \leq m$.

The main theorem of [1] can now be stated:

Theorem 4. *The vector subspace V is a lattice subspace of $\mathbb{R}^n$ if and only if the vectors $x_1, \ldots, x_m$ admit a set of fundamental indices $\{n_1, \ldots, n_m\} = I$.*

We shall not supply a proof of this result since the proof is quite technical and it is off-topic. For a complete proof of this, see [1].

In consideration of applications of this theorem, it gives us a direct and expedient way to determine whether or not a given collection of vectors $x_1, \ldots, x_m$ generates a lattice-subspace. We now give an example.

Example 3.1. Let x_1, x_2 and x_3 be the three positive vectors in $\mathbb{R}^4$ defined as

$$x_1 = \begin{pmatrix} 4 \\ 2 \\ 7 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

A quick calculation shows that these vectors are linearly independent. Following the method outlined above, we form the four vectors in $\mathbb{R}^3$:

$$y_1 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}, \quad y_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The first possible set of positive fundamental indices is $I_1 = \{1, 2, 3\}$. We obtain the identity

$$y_4 = -y_1 - y_2 + y_3$$

Since not all coefficients are non-negative in the expansion of y_4, I_1 is not a fundamental set of indices.

Continuing in this manner, we obtain for $I_2 = \{1, 2, 4\}$:

$$y_3 = y_1 + y_2 + y_4$$

Since each coefficient in the expansion of y_3 is non-negative (and the set $\{y_1, y_2, y_4\}$ is linearly independent), by Definition 3.5 we see that I_2 is a fundamental set of indices. Thus the subspace generated by x_1, x_2 and x_3 is a lattice subspace.

The power of the above theorem and its corresponding algorithm is that we are able to easily determine whether or not a subspace of $\mathbb{R}^n$ is a lattice subspace. It may be very difficult to determine otherwise. This is highly conducive to applications of this theory. In the next section, we develop the main result of this thesis in the spirit of this method. We will see that we are able to determine when an arbitrary collection of m linearly independent vectors in $\mathbb{R}^n$ span a directly ordered subspace.

Chapter 4

Directly Ordered Subspaces of $\mathbb{R}^n$

In this chapter we present our principle theorem regarding when a subspace of $\mathbb{R}^n$ is directly ordered. The applications of this theory are wide not only within the mathematical subfields of Linear Algebra and Order Theory, but also in fields such as Economics. We will first develop some supporting material in Section 1 and then move to the statement and proof of the main result in Section 2. Section 3 contains three examples of the application of our result.

4.1 Supporting Lemmas

We first specify a few additional results from linear algebra.

Recall that a linear equality is any equation of the form

$$\sum_{i=1}^{k} \alpha_i y_i = v \tag{4.1}$$

where $\alpha_i \in \mathbb{R}$ and $v, y_i \in \mathbb{R}^n$ for each i.

A solution of coefficients $(\alpha_1, \ldots, \alpha_k)$ to Equation 4.1 is said to *depend* on a set $Y = \{y_1, \ldots, y_r\}$, where $Y \subset \{y_1, \ldots, y_k\}$ and $1 \leq r < k$ if $\alpha_j = 0$ for all $y_j \notin Y$.

Definition 4.1. A solution of coefficients $(\alpha_1, \ldots, \alpha_k)$ to Equation 4.1 is called a *basic solution* if it depends on a linearly independent set $\{y_1, \ldots, y_r\}$ of the y_i.

The following lemma is adapted from [3].

Lemma 4.1. *If a linear equality of the form 4.1 has a non-negative solution $(\alpha_1, \ldots, \alpha_k)$, then it has a non-negative basic solution.*

Proof. The proof is by induction on k. If $k = 1$ then 4.1 reduces to

$$\alpha y = v.$$

If $y = \mathbf{0}$, then $v = \mathbf{0}$, so we can find some $y' \neq \mathbf{0}$ and let $\alpha = 0$, which satisfies the condition. If $y \neq \mathbf{0}$, then clearly the set $\{y\}$ is linearly independent, so again the claim holds. Assume that the claim holds for $n < k$. Let $(\alpha_1, \ldots, \alpha_k)$ be a non-negative solution of 4.1. If for any $i = 1, \ldots, k$ we have that $\alpha_i = 0$, then by the induction hypothesis the claim is satisfied. Instead, assume that $\alpha_i \neq 0$ for each i. We also assume that $\{y_1, \ldots, y_k\}$ is not a linearly independent set, for otherwise the claim is trivially satisfied. Then, for some scalars $\beta_1, \ldots, \beta_k$ we have that

$$\sum_{i=1}^{k} \beta_i y_i = 0 \tag{4.2}$$

where at least one $\beta_i \neq 0$ by the linear dependence of the y_i. Consider the set $S = \left\{\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_k}{\alpha_k}\right\}$ and let $\sigma = \max(S)$. Since at least one $\beta_i \neq 0$ and $\alpha_i > 0$ for each i, it is clear that $\sigma \neq 0$. If we multiply 4.1 by $\frac{1}{\sigma}$ and subtract 4.2 from that, we obtain:

$$\frac{1}{\sigma} \sum_{i=1}^{k} \left(\sigma - \frac{\beta_i}{\alpha_i}\right) \alpha_i y_i = v$$

But for some $i = 1, \ldots, k$, we have that $\sigma - \frac{\beta_i}{\alpha_i} = 0$ and the remaining coefficients are non-negative. Therefore by the induction hypothesis, the claim is satisfied. $\square$

From this result we immediately obtain the following.

Lemma 4.2. *If $\alpha_1 y_1 + \ldots + \alpha_n y_n = 0$, with all $\alpha_i \neq 0$, and all $y_i \neq 0$, then there exists a subset $L \subseteq \{1, \ldots, n\}$ such that the set S of vectors $\{y_j : j \in L\}$ is linearly independent and there is $i \notin L$ such that y_i is a negative linear combination of the vectors from S.*

Proof. Without loss of generality we can assume that $\alpha_1 \neq 0$ and consider the equation

$$\alpha_2 y_2 + \ldots + \alpha_n y_n = -\alpha_1 y_1.$$

By assumption this is a non-negative solution of the equation and therefore, by Lemma 4.1, there exists a basic non-negative solution, i.e., there exists a subset $L \subseteq \{1. \ldots, n\}$ and scalars β_j where $j \in L$ such that the set of vectors $S = \{y_j : j \in L\}$ is linearly independent, and $\beta_j > 0$ with

$$y_1 = -\frac{1}{\alpha_1} \sum_{j \in L} \beta_j y_j.$$

□

For a given subspace V, we can characterize the vectors of $V^\perp$ in terms of linear equalities. As above, we form the matrix A whose columns are given by a collection $x_1, \ldots, x_m$ of linearly independent vectors in $\mathbb{R}^n$. Taking the transpose of A, we form the collection of vectors $y_1, \ldots, y_n$ which are the columns of A^T. We use these definitions in the following remark.

Remark 4.1. *A vector v is an element of $V^\perp$ if and only if it is of the form $v = (\alpha_1, \ldots, \alpha_n)$ where the α_i satisfy*

$$\sum_{i=1}^{n} \alpha_i y_i = 0.$$

Proof. Say $v = (\alpha_1, \ldots, \alpha_n)$. We know by Theorem 2 that $v \in V^\perp$ if and only if $A^T v = 0$. By the definition of matrix multiplication, this is equivalent to

$$\sum_{i=1}^{n} \alpha_i y_i = 0.$$

□

4.2 Main Result

Definition 4.2. A set of m indices $\{n_1, \ldots, n_m\}$ is called a *negative fundamental* set of indices for the vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ whenever

(1) the m vectors $y_{n_1}, \ldots, y_{n_m}$ are linearly independent; and

(2) for at least one $j \notin \{n_1, \dots, n_m\}$, all the coefficients in the expansion

$$y_j = \sum_{r=1}^{m} \alpha_{j,r} y_{n_r}$$

are non-positive.

Notice that this definition differs from the definition of a *fundamental set of indices* from [1] in two ways. The first is that we require the coefficients $\alpha_{j,r}$ to be non-positive. The second, more subtle difference is that we require such a comination of non-positive coefficients *for only one* $j \notin \{n_1, \dots, n_m\}$. It may happen that more than one $j \notin \{n_1, \dots, n_m\}$ satisfies this condition, but we require only one such j.

We now present the main result of this work.

Theorem 5. *The vector subspace V of $\mathbb{R}^n$ is directly ordered if and only if the vectors $x_1, \dots, x_m$ do not admit a negative fundamental set of indices.*

Proof. First, assume that there exists a negative fundamental set of indices, $\{n_1, \dots, n_m\}$. Without loss of generality, we may assume that $\{n_1, \dots, n_m\} = \{1, \dots, m\} = I$. Then there exists $j \notin I$ such that

$$y_j = \sum_{i=1}^{m} \alpha_i y_i$$

where $\alpha_i \leq 0$ for each i. This is equivalent to

$$y_j - \sum_{i=1}^{m} \alpha_i y_i = 0,$$

which implies the existence of the vector $v = (-\alpha_1, -\alpha_2, \dots, -\alpha_m, 0, \dots, 0, 1, 0, \dots, 0)^T$ in $V^\perp$, where 1 occurs at the jth component. But $v \geq 0$, so by Corollary 2, V^+ is not generating.

For the converse, assume that V^+ is not generating. By Corollary 2, there exists $v \in V^\perp$

such that $v \geq 0$. That is, v is of the form

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

with $\alpha_k \geq 0$ for each k. These coefficients α_k and the column vectors y_k satisfy the equation

$$\sum_{k=1}^{n} \alpha_k y_k = 0.$$

Hence by Lemma 4.1 there exists a basic set of indices $\{n_1, \ldots, n_m\} = I$ such that for some $j \notin I$ we have

$$y_j = \sum_{i=1}^{m} \beta_i y_i,$$

where $\beta_i \leq 0$ for each i. The set $\{n_1, \ldots, n_m\}$ is the desired set of negative fundamental indices, and this completes the proof. □

4.3 Examples

Next, we illustrate Theorem 5 with three examples.

Example 4.1. Consider the following three vectors $x_1, x_2, x_3 \in \mathbb{R}^4$. Let

$$x_1 = \begin{pmatrix} -1 \\ -2 \\ -1 \\ 2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ -5 \\ 3 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} -3 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

A quick calculation shows that these vectors are linearly independent. Following the method outlined above, we form the four vectors in $\mathbb{R}^3$

$$y_1 = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}, \quad y_2 = \begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}, \quad y_4 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Next, we check that there does not exist a fundamental negative set of indices. There are four possible sets, which give rise to the following equations.

$I_1 = \{1, 2, 3\}$:

$$y_4 = \frac{8}{21}y_1 - \frac{11}{21}y_2 - \frac{4}{3}y_3$$

$I_2 = \{1, 2, 4\}$:

$$y_3 = \frac{2}{7}y_1 - \frac{11}{28}y_2 - \frac{3}{4}y_4$$

$I_3 = \{1, 3, 4\}$:

$$y_2 = \frac{8}{11}y_1 - \frac{28}{11}y_3 - \frac{21}{11}y_4$$

$I_4 = \{2, 3, 4\}$:

$$y_1 = \frac{11}{8}y_2 + \frac{7}{2}y_3 + \frac{21}{8}y_4$$

Inspection shows that not one of the $I_i, i = 1, \ldots, 4$, is a negative fundamental set. Therefore V_+ is generating. In fact, notice that

$$3x_1 - 3x_2 - 7x_3 = \begin{pmatrix} 15 \\ 2 \\ 2 \\ 2 \end{pmatrix} \in V_+.$$

Example 4.2. Let x_1, x_2, x_3 be the three vectors in $\mathbb{R}^6$ defined as

$$x_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -4 \\ 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ -3 \\ -2 \end{pmatrix}.$$

These vectors are linearly independent. We form the vectors

$$y_1 = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

$$y_4 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad y_6 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

Next we investigate the sets of indices:

$I_1 = \{1, 2, 3\}$:

$$y_4 = -\frac{7}{17}y_1 + \frac{15}{17}y_2 - \frac{13}{176}y_3$$
$$y_5 = -2y_2 + y_3$$
$$y_6 = \frac{3}{17}y_1 - \frac{21}{17}y_2 + \frac{8}{17}y_3$$

$I_2 = \{1, 2, 4\}$:

$$y_3 = -\frac{7}{13}y_1 + \frac{15}{13}y_2 - \frac{17}{13}y_4$$
$$y_5 = -\frac{7}{13}y_1 - \frac{11}{13}y_2 - \frac{17}{13}y_4$$
$$y_6 = -\frac{1}{13}y_1 - \frac{9}{13}y_2 - \frac{8}{13}y_4$$

Hence $I_2 = \{1, 2, 4\}$ is a negative set of fundamental indices. Furthermore, from the equations of y_5 and y_6 in terms of the set I_2, it is clear that the vectors

$$\begin{pmatrix} \frac{7}{13} \\ \frac{11}{13} \\ 0 \\ \frac{17}{13} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{13} \\ \frac{9}{13} \\ 0 \\ \frac{8}{13} \\ 0 \\ 1 \end{pmatrix}$$

lie in the orthogonal complement of V.

Example 4.3. In this example we present a case in which we "compress" the matrix formed by the x_i and thus reduce the problem from $\mathbb{R}^7$ to $\mathbb{R}^5$. Here, let

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -2 \\ 0 \\ 5 \\ 6 \\ 0 \\ 4 \\ -2 \end{pmatrix}, \quad x_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

These vectors are linearly independent and form the matrix

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ -1 & 5 & 1 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \\ 0 & -2 & -1 \end{pmatrix}$$

which can be "compressed" by removing rows 2 and 5 to form the matrix

$$\begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 1 \\ 2 & 6 & 0 \\ -1 & 4 & 1 \\ 0 & -2 & -1 \end{pmatrix}.$$

Now, the relevant vectors we form from the transpose of this matrix are

$$y_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad y_2 = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix},$$

$$y_4 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

There are ten possibilities of negative fundamental indices:

$I_1 = \{1, 2, 3\}$:

$$y_4 = -\frac{1}{3}y_1 + \frac{2}{3}y_2$$
$$y_5 = 2y_1 + y_2 - \frac{1}{2}y_3$$

$I_2 = \{1, 2, 4\}$:

This set is not linearly independent.

$I_3 = \{1, 2, 5\}$:

$$y_3 = 4y_1 + 2y_2 - 2y_5$$
$$y_4 = -\frac{1}{3}y_1 + \frac{2}{3}y_2$$

$I_4 = \{1, 3, 4\}$:

$$y_2 = \frac{1}{2}y_1 + \frac{3}{2}y_4$$
$$y_5 = \frac{5}{2}y_1 - \frac{1}{2}y_3 + \frac{3}{2}y_4$$

$I_5 = \{1, 3, 5\}$:

$$y_2 = -2y_1 + \frac{1}{2}y_3 + y_5$$
$$y_4 = -\frac{5}{3}y_1 + \frac{1}{3}y_3 + \frac{2}{3}y_5$$

$I_6 = \{1, 4, 5\}$:

$$y_2 = \frac{1}{2}y_1 + \frac{3}{2}y_4$$
$$y_3 = 5y_1 + 3y_4 - 2y_5$$

$I_7 = \{2, 3, 4\}$:

$$y_1 = 2y_2 - 3y_4$$
$$y_5 = 5y_2 - \frac{1}{2}y_3 - 6y_4$$

$I_8 = \{2, 3, 5\}$:

$$y_1 = -\frac{1}{2}y_2 + \frac{1}{4}y_3 + \frac{1}{2}y_5$$
$$y_4 = \frac{5}{6}y_2 - \frac{1}{12}y_3 - \frac{1}{6}y_5$$

$I_9 = \{2, 4, 5\}$:

$$y_1 = 2y_2 - 3y_4$$
$$y_3 = 10y_2 - 12y_4 - 2y_5$$

$I_{10} = \{3, 4, 5\}$:

$$y_1 = \frac{1}{5}y_3 - \frac{3}{5}y_4 + \frac{2}{5}y_5$$
$$y_2 = \frac{1}{10}y_3 + \frac{6}{5}y_4 + \frac{1}{5}y_5$$

Thus, no negative set of fundmental indices exists, so V_+ is generating. In fact, calculations show that

$$-2x_1 + x_2 - 5x_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \in V.$$

This vector is an interior point of the cone V_+.

Chapter 5

Concluding Remarks and Future Work

We have presented a characterization of directly ordered subspaces of $\mathbb{R}^n$. This characterization was seen to naturally give rise to an algorithm with which it is relatively simple to determine when a collection of linearly independent vectors generate a directly ordered subspace. However, many related questions remain.

We saw in Chapter 3 that the AAP method on which this work is loosely based works only for collections of positive vectors. It may be desirable to extend that result to arbitrary collections of vectors, without the restriction of positivity.

Closely related to the notions of lattices and directly ordered vector spaces are a class of spaces which possess the *Riesz Decomposition Property*: If $0 \leq u, v, w \in V$ and $w \leq u + v$ then there exist vectors $u', v' \in V$ such that $0 \leq u' \leq u$, $0 \leq v' \leq v$ and $w = u' + v'$. This property is of fundamental importance in ordered vector spaces, partially ordered groups and related areas. It is known that in $\mathbb{R}^n$, a directly ordered vector subspace, V, is a lattice-subspace if and only if V has the Riesz Decomposition Property (RDP); see [4], for example. However, it is not known when a subspace of $\mathbb{R}^n$ has the RDP if the subspace is not directly ordered. Perhaps the theorem presented in this work could lead to a solution of this problem.

In our work, we only investigated the finite dimensional space $\mathbb{R}^n$. However, as directed orders are also relevant in other spaces, it is desirable to characterize generating cones in general and we hope our method may prove useful.

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Curriculum Vitae

Jennifer Del Valle entered the University of Texas at El Paso in the fall of 2006 and earned her Bachelor of Science in Mathematics in August, 2009. While pursuing this degree she worked in the Mathematics Department as a Peer-Leader for Pre-calculus in the spring of 2008 and as a Supplemental Instructor for Calculus in the Fall of 2008. In the fall of 2009, she entered the Graduate School of the University of Texas at El Paso to pursue a master's degree in Mathematics. Mrs. Del Valle received the LSAMP Bridge to the Doctorate Fellowship for her graduate studies. She is a member of the honor societies Phi Kappa Phi and Alpha Chi. Her professional membership includes the American Mathematical Society, the Mathematical Association of America, and the Association for Women in Mathematics.

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