Solving the Partial Differential Equation of Vibrations with Interval Parameters using the Interval Finite Difference Method

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SOLVING THE PARTIAL DIFFERENTIAL EQUATION OF VIBRATIONS
WITH INTERVAL PARAMETERS USING THE INTERVAL
FINITE DIFFERENCE METHOD

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FINITE DIFFERENCE METHOD

by

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Abstract

Accuracy and efficiency are among the main factors that drive today’s innovative disciplines. As technology rapidly advances, efficiency takes on new meanings but what about accuracy? *How accurate is accurate?* Human error, uncertainties in measurement, and rounding errors are just some causes of inaccuracy. *Interval Computations* is an area that allows for such issues to be taken into account; for each measurement attained (for example), an interval can be built by considering the error associated with the measurement, and such an interval can be utilized in the mathematical computations of interest.

It is then natural to state that one of the disciplines that would greatly benefit from the above-mentioned approach is Engineering. Consider the following partial differential equation (PDE) which has many applications in structural mechanics including modeling structures such as bridges, towers and other buildings:

\[-\frac{\partial^2}{\partial x^2} \left( EJ \frac{\partial^2 w}{\partial x^2} \right) + q = \rho A \frac{\partial^2 w}{\partial t^2} \]

Converting the above equation into an interval-parametrized equation, and applying interval computation techniques to accurately and efficiently find a solution would be of great value to the Engineering community.

The purpose of this thesis is to present the application of the interval finite difference method towards solving interval-parametrized equations of dynamics, and verify the efficiency and accuracy of the approach.
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Chapter 1

Introduction

1.1 Need for Interval Mathematics

Accuracy and efficiency are among the main factors that drive today’s innovative disciplines. As technology rapidly advances, efficiency takes on new meanings but what about accuracy? *How accurate is accurate?* Human error, uncertainties in measurement, and rounding errors are just some causes of inaccuracy, but how can we account for such issues during mathematical computations? *Interval Computations* is an area that allows for such issues to be taken into account which can be exemplified as follows.

When a measurement is taken, resulting in the value \( \tilde{x} \in \mathbb{R} \), there are methodologies to estimate the measurement error \( \Delta x = \tilde{x} - x \), i.e., the difference between the measurement result and the actual (unknown) value of the quantity. These methodologies usually produce an upper bound \( \Delta \in \mathbb{R} \) on the absolute value of the measurement error. In some practical situations, we also know the probabilities of different values of the measurement error, but in many cases, we only know this upper bound. In such cases, the only information that we have about the actual (unknown) value of the corresponding quantity is that it belongs to the interval \( x = [\tilde{x} - \Delta, \tilde{x} + \Delta] \).

In practice, we are often interested not only in the values \( x_1, \ldots, x_n \) of the directly measured quantities, but also in the values of other quantities \( y \) which depend on \( x_i \): \( y = f(x_1, \ldots, x_n) \). Based on the measurement results, we get intervals \( x_1 = [\tilde{x}_1 - \Delta_1, \tilde{x}_1 + \Delta_1] \), \( \ldots \), \( x_n = [\tilde{x}_n - \Delta_n, \tilde{x}_n + \Delta_n] \) of possible values of these quantities. Different values \( x_i \) from the corresponding intervals \( x_i \) lead, in general, to different values \( y \). It is therefore desirable
to find the range of possible values of $y$ when $x_i \in x_i$:

$$y = \{ f(x_1, \ldots, x_n) : x_1 \in x_1, \ldots, x_n \in x_n \}.$$ 

Computing this interval range based on interval inputs is called *interval computation*, and the corresponding mathematical analysis is called *interval mathematics*; see, e.g., [2, 18, 19, 20, 34, 23, 35].

In situations when the algorithm $f(x_1, \ldots, x_n)$ solves an equation or a sequence of equations, this range is called a *united solution set*; see, e.g., [16, 23].

Interval computations has been successfully applied to areas such as optimization, constraint logic programming, chemical engineering, and robotics among other areas.

The problem of exactly computing the range $y$ is, in general, NP-hard [11], which means, crudely speaking, that there is no feasible algorithm that always computes the exact range. Most interval computation techniques produce an interval which is guaranteed to enclose the desired range $y$. However, when we restrict ourselves to feasible algorithms, the resulting enclosure intervals are significantly wider than the actual range.

### 1.2 Mathematical Example Motivated by an Engineering Application

In engineering problems, it is usually desirable to feasibly produce an accurate approximation to the range – even when this approximation is not an enclosure. This is what we do in this thesis, for a specific engineering problem of estimating the displacement $w(x, t)$ of a beam with non-uniform but symmetric load $q(x, t)$ (where $x$ is the spatial coordinate and $t$ is time); see the following Figure.
The change of displacement is described by the following partial differential equation:

\[-\frac{\partial^2}{\partial x^2} \left( EJ \frac{\partial^2 w}{\partial x^2} \right) + q = \rho A \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (1.1)

where \(E(x)\) is the Young modulus, \(J(x)\) is the moment of inertia, \(\rho(x)\) is the density of the material, and \(A(x)\) is the area of a cross-section.

The boundary conditions are \(w(0, t) = w(L, t) = 0\), where \(L\) is the length of the beam. These conditions reflect the fact that the beam is attached at both ends. Another physical consequence is that \(\frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(L, t) = 0\).

The initial displacement is usually denoted by \(w^*(x)\), and the initial velocity by \(v^*(x)\), so initial conditions have the form \(w(x, 0) = w^*(x)\) and \(\frac{\partial w}{\partial t}(x, 0) = v^*(x)\).

The equation (1.1) provides a good description of a bridge, and it is has many applications in structural mechanics of towers and other buildings to describe the displacement of the corresponding beams.

In this thesis, we assume that in the beginning, there was no prior load and no displacement \(w^*(x) = v^*(x) = 0\). At the initial moment \(t = 0\), we add a piece-wise constant load:
the load $q_1$ is applied to the first part of the beam, the load $q_2$ is applied to the second part of the beam, etc. Each of the loads $q_i$ remains until a moment $t = T_i$, after which there is no load. We also assume that the values $E$, $J$, $\rho$, and $A$ are piece-wise constant: e.g., $E = E_1$ on the first part of the beam, etc.

On each segment where $E$ and $J$ are both constant – and thus, the product $EJ$ is constant – the above equation takes the simplified form:

$$-EJ \frac{\partial^4 w}{\partial x^4} + q = \rho A \frac{\partial^2 w}{\partial t^2}$$

(1.2)

In principle, the values of all corresponding parameters come from measurements, so they are all known with uncertainty. In practice, we rarely know the corresponding probabilities, so it is reasonable to assume that we know only the intervals that contain the actual values of all these parameters.

Our objective is to find, for each location $x$ and each moment of time $t$, the range $w(x,t) = [w(x,t), \bar{w}(x,t)]$ of possible values of displacement $w(x,t)$. It is also desirable to find the values of the parameters for which the corresponding extreme values $w(x,t)$ and $\bar{w}(x,t)$ are attained — because these values provide the worst case scenario for the beam.

In principle, from the engineering viewpoint, we have to consider interval uncertainty in all the values $E_i$, $J_i$, $q_i$, $\rho_i$, and $A_i$. However, from the mathematical viewpoint, we can somewhat simplify the problem by taking into account that the parameters $E_i$ and $J_i$ enter into the above equation only as a combination $E_i \cdot J_i$. Thus, instead of considering the actual ranges of $E_i$ and $J_i$, we can take the nominal values $\tilde{J}_i$ and use the modified values $E_i' = E_i \cdot \frac{\tilde{J}_i}{J_i}$, without changing the equations. Because of this possibility, in this thesis, we will assume that we know the exact values of $J_i$, and we know the intervals $E_i$, $q_i$, $[\rho_i, \bar{\rho}_i]$, and $A_i$ of possible values of all other parameters.
1.3 Interval Methods for Solving Partial Differential Equations: What Is Known

Nedialkov et al. applied special integration techniques which find guaranteed bounds for the solution of differential equations with interval parameters [22].

There is a very efficient family of methods that was developed by the Makino and Berz [12, 14]. The methods are based on special techniques for bounding remainder of the Taylor polynomial. The authors have published papers on the guaranteed solution of some selected partial differential equations [13, 15].

Several authors worked on a guaranteed version of the interval Finite Element Method [21] and the Boundary Element Method [5, 25, 37].

Neumaier published a paper about a general certified error bound for the solution of partial differential equations [24].

However, in general, as we have mentioned earlier, the known interval methods – because of their desire to provide an enclosure – lead to excess width. It is therefore desirable to develop methods which produce more accurate bounds for the desired ranges.

Some such methods have been proposed before. For example, Akpan et al. used the concept of the response surface method in order to approximate the upper and lower bounds of the solutions $w(x,t)$, $\bar{w}(x,t)$ [1, 17]. Specifically, they approximate the dependence of $w(x,t)$ on the parameters $p_i$ by a simple surface, and then use this surface to find the desired bounds. The problem with this method is that sometimes, when the actual dependence of $w(x,t)$ on $p_i$ is simple, this method works well, but in other practical situations, when the dependence is more complex, the approximation by a simple surface is not very accurate, and leads to not very accurate estimates for the desired bounds.

Several authors applied the vertex method [7] for the solution of complex partial differential equations of structural mechanics with uncertain parameters [33, 32]. In this methods, we try all possible combinations of the endpoints of the corresponding intervals. For $n$ inputs known with interval uncertainty, this method requires that we try $2^n$ combinations;
for large \( n \), this requires too much computation time.

It is therefore desirable to develop faster methods which would provide an accurate approximation to the desired range.

### 1.4 Our Main Objective

The purpose of this thesis is to use sensitivity analysis to find an approximation to the desired range \( w(x,t) \), an approximation which is (almost) exact in the linear case. Specifically, we provide such as approximation by using the finite differences method. The accuracy of the finite differences method is well known, and it is known that the corresponding approximation error tends to 0 fast as the time step decreases. So, one can easily produce an approximation which is as close to the linearized solution as possible.

### 1.5 Linearization Technique

To find a good approximation to the range, we use the following techniques. Each value \( w(x,t) \) is a function of the corresponding parameters \( p_1, \ldots, p_n \): \( w(x,t) = f(p_1, \ldots, p_n) \).

Each parameter \( p_i \) is only known with an interval uncertainty, i.e., we only know the interval \( p_i = [p_i, \overline{p}_i] \) that contains the actual (unknown) value of \( p_i \). This interval can be equivalently described as \( [p_{0i} - \Delta_i, p_{0i} + \Delta_i] \), where \( p_{0i} = \frac{p_i + \overline{p}_i}{2} \) and \( \Delta_i = \frac{\overline{p}_i - p_i}{2} \). In these terms, each value \( p_i \in p_i \) can be represented as \( p_{0i} + \Delta p_i \), where \( \Delta p_i = p_i - p_{0i} \) takes all possible values from the interval \( [-\Delta_i, \Delta_i] \).

Our objective is to find a good approximation for the range

\[
\{ f(p_1, \ldots, p_n) : p_i \in [p_{0i} - \Delta_i, p_{0i} + \Delta_i] \}.
\]

To find such an approximation, we expand the function \( f(p_1, \ldots, p_n) \) in Taylor series and keep only linear terms in this expansion:

\[
f(p_1, \ldots, p_n) \approx f(p_{01}, \ldots, p_{0n}) + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \cdot \Delta p_i.
\]
We want to find the range of this function under the assumption that $\Delta p_i \in [-\Delta_i, \Delta_i]$.

With respect to each unknown, the linear function attains its maximum either for $\Delta p_i = \Delta_i$ (when the corresponding partial derivative is non-negative) or for $\Delta p_i = -\Delta_i$ (when the corresponding partial derivative is non-positive). In both cases, the largest value of the corresponding term $\frac{\partial f}{\partial p_i} \cdot \Delta p_i$ is equal to $\left| \frac{\partial f}{\partial p_i} \right| \cdot \Delta_i$ and the smallest value is equal to $- \left| \frac{\partial f}{\partial p_i} \right| \cdot \Delta_i$. Thus, the desired range can be estimated as

$$[f(p_{01}, \ldots, p_{0n}) - \Delta, f(p_{01}, \ldots, p_{0n}) + \Delta],$$

where

$$\Delta = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial p_i} \right| \cdot \Delta_i.$$

To find the values of the partial derivatives, we use numerical differentiation

$$\frac{\partial f}{\partial p_i} = \frac{f(p_{01}, \ldots, p_{0,i-1}, p_{0,i} + h, p_{0,i+1}, \ldots, p_{0n}) - f(p_{01}, \ldots, p_{0,i-1}, p_{0,i} - h, p_{0,i+1}, \ldots, p_{0n})}{2h}$$

for some small $h$, a method which is also exact in the linearized case.

In the linear approximation, the values of $p_i$ corresponding to the maximum displacement $\overline{w}(x, t)$ can be obtained as follows:

- for the parameters $p_i$ for which $\frac{\partial f}{\partial p_i} \geq 0$, we take $p_i = \overline{p}_i$; and
- for the parameters $p_i$ for which $\frac{\partial f}{\partial p_i} \leq 0$, we take $p_i = \underline{p}_i$.

Similarly, the values of $p_i$ corresponding to the minimum displacement $\underline{w}(x, t)$ can be obtained as follows:

- for the parameters $p_i$ for which $\frac{\partial f}{\partial p_i} \geq 0$, we take $p_i = \underline{p}_i$; and
- for the parameters $p_i$ for which $\frac{\partial f}{\partial p_i} \leq 0$, we take $p_i = \overline{p}_i$.  

7
1.6 Verification of the Results

Since the technique is approximate, it is important to check how accurate the results of this linearized technique are. To verify the accuracy of our computations, we selected several equally spaced points in each interval \([p_i, \bar{p}_i]\), i.e., selected points

\[
p_{ik} = p_i + \frac{\bar{p}_i - p_i}{n_i} \cdot k, \quad k = 0, 1, \ldots, n_i,
\]

and then computed the value \(f(p_1, \ldots, p_n)\) for all \((n_1 + 1) \cdot (n_2 + 1) \cdots (n_n + 1)\) possible combinations of selected values. The interval formed by the smallest and the largest of the computed values \(f(p_1, \ldots, p_n)\) provides a good approximation to the actual range of the function \(f(p_1, \ldots, p_n)\). So, to verify our technique, we compare the range obtained by our linearized technique with this approximation.

We also find out for which combination of parameters the resulting values of \(w(x, t)\) are the largest and the smallest, and compare the results with the predictions of the linearized model.

1.7 The Organization of the Thesis

This thesis proceeds as follows: **Chapter 2** describes the finite differences techniques that we use to solve the above partial differential equation. **Chapter 3** presents the results of applying our approach. Conclusions and future work form the last **Chapter 4**.
Chapter 2

Finite Difference Method for the Beam Equation

2.1 Need for Numerical Methods

In the general case, no analytical solution is known for the beam equation. Hence, it is necessary to apply numerical methods. There are several approximation methods which can be applied in this case, such as Finite Element Methods (FEM) and Finite Difference Methods (FDM). Historically, Finite Difference Methods have been used for the 1-D beam equation. Because of this, in this thesis, the Finite Difference Method will be considered [30, 36].

In order to apply FDM, it is necessary to introduce a set of grid points in the set $\Omega = [0, L] \times [0, T]$. In the spatial dimension, let’s introduce $n_x$ intervals with length $\Delta x = \frac{L}{n_x}$. The spatial coordinates of the grid points can be calculated as follows:

$$x_i = i \cdot \Delta x.$$  \hspace{1cm} (2.1)

It is interesting to note that there are $n_x + 1$ points in the $x$-spatial dimension. In the time dimension, $n_t$ time steps will be introduced. The length of each time step is equal to $\Delta t = \frac{T}{n_t}$. The discrete values of time can be calculated as follows:

$$t_i = i \cdot \Delta t.$$  \hspace{1cm} (2.2)

There are $n_t + 1$ discrete values of time. In calculations, it is necessary to find $(n_x + 1)(n_t + 1)$ values of the solution $w_{i,j}$. Partial derivatives in the space and time can be approximated
by the following finite differences:

\[
\left( \frac{\partial^4 w}{\partial x^4} \right)_{i,j} \approx \frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{\Delta x^4}; \quad (2.3)
\]

\[
\left( \frac{\partial^2 w}{\partial t^2} \right)_{i,j} \approx \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2}. \quad (2.4)
\]

### 2.2 The Explicit Finite Difference Method for the Beam Equation

Let us start with the explicit finite difference method. In this method, we use the above discretized expressions instead of the actual partial derivatives, and we arrive at the following finite difference approximation of the original partial differential equation:

\[
\left( EJ \frac{\partial^4 w}{\partial x^4} \right)_{i,j} = \left( q - \rho A \frac{\partial^2 w}{\partial t^2} \right)_{i,j}, \quad (2.5)
\]

where

\[
E_{i,j} J_{i,j} \frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{\Delta x^4} = q_{i,j} - \rho_{i,j} A_{i,j} \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2}; \quad (2.6)
\]

Dividing both sides of this equation by \( \rho_{i,j} A_{i,j} \), we get the following formula:

\[
\left( \frac{EJ}{\rho A} \right)_{i,j} \frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{\Delta x^4} =
\]

\[
\left( \frac{q}{\rho A} \right)_{i,j} - \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2}. \quad (2.7)
\]

Using this formula, we can now explicitly describe the value \( w_{i,j+1} \) at the next moment of time in terms of the previous values:

\[
w_{i,j+1} = 2w_{i,j} - w_{i,j-1} + \left( \frac{q\Delta t^2}{\rho A} \right)_{i,j}
\]

\[
- \left( \frac{EJ\Delta t^2}{\rho A \Delta x^4} \right)_{i,j} (w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}) \quad (2.8)
\]
Discretization of the boundary conditions:

\[ w(0, t) = 0 \Rightarrow w_{0,j} = 0; \] (2.9)

\[ w(L, t) = 0 \Rightarrow w_{n,j} = 0; \] (2.10)

\[ \frac{\partial^2 w(0, t)}{\partial x^2} = 0 \Rightarrow \frac{w_{0,j} - 2w_{1,j} + w_{2,j}}{\Delta x^2} = 0; \] (2.11)

\[ \frac{\partial^2 w(L, t)}{\partial x^2} = 0 \Rightarrow \frac{w_{n,j} - 2w_{n-1,j} + w_{n-2,j}}{\Delta x^2} = 0; \] (2.12)

\[ w(x, 0) = w^*(x) \Rightarrow w_{i,0} = w_i^*; \] (2.13)

\[ v(x, 0) = v^*(x) \Rightarrow v_{i,0} = v_i^* \Rightarrow \frac{w_{i,1} - w_{i,0}}{\Delta t} = v_i^*. \] (2.14)

All these equations can be rewritten into the following form:

\[
\begin{cases}
  w_{i,0} = w_i^* \\
  w_{i,1} = w_{i,0} + v_i^* \Delta t \\
  w_{0,j} = 0 \\
  w_{0,j} - 2w_{1,j} + w_{2,j} = 0 \\
  w_{i,j+1} = 2w_{i,j} - w_{i,j-1} - \\
  \left( \frac{EJ\Delta t^2}{\rho A} \right)_{i,j} w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j} \Delta x^4 + \left( \frac{q\Delta t^2}{\rho A} \right)_{i,j} \\
  w_{n-2,j} - 2w_{n-1,j} + w_{n,j} = 0 \\
  w_{n,j} = 0
\end{cases}
\] (2.15)

It is now possible to formulate the computational algorithm:

- Calculate \( w \) for \( j = 0 \) – since \( w(x, 0) = w^*(x) \):

  \[ w_{i,0} = w_i^*, i = 0, \ldots, n \] (2.16)

- Calculate \( w \) for \( j = 1 \) – since \( v(x, 0) = \frac{\partial w(x, 0)}{\partial t} = v_0(x) \):

  \[ w_{i,1} = w_{i,0} + v_i^* \Delta t. \] (2.17)
• Calculate $w$ for $j = 2, \ldots, m$:

$$
\begin{align*}
&w_{0,j+1} = 0 \\
&w_{1,j+1} = \frac{w_{2,j+1}}{2} \\
&w_{i,j+1} = 2w_{i,j} - w_{i,j-1} + \left( \frac{EJ\Delta t^2}{\rho A} \right)_{i,j} \\
&\quad - \left( \frac{EJ\Delta t^2}{\rho A\Delta x^4} \right)_{i,j} (w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}) \\
&w_{n-1,j+1} = \frac{w_{n-2,j+1}}{2} \\
&w_{n,j+1} = 0
\end{align*}
$$

(2.18)

2.3 The Implicit Finite Difference Method for the Beam Equation

The accuracy and stability of the explicit finite difference method is not always acceptable. Thus it is better to use the implicit FDM.

$$
\left( EJ \frac{\partial^4 w}{\partial x^4} \right)_{i,j+1} = q_{i,j+1} - \left( \rho A \frac{\partial^2 w}{\partial t^2} \right)_{i,j+1}
$$

(2.19)

The appropriate finite difference equation has the following form:

$$
E_{i,j+1} J_{i,j+1} \frac{w_{i+2,j+1} - 4w_{i+1,j+1} + 6w_{i,j+1} - 4w_{i-1,j+1} + w_{i-2,j+1}}{\Delta x^4} = q_{i,j+1} - \rho_{i,j+1} A_{i,j+1} \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2}
$$

(2.20)

The solution is assumed to be known for time step $j$ and $j - 1$. It is then necessary to find the solution for the time step $j + 1$. In order to do that, it is necessary to rewrite the equation (2.20) into a more convenient form with respect to matrix operations:

$$
\begin{align*}
&\frac{E_{i,j+1} J_{i,j+1}}{\Delta x^4} w_{i+2,j+1} - 4 \frac{E_{i,j+1} J_{i,j+1}}{\Delta x^4} w_{i+1,j+1} + 6 \frac{E_{i,j+1} J_{i,j+1}}{\Delta x^4} w_{i,j+1} - 4 \frac{E_{i,j+1} J_{i,j+1}}{\Delta x^4} w_{i-1,j+1} + \rho_{i,j+1} A_{i,j+1} w_{i,j+1} = \\
&\quad q_{i,j+1} + \rho_{i,j+1} A_{i,j+1} \frac{2w_{i,j} - w_{i,j-1}}{\Delta t^2}.
\end{align*}
$$

(2.21)
In order to solve the equation (2.21) it is necessary to add appropriate boundary conditions.

\[ w_{0,j+1} = 0; \] (2.22)

\[ w_{0,j+1} - 2w_{1,j+1} + w_{2,j+1} = 0; \] (2.23)

\[
\frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4} w_{i+2,j+1} - 4 \frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4} w_{i+1,j+1} + \left( \frac{6 \frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4} + \rho_{i,j+1}A_{i,j+1}}{\Delta t^2} \right) w_{i,j+1} - \\
4 \frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4} w_{i-1,j+1} + \frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4} w_{i-2,j+1} = q_{i,j+1} + \rho_{i,j+1}A_{i,j+1} \frac{2w_{i,j} - w_{i,j-1}}{\Delta t^2}; \] (2.24)

\[ w_{n-2,j+1} - 2w_{n-1,j+1} + w_{n,j+1} = 0; \] (2.25)

\[ w_{n,j+1} = 0; \] (2.26)

The above equations can be written in a matrix form as:

\[
A(E^{j+1}, J^{j+1}, A^{j+1}, \rho^{j+1})w^{j+1} = b(q^{j+1}, \rho^{j+1}, A^{j+1}, w^j, w^{j-1}), \] (2.27)

where

\[
w^{j+1} = \begin{bmatrix}
w_{1,j+1} \\
w_{2,j+1} \\
\vdots \\
w_{n,j+1}
\end{bmatrix}; \] (2.28)

\[
b = \begin{bmatrix}
0 \\
0 \\
q_{2,j+1} + \left( \frac{\rho A}{\Delta t^2} \right)_{i,j+1} (2w_{2,j} - w_{2,j-1}) \\
q_{3,j+1} + \left( \frac{\rho A}{\Delta t^2} \right)_{i,j+1} (2w_{3,j} - w_{3,j-1}) \\
\vdots \\
q_{n-2,j+1} + \left( \frac{\rho A}{\Delta t^2} \right)_{i,j+1} (2w_{n-2,j} - w_{n-2,j-1}) \\
0 \\
0
\end{bmatrix}. \] (2.29)
The nonzero elements of the matrix $A$ are:

\begin{align*}
A_{1,1} &= 1; \\
A_{2,1} &= 1, A_{2,2} = -2, A_{2,3} = 1; \\
A_{3,1} &= \frac{E_{3,j+1}J_{i,j+1}}{\Delta x^4}, A_{3,2} = -4\frac{E_{3,j+1}J_{i,j+1}}{\Delta x^4}, \\
A_{3,3} &= \left(6\frac{E_{3,j+1}J_{i,j+1}}{\Delta x^4} + \frac{\rho_{3,j+1}A_{3,j+1}}{\Delta t^2}\right), \\
A_{3,4} &= -4\frac{E_{3,j+1}J_{3,j+1}}{\Delta x^4}, A_{3,5} = \frac{E_{3,j+1}J_{3,j+1}}{\Delta x^4}, \\
A_{4,2} &= \frac{E_{4,j+1}J_{i,j+1}}{\Delta x^4}, A_{4,3} = -4\frac{E_{4,j+1}J_{i,j+1}}{\Delta x^4}, \\
A_{4,4} &= \left(6\frac{E_{4,j+1}J_{i,j+1}}{\Delta x^4} + \frac{\rho_{4,j+1}A_{i,j+1}}{\Delta t^2}\right), \\
A_{4,5} &= -4\frac{E_{4,j+1}J_{4,j+1}}{\Delta x^4}, A_{4,6} = \frac{E_{4,j+1}J_{4,j+1}}{\Delta x^4},
\end{align*}

and so forth.

The first two iterations can be found by using initial conditions.

\begin{align*}
w_{i,0} &= w_i^*; \\
w_{i,1} &= w_{i,0} + v_i^*\Delta t.
\end{align*}

2.4 A Numerical Example

Let us consider the equation (1.1) with the following numerical data $E = 200 \cdot 10^9 \frac{N}{m^2}$, $\rho = 7850 \frac{kg}{m^3}$, $A = 0.1 \ m^2$, $L = 10 \ m$, $n_x = 100 \ \left(\Delta x = \frac{L}{n_x}\right)$, $\Delta t = 0.001 \ s$, $n_t = 300$ (number of time steps), $q_1 = 100 \ \frac{N}{m}$ for $x \in [2, 3] \ m$ (the load is applied for first 6 time steps), and $q_2 = -100 \ \frac{N}{m}$ for $x \in [6, 8] \ m$ (the load is applied for first 4 time steps). This example’s solution is shown in Fig. 2.1, 2.2, 2.3, 2.4, and in Table 2.4.
Figure 2.1: The Solution of the equation (1.1) for time step 7.

Figure 2.2: The Solution of the equation (1.1) for time step 70.
**Figure 2.3:** The Solution of the equation (1.1) for all 100 time steps.

**Figure 2.4:** The Solution of the equation (1.1) for all 100 time steps.
Table 2.1: The Solution $w_i$ (in m) for time step 70

<table>
<thead>
<tr>
<th>$i$</th>
<th>$w_i$ (in m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>$-4.45 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>46</td>
<td>$-4.26 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>47</td>
<td>$-4.06 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>48</td>
<td>$-3.86 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>49</td>
<td>$-3.65 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>50</td>
<td>$-3.43 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>51</td>
<td>$-3.21 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>52</td>
<td>$-2.98 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>53</td>
<td>$-2.75 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>54</td>
<td>$-2.51 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>55</td>
<td>$-2.27 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>
Chapter 3

Testing Our Method: Examples

In this chapter, we describe the results of testing our method on several numerical examples. Following the techniques described earlier, for each of these examples, we will provide the approximate range by using numerical differentiation, and the (almost) actual range – by selecting finitely many points within each interval and by trying all possible combinations of these points.

3.1 Example 1: Narrow Intervals, Homogeneous Beam, Non-Uniform Load

In the previous chapter, we considered an example when the beam is homogeneous and all its parameters are known exactly. Let us now consider the case when these parameters are only known with interval uncertainty. Specifically, we assume that the values of \( A = 0.01 \text{ m}^2 \) and \( L = 10 \text{ m} \) are known exactly, but other parameters are known with interval uncertainty: \( E \in [200 \cdot 10^9, 202 \cdot 10^9] \text{ N m}^2/m \), \( \rho \in [7850, 7852] \text{ kg m}^3 \), \( q_1 \in [100, 102] \text{ N m} \) for \( x \in [2, 3] \text{ m} \) (the load is applied for first 6 time steps), and \( q_2 \in [−102, −100] \text{ N m} \) for \( x \in [6, 8] \text{ m} \) (the load is applied for first 4 time steps). We use the same discretization as before: \( n_x = 100 \) (\( \Delta x = \frac{L}{n_x} \)), and \( \Delta t = 0.001 \text{ s} \) with \( n_t = 100 \) (number of time steps).

In this example, the vector of interval parameters is 4-dimensional:

\[ p = [E, \rho, q_1, q_2] = [p_1, p_2, p_3, p_4] \in \mathbb{R}^4. \]

Since the intervals are narrow, we selected only two points per each interval – its endpoints. As a result, we tried \( 2^4 = 16 \) different combinations. The corresponding 16 solutions after
100 time steps are shown on Fig. 3.1. The resulting interval range is shown in Fig. 3.2.

Figure 3.1: The solutions after 100 time steps, 16 combinations of parameters
The widths $\bar{w}(x,t) - \underline{w}(x,t)$ of the interval range $w(x,t) = [\underline{w}(x,t), \bar{w}(x,t)]$ for all moments of time from 0 to 100 are described in Fig. 3.3 and 3.4.
Figure 3.3: The width of the interval range for all 100 time steps.

Figure 3.4: The width of the interval range for all 100 time steps.
In Fig. 3.4 we can observe that the uncertainty of the solution increases with time.

The combinations of parameters which generate appropriate bounds \( w(t, x) \) and \( \overline{w}(t, x) \) are shown in Fig. 3.5 and 3.6. Here, each number \( i \) from 1 to 16 means the following: we transform \( i - 1 \) into binary code, and interpret 0 as meaning the lower endpoint and 1 as meaning the upper endpoint of the corresponding interval. For example, \( i = 11 \) means that we consider \( i - 1 = 1010_2 \). With respect to the 4-tuple \((E, \rho, q_1, q_2)\), this means that we consider the upper endpoint \( E \) of the interval \( E \), the lower endpoint \( \rho \), the upper endpoint \( q_1 \), and the lower endpoint \( q_2 \).

![Figure 3.5: Combinations of parameters for which the lower bound is attained](image)

**Figure 3.5:** Combinations of parameters for which the lower bound is attained
To confirm these estimates, we also tried 4 and 7 points per interval. In the first case, we got $4^4 = 256$ combinations of parameters, in the second case, we got $7^4 = 2401$ combinations. The maximum differences between the results of the corresponding range estimates are as follows: $\max_{i,j} |w_{i,j}^{16} - w_{i,j}^{256}| = 3.26 \cdot 10^{-9}$, and $\max_{i,j} |\bar{w}_{i,j}^{16} - \bar{w}_{i,j}^{256}| = 2.43 \cdot 10^{-9}$. Only 0.24% of the coefficients of the matrix $w_{i,j}^{16} - w_{i,j}^{256}$ and 0.28% of the coefficients of the matrix $\bar{w}_{i,j}^{16} - \bar{w}_{i,j}^{256}$ are non-zero. Similar result can be calculated for 2041 combinations: $\max_{i,j} |w_{i,j}^{16} - w_{i,j}^{2041}| = 3.55 \cdot 10^{-9}$, and $\max_{i,j} |\bar{w}_{i,j}^{16} - \bar{w}_{i,j}^{2041}| = 2.73 \cdot 10^{-9}$. In this case only 0.32% of the coefficients of the matrix $w_{i,j}^{16} - w_{i,j}^{2041}$ and 0.30% of the coefficients of the matrix $\bar{w}_{i,j}^{16} - \bar{w}_{i,j}^{2041}$ are non-zero. The results of the calculations indicate that for narrow intervals, it is sufficient to only consider two points per interval.

The difference between the ranges $w_{i,j}^T = [w_{i,j}^{T}, \bar{w}_{i,j}^{T}]$ produced by numerical differentiation and the (almost) actual range (calculated based on 2401 combinations of parameters) is shown in Fig. 3.7 and 3.8.
Figure 3.7: The difference between the range produced by numerical differentiation and the (almost) actual range (calculated based on 2401 combinations of parameters): Lower bounds.
The maximum approximation errors of the numerical differentiation method are

\[
\max_{i,j} |w_{i,j}^T - w_{2401}^{i,j}| = 5.45 \cdot 10^{-9} \quad \text{and} \quad \max_{i,j} |\bar{w}_{i,j}^T - \bar{w}_{2401}^{i,j}| = 5.45 \cdot 10^{-9}.
\]

3.2 Example 2: Wide Intervals, Homogeneous Beam, Non-Uniform Load

In this example, we consider wider intervals \( E \in [200 \cdot 10^9, 240 \cdot 10^9] \ \frac{N}{m^2}, \rho \in [7850, 9420] \ \frac{kg}{m^3}, \)
\( q_1 \in [100, 120] \ \frac{N}{m} \) for \( x \in [2, 3] \ m \) (the load is applied for first 6 time steps), and \( q_2 \in [-120, -100] \ \frac{N}{m} \) for \( x \in [6, 8] \ m \) (the load is applied for first 4 time steps).

Here, we also have a 4-dimensional vector of interval parameters \( p = [E, \rho, q_1, q_2] = [p_1, p_2, p_3, p_4] \in R^4. \) Since the intervals are wider, in addition to trying two points per interval, we also tried to select 5 points in each interval (which leads to \( 5^4 = 625 \) combina-
tions). For the endpoint combination method there are 16 combinations of endpoints. The corresponding solutions after 100 time steps are shown in Fig. 3.11. The resulting range is shown in Fig. 3.2.

Figure 3.9: Solutions after 100 time steps, 16 combinations of parameters
Figure 3.10: Solutions after 100 time steps, 625 combinations of parameters
Figure 3.11: The interval range after 100 time steps.

The difference between upper and lower bounds $\overline{w}(t, x) - \underline{w}(x, t)$ is shown in Fig. 3.12.
To validate these computations, we also tried to take 7 points per interval, to the total of $7^4 = 2041$ combinations of parameters. The difference between the ranges produced by numerical differentiation and the (almost) actual range (calculated based on 2401 combinations of parameters) is shown in Fig. 3.13 and 3.14:
Figure 3.13: The difference between the range produced by numerical differentiation and the (almost) actual range (calculated based on 2401 combinations of parameters): Lower bounds.
Figure 3.14: The difference between the range produced by numerical differentiation and
the (almost) actual range (calculated based on 2401 combinations of parameters): Upper
bounds.

For wide intervals, the numerical differentiation method is not as accurate as in the case
of narrow intervals.

3.3 Example 3: Narrow Interval, Inhomogeneous Beam,
Non-Uniform Load

Let us now consider a more realistic example, in which the two parts of the 10 m beam
may have different values of parameters.

- For the first half $x \in [5, 10] \ m$, we have $E_1 \in [200 \cdot 10^9, 202 \cdot 10^9] \ \frac{N}{m^2}$ and $\rho_1 \in [7850, 7852] \ \frac{kg}{m^3}$.

- On the second half $x \in [5, 10] \ m$, we have values which may be different but which
belong to the same intervals $E_2 \in [200 \cdot 10^9, 202 \cdot 10^9] \frac{N}{m^2}$ and $\rho_2 \in [7850, 7852] \frac{kg}{m^3}$.

The loads are the same as in the homogeneous narrow intervals example: $q_1 \in [100, 102] \frac{N}{m}$ for $x \in [2, 3] m$ (the load is applied for first 6 time steps), and $q_2 \in [-102, -100] \frac{N}{m}$ for $x \in [6, 8] m$ (the load is applied for first 4 time steps).

In this example, the vector of interval parameters is 6-dimensional

$$p = [E_1, E_2, \rho_1, \rho_2, q_1, q_2] = [p_1, p_2, p_3, p_4, p_5, p_6] \in \mathbb{R}^6.$$  

We took two points per interval, resulting in $2^6 = 64$ combinations of parameters. Solutions after 100 time steps are shown in Fig. 3.15. The resulting interval range is shown in Fig. 3.16.

![Figure 3.15: Solution after 100 time steps, 64 combinations of parameters](image)
Figure 3.16: The interval range: Solution after 100 time steps.

The difference between the ranges corresponding to the homogeneous and inhomogeneous beams is shown in Fig. 3.17.
3.4 Example 4: Wide Intervals, Inhomogeneous Beam, Non-Uniform Load

In this example, for the first half $x \in [0, 5] \text{ m}$, we have $E_1 \in [200 \cdot 10^9, 240 \cdot 10^9] \frac{N}{m^2}$ and $\rho_1 \in [7850, 9420] \frac{kg}{m^3}$. For the second half $x \in [5, 10] \text{ m}$, we have values which may be different but which belong to the same intervals $E_2 \in [200 \cdot 10^9, 240 \cdot 10^9] \frac{N}{m^2}$ and $\rho_2 \in [7850, 9420] \frac{kg}{m^3}$. The loads are the same as in the homogeneous wide intervals example: $q_1 \in [100, 120] \frac{N}{m}$ for $x \in [2, 3] \text{ m}$ (the load is applied for first 6 time steps), and $q_2 \in [-120, -100] \frac{N}{m}$ for $x \in [6, 8] \text{ m}$ (the load is applied for first 4 time steps).

In this example, we also have a 6-dimensional vector of interval parameters

\[ p = [E_1, E_2, \rho_1, \rho_2, q_1, q_2] = [p_1, p_2, p_3, p_4, p_5, p_6] \in R^6. \]
In this example, we also considered two points per interval, to the total of $2^6 = 64$ combinations. Solutions after 100 time steps are shown in Fig. 3.18. The resulting interval range is shown in Fig. 3.2.

**Figure 3.18:** Solution after 100 time steps, 64 combinations of parameters
Figure 3.19: The interval range: Solution after 100 time steps.

The difference between the ranges corresponding to the homogeneous and inhomogeneous beams is shown in Fig. 3.20.
3.5 Example 5: Narrow Intervals, Homogeneous Beam, Uniform Load

It is interesting to observe the behavior of the results if we consider a uniform distribution load on the beam (instead of that depicted in Fig. 1.1). See Fig. 3.21 for an illustration of a beam with a uniform distribution load.
We consider the same values and narrow intervals as before: $A = 0.01 \, m^2$, $L = 10 \, m$, $E \in [200 \cdot 10^9, 202 \cdot 10^9] \, \frac{N}{m^2}$, $\rho \in [7850, 7852] \, \frac{kg}{m^3}$, but now $q_1 \in [100, 102] \, \frac{N}{m}$ for all $x \in [0, 10] \, m$ (the load is applied for first 6 time steps). Here, the vector of interval parameters is 3-dimensional: $p = [E, \rho, q_1] = [p_1, p_2, p_3] \in \mathbb{R}^3$.

We selected 7 points on each interval, resulting in $7^3 = 343$ combinations of parameters. The combinations that correspond to upper and lower bounds are shown in Fig. 3.22 and 3.23.

---

**Figure 3.21**: Beam with uniform load
Figure 3.22: Combinations of parameters for which the lower bound is attained.

Figure 3.23: Combinations of parameters for which the upper bound is attained.
Chapter 4

Conclusions and Future Work

The purpose of this work was to estimate a range of possible solutions for the interval-parametrized partial differential equation describing vibrations of beams under loads – by applying interval estimates to the linear approximation of the problem. In order to verify the accuracy of the resulting estimates, we compared the estimated range with an (almost) accurate range which was obtained by taking multiple combinations of parameters. For narrow intervals, as expected, the results are very accurate, the difference between the estimated and the actual ranges is about $10^{-9}$.

For wider intervals, the difference is larger, about $10^{-6}$, which is about 10% of the actual range. In many practical cases, this is good enough for engineering estimations. In situations when we need more accurate estimates for the range, we need to go beyond the linearized case.
References


Curriculum Vitae

Brenda G. Medina was born on November 14, 1986, and was raised in the beautiful town of Santa Fe, New Mexico. She is the second of three daughters born to Juliana Gonzalez-Medina and Jesus M. Medina. She entered the University of Texas at El Paso in the Fall of 2004, and graduated five years later with a Bachelor’s Degree of Science in Computer Science, Mathematics, and Biology with a minor in Chemistry. While completing her undergraduate studies, she worked as a research assistant in the Computer Science Department and was selected for a Distributed Mentor Project Award which included funding for a research internship at the University of Delaware.

She began her graduate studies in the Fall of 2009 pursuing a Masters Degree of Science in Mathematics at the University of Texas at El Paso. While completing her graduate studies she worked as a research- and teaching- assistant in the Department of Mathematical Sciences. She was selected for internships at the Institute for Creative Technologies and NASA’s Jet Propulsion Laboratory. She also completed a co-op with AT&T and the U.S. Army Research Laboratory.

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