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Reconstructing an Open Order from Its Closure, with Applications to Space-Time Physics and to Logic

Abstract. In his logical papers, Leo Esakia studied corresponding ordered topological spaces and order-preserving mappings. Similar spaces and mappings appear in many other application areas such the analysis of causality in space-time. It is known that under reasonable conditions, both the topology and the original order relation $\leq$ can be uniquely reconstructed if we know the interior $<$ of the order relation. It is also known that in some cases, we can uniquely reconstruct $<$ (and hence, topology) from $\leq$. In this paper, we show that, in general, under reasonable conditions, the open order $<$ (and hence, the corresponding topology) can be uniquely determined from its closure $\leq$.

Keywords: ordered topological space, order-preserving mappings, open and closed orders, space-time geometry, logic

1. Formulation of the Problem

Order-preserving mappings of topological spaces in logic and in physics: general reminder. Many interesting mathematical results appear when we are able to find connection between two seemingly different areas of mathematics – and thus, use known results and techniques from one area to study techniques from another area. In particular, for Heyting algebras – models of intuitionistic logics – many results originated from a relation between Heyting algebras and a special class of (partially) ordered topological spaces called *Esakia spaces*, a relation that was discovered and actively explored by Leo Esakia in [10, 11]. In his research, L. Esakia paid special attention to studying order-preserving maps between the corresponding partially ordered spaces.

Esakia's work was not the first application of ordered topological spaces and order-preserving mappings: such spaces and mappings also naturally appear in space-time physics and in other areas of logic.

In physics, a natural ordering relation is the *causality* relation between events, when $a \leq b$ means that an event a can influence the event b. Here,

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topology corresponds to closeness of events. It is usually assumed that the relation $\le$ is closed, i.e., if $b_n \to b$ and $a \le b_n$, then $a \le b$. Indeed, if $b_n \to b$, this means, crudely speaking, that for every given measurement accuracy, when n is sufficiently large, we cannot distinguish events b_n and b. So,

- if $a \le b_n$, i.e., if physical evidence shows a can influence all events b_n, and
- $b_n \to b$, meaning that b is indistinguishable from b_n,

then this same evidence shows that a can influence b ($a \le b$).

In Newton's physics, in which signals can potentially travel with an arbitrarily large speed, the causality relation is trivial: an event $a = (t, x)$ occurring at the spatial location x at time t can influence the event $a' = (t', x')$ if and only if $t < t'$ (or $t = t'$ and $x = x'$).

The fundamental role of the non-trivial causality relation emerged with the Special Relativity, according to which the speed of all the signals is limited by the speed of light c, as a result of which $a = (t, x) \le a' = (t', x')$ if and only if $t' \ge t$ and $\dfrac{d(x, x')}{t' - t} \le c$; see, e.g., [12].

In the original relativity theory, causality was one of the main concepts. Its central role was revealed in the 1950s, when A. D. Alexandrov proved that in special relativity, causality implied Lorenz group: every order-preserving transforming of the corresponding partial ordered set is linear, and can be represented as a composition of spatial rotations, Lorentz transformations (describing a transition to a moving reference frame), and re-scaling $x \to \lambda \cdot x$ (corresponding to a change of unit for measuring space and time) [1, 5]. This theorem was later generalized by E. Zeeman (and by many others) and is currently known as the *Alexandrov-Zeeman theorem*; see, e.g., [2, 3, 4, 6, 7, 13, 14, 15, 16, 17, 18, 21, 23, 25, 27, 28, 29, 30, 31, 32, 35, 43].

Special relativity theory is an approximate description of space-time, a description that does not take into account that space-time is curved. To describe curved space-times, we need General Relativity Theory and its generalizations. The notion of causality is the basis of several formalizations of space-time physics, both as foundations of the General Relativity Theory and as a way to describe its generalizations; see, e.g., [8, 26, 24, 25, 27, 39].

In logic, partial orders are used when we formalize commonsense and expert reasoning. In this application, to each statement, we assign the expert's degree of certainty that this statement is true. A natural partial ordering relation $a \le b$ describes the fact that we are more certain in b than in a; see, e.g., [20, 34, 37, 38, 41, 42].

In this application, topology represents the closeness of the corresponding degrees of certainty. In this case, it is also reasonable to require that the relation $\le$ is closed.

Another important application of partial orders is decision making, when we need to describe human preferences [9, 22, 33, 36, 40].

Need for open partial orders. In many applications, we only observe an event b with some accuracy. For example, in physics, we may want to check what is happening exactly 1 second after a certain reaction. However, in practice, we cannot measure time exactly, so, we can only observe an event which is close to b – e.g., an event that occurs 1 ± 0.001 sec after the reaction. In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b.

Because of this uncertainty, the only possibility to experimentally confirm that a can influence b is when for some neighborhood U_b of the event b, we have $a \le \widetilde{b}$ for all $\widetilde{b} \in U_b$. In topological terms, this "experimentally confirmable" relation $a < b$ means that b contains in the *future cone* $C_a^+ \stackrel{\text{def}}{=} \{b : b \ge a\}$ of the event a together with some neighborhood, i.e., that b belongs to the *interior* of the closed cone C_a^+.

Comment. To avoid confusion, please note that here $a < b$ *does not* mean $a \le b$ and $a \ne b$.

Similar arguments justify the need to consider open cones also in case of uncertainty.

In physics, there is another motivation for open cones: open cones correspond to influences with speeds smaller than the speed of light. This is important because, according to modern physics, there are two types of objects (see, e.g., [12]):

- objects with non-zero rest mass that can travel with any possible speed which is smaller than the speed of light – but not with the speed of light, and
- objects with zero rest mass (like photons), that can travel only with the speed of light, but not with any smaller speed.

Thus, open cones correspond to causality by traditional (kinematic) objects. Because of this, the open relation $a < b$ is also known as *kinematic* causality, and spaces with this open relation $<$ are known as *kinematic spaces* [39].

Natural questions: what is the relation between open and closed partial orders? In all the above applications, on the same space, we have three things – topology and two different ordering relations

- the original (closed) partial order $\le$ and
- the open partial order $<$.

It is reasonable to ask to what extent knowing only *some* of these things enables us to reconstruct the others.

Relation between open and closed partial orders: what is known. It is known that under some physically (and logically) reasonable assumptions, the open relation uniquely determines the topology and the closed relation.

The corresponding topology was first introduced by A. D. Alexandrov and is thus known as *Alexandrov topology.* It is a topology whose base are *open intervals* $(a,b) \stackrel{\text{def}}{=} \{c : a < c < b\}$. For this definition to be valid, we need to make sure that intervals do form a base of a topology, i.e., when a point x belongs to the intersection of two open intervals, there a whole open interval containing x is contained in this intersection.

Once this topology is defined, we can define $a \le b$ as b belonging to the closure $\overline{K_a^+}$ of the open cone $K_a^+ = \{b : a < b\}$. Of course, we need to make sure that a dual definition $a \in \overline{K_b^-}$, where $K_b^- = \{b : b < a\}$ leads to the exact same ordering.

It is also usually assumed that for every element a, there are elements larger than a and smaller than a, and that if $a < b$, then there is a point in between a and b.

Under these conditions, the above description determines the topology and the closed order in terms of the open order $<$. Thus, the open order uniquely determines both the topology and the closed order.

In the case of special relativity, the inverse is also true: if we know the closed partial order, then we can uniquely reconstruct the open order as well – and so, the topology. Hence, every 1-1 transformation preserving a closed order also preserves the open order and the topology. This conclusion is used in many proofs that every order-preserving transformation is linear. The proof of this conclusion is based on the easy-to-check observation that when $a \le b$, we have $a < b$ if and only if the relation $\le$ restricted to the closed interval $[a,b] = \{c : a \le c \le b\}$ is *not* a total (linear) order, i.e., if and only if there exist c and c' for which $a \le c \le b$, $a \le c' \le b$, $c \not\le c'$, and $c' \not\le c$.

It is known, however, that this observation does not hold in general. For example, on the 3-D space $\mathbb{R}^3$ with a standard topology, we can define a component-wise partial order as follows: $a = (a_1, a_2, a_3) \le b = (b_1, b_2, b_3)$ if and only if $a_1 \le b_1$, $a_2 \le b_2$, and $a_3 \le b_3$. In this space, the corresponding open order is also easy to describe: $a = (a_1, a_2, a_3) < b = (b_1, b_2, b_3)$ if and only if $a_1 < b_1$, $a_2 < b_2$, and $a_3 \le b_3$. Here, however, we can have $a = (0, 0, 0) \le b = (0, 1, 1)$, $a \not< b$, but for $c = (0, 0, 1)$ and $c' = (0, 1, 0)$, we have $a \le c \le b$, $a \le c' \le b$, $c \not\le c'$, and $c' \not\le c$.

Remaining problem – that we solve in this paper. A natural question is: what happens in the general case? Can we uniquely reconstruct an open order if we know the corresponding closed order? In this paper, we show that under reasonable assumptions, such a reconstruction is indeed possible.

This work was motivated by our discussions with Leo Esakia during his visit to Las Cruces, New Mexico.

2. Definitions and the Main Result

Definition 1. [39] *A set X with a partial order $<$ is called a* kinematic space *if is satisfies the following conditions:*

$$\forall a\, \exists a_-, a_+\, (a_- < a < a_+);$$

$$\forall a, b\, (a < b \to \exists c\, (a < c < b));$$

$$\forall a, b, c\, (a < b, c \to \exists d\, (a < d < b, c));$$

$$\forall a, b, c\, (b, c < a \to \exists d\, (b, c < d < a)).$$

Definition 2. *For every partial ordered set, and every $a < b$, by an* interval *(or open interval), we mean the set $(a, b) \stackrel{\text{def}}{=} \{x : a < c < b\}$.*

Definition 3. *A kinematic space is called* separable *if there exists a countable set $\{x_n\}$ such that every open interval contains one of the elements x_i.*

Definition 4. [39] *For every separable kinematic space, we define* convergence $a_n \to a$ *as follows:*

$$a_n \to a \Leftrightarrow \forall a_-, a_+ \,(a_- < a < a_+ \Rightarrow \exists N \,\forall n \,(n \geq N \Rightarrow a_- < a_n < a_+))).$$

For each set S, it closure $\overline{S}$ is defined as the set of all the the points s for which $s_n \to s$ for some $s_n \in S$.

Comment. In other words, $a_n \to a$ if and only if every interval (a_-, a_+) containing a also contains almost all elements of a_n – i.e., all but finitely many of them.

Mathematical comment. In this paper, we consider separable kinematic spaces, i.e., spaces in which there is a countable set $\{x_n\}$ which is everywhere dense in X. In such spaces, to describe topology, it is sufficient to consider convergence of sequences. Our result, however, can be easily extended to general (not necessarily separable) kinematic spaces if, instead of sequences $\{a_n\}$, we consider *nets* $\{a_\alpha\}_{\alpha \in A}$ corresponding to directed sets A; see, e.g., [19].

Definition 5. [39] *A kinematic space is called* normal *if*

$$b \in \overline{\{c : c > a\}} \Leftrightarrow a \in \overline{\{c : c < b\}}.$$

Notation. For a normal kinematic space, we denote $b \in \overline{\{c : c > a\}}$ by $a \leq b$. For every $a \leq b$, the set $[a, b] \stackrel{\text{def}}{=} \{c : a \leq c \leq b\}$ is called a *closed interval.*

The following transitivity and closure properties hold for this relation:

Proposition 1. [39] *For every separable normal kinematic space and for every elements a, b, and c, the following holds:*

- *$a \leq a$;*
- *if $a < b$, then $a \leq b$;*
- *if $a \leq b$ and $b < c$, then $a < c$;*
- *if $a < b$ and $b \leq c$, then $a < c$.*

The proof of the first part of Proposition 1 is based on the following lemma:

Definition 6. *We say that a sequence* $\{a_n\}$ *is* $<$-decreasing *if* $a_n > a_{n+1}$ *for all* n.

Lemma 1. *For every separable kinematic space, for every element* a*, there exists an* $<$*-decreasing sequence* a_n *of element* $a_n > a$ *for which* $a_n \to a$.

Comment. For readers' convenience, all the proofs are placed in the special (final) Proofs section.

A dual lemma also holds:

Definition 7. *We say that a sequence* $\{a_n\}$ *is* $<$-increasing *if* $a_n < a_{n+1}$ *for all* n.

Lemma 2. *For every separable kinematic space, for every element* a*, there exists an* $<$*-increasing sequence* a_n *of element* $a_n < a$ *for which* $a_n \to a$.

Proposition 2. [39] *For every separable normal kinematic space:*

- *if* $a_n \geq b$ *and* $a_n \to a$*, then* $a \geq b$*;*
- *if* $a_n \leq b$ *and* $a_n \to a$*, then* $a \leq b$*;*
- *if* $a \leq b$ *and* $b \leq c$*, then* $a \leq c$.

Comment. Please note that the limit relation $\leq$ is not necessarily an order, since we may have $a \leq b$, $b \leq a$, and $a \neq b$. For example, in the closure of the above Newtonian causality order, $e = (t, x) \leq e' = (t', x') \Leftrightarrow t \leq t'$. In this case, $(0, (0, 0, 0)) \leq (0, (1, 0, 0))$ and $(0, (1, 0, 0)) \leq (0, (0, 0, 0))$, but $(0, (0, 0, 0)) \neq (0, (1, 0, 0))$. In such cases, we have a non-trivial equivalence relation $a \equiv b \Leftrightarrow (a \leq b \,\&\, b \leq a)$. For each element a, its equivalence class $\{b : b \equiv a\}$ is equal to $[a, a]$.

To formulate our result, we need to introduce the an additional completeness property.

Definition 8. *We say that a sequence* $\{a_n\}$ *is* $\leq$-decreasing *if* $a_n \geq a_{n+1}$ *for all* n.

Definition 9. *We say that a sequence* $\{a_n\}$ *is* bounded from below *if there exists an element* b *for which* $b \leq a_n$ *for all* n.

Definition 10. *We say that a separable kinematic space is* complete *if every $\le$-decreasing bounded sequence has a limit.*

Physical comment. From the application viewpoint, this requirement does not change much. Indeed, the events are only approximately known anyway, so explicitly adding a limit event $a = \lim a_n$ does not affect the physical picture: for all practical purposes, the limit a is indistinguishable from a_n for large n.

The following result shows that completeness holds in most physically interesting cases, for example, in the space-time corresponding to special relativity.

Definition 11. *A kinematic space is called intervally compact if in this space, every closed interval is compact.*

Proposition 3. *Every intervally-compact separable normal kinematic space is complete.*

The following result states that a complete (open) kinematic order can indeed be uniquely reconstructed from the corresponding closed order:

Theorem. *If two complete separable normal kinematic orders $<$ and $<'$ on the same set X lead to the same closed order $\le = \le'$, then $< = <'$.*

3. Proofs

3.1. Proof of Lemma 1

Since the kinematic space is separable, there exists a sequence x_n that has elements in every open interval. We will construct a sequence a_n with the following additional property: for every n, if $x_n > a$, then $x_n > a_n$.

By definition of a kinematic metric, there exists an element $a_+ > a$; we will take this element as a_1.

Let us now assume that the values $a_1 > \ldots > a_{n-1} > a$ have already been constructed. The construction of the next element a_n will depend on whether $x_n > a$ or not. If $x_n > a$, then we have $a < x_n, a_{n-1}$. So, by definition of a kinematic space, there exists an element c for which $a < c < x_n, a_{n-1}$. We will take one of these elements c as a_n.

If $x_n \not> a$, then we have $a < a_{n-1}$. So, by definition of a kinematic space, there exists an element c for which $a < c < a_{n-1}$. We will take one of these elements c as a_n.

We have constructed a $<$-decreasing sequence. Let us prove that this sequence converges to a, i.e., that for every $a_- < a < a_+$, there exists an N such that for all $n \geq N$, we have $a_- < a < a_+$. Indeed, since $a < a_+$, the sequence x_n has an element x_N in an open interval (a, a_+): $a < x_N < a_+$. By our construction, $x_N > a$ implies that $a < a_N < x_N$. By transitivity, we conclude that $a_- < a_N < a_+$. Since the sequence a_n is $<$-decreasing, we conclude that for $n > N$, we have $a < a_n < a_N$, so, by transitivity, $a_- < a_n < a_+$. Convergence is proven.

3.2. Proof of Lemma 2

This proof is similar to the proof of Lemma 1.

3.3. Proof of Proposition 1

Let us first prove that $a \geq a$. Indeed, by Lemma 1, there exists a sequence a_n for which $a_n > a$ and $a_n \to a$. Thus, $a \geq a$.

Let us prove that if $a < b$, then $a \leq b$. Indeed, we can take $b_n = b$. Each open interval neighborhood of b contains b and thus, contains all elements of the sequence b_n. Thus, $b_n \to b$ and hence, $a \leq b$.

Let us now prove that if $a \leq b$ and $b < c$, then $a < c$. Indeed, by definition, $a \leq b$ means that there is a sequence $b_n \to b$ for which $a < b_n$ for all n. By definition of convergence, $b_n \to b$ means that for every two elements $b_- < b < b_+$, there exists N for which, for all $n \geq N$, $b_- < b_n < b_+$. By definition of a kinematic space, there is an element $b_- < b$. As b_+, we take $b_+ = c$. In this case, for sufficiently large n, we have $b_n < c$, so $a < b_n$ and transitivity imply that $a < c$.

Finally, let us prove that if $a < b$ and $b \leq c$, then $a < c$. Indeed, since the kinematic space is normal, $b \leq c$ means that there exists a sequence $b_n \to b$ for which $b_n < c$ for all n. By definition of convergence, $b_n \to b$ means that for every two elements $b_- < b < b_+$, there exists N for which, for all $n \geq N$, $b_- < b_n < b_+$. By definition of a kinematic space, there is an element $b_+ > b$. As b_-, we take $b_- = a$. In this case, for sufficiently large n, we have $b_n > a$, so $b_n < c$ and transitivity imply that $a < c$.

Finally, let us prove that the relation $\geq$ is transitive. Let $a \geq b$ and $b \geq c$. By definition, $a \geq b$ means that there exists a sequence $a_n > b$ for which $a_n \to a$. As we have shown, from $a_n > b$ and $b \geq c$, we conclude that

$a_n > c$. Thus, $a_n > c$ for some sequence $a_n \to a$. This is exactly what is means to have $a \geq c$. The statement is proven.

The proposition is proven.

3.4. Proof of Proposition 2

Without losing generality, let us prove the first statement, i.e., let us assume that $a_n \to a$ and $a_n \geq b$, and let us prove that $a \geq b$. For that, we will need to prove that there exists a sequence $a'_n > b$ for which $a'_n \to a$. As such a sequence, we will take a $<$-decreasing sequence a'_n for which $a'_n > a$ and $a'_n \to a$, a sequence whose existence was proved in Lemma 1. Since $a'_n \to a$, to complete our proof, it is sufficient to prove that $a'_n > b$ for all n.

Indeed, let n be an arbitrary natural number. By definition of a kinematic space, there exists an element $a_- < a$, so we have $a_- < a < a'_n$. Since the element a is contained in the open interval (a_-, a'_n) and $a_n \to a$, by definition of convergence, there exists an N for which $a_- < a_N < a'_n$. By definition, $a_N \geq b$ means that there exists a sequence of elements $a_{N1}, a_{N2}, \ldots$ for which $a_{Nk} > b$ and $a_{Nk} \to a_N$. Since $a_N \in (a_-, a'_n)$, by definition of convergence, this implies that for some K, we have $a_{NK} \in (a_-, a'_n)$. From $a_{NK} < a'_n$ and $a_{NK} > b$, we conclude that $a'_n > b$. The proposition is proven.

3.5. Proof of Proposition 3

If $\{a_n\}$ is a $\leq$-decreasing bounded sequence, with a bound b, then all its elements belong to the interval $[b, a_1]$. Since the kinematic space is intervally-compact, this interval is compact. Thus, by known properties of compactness, the sequence $\{a_n\}$ has a convergent subsequence $a_{n_k} \to a$, where $n_k \to \infty$. By definition of the Alexandrov topology on a kinematic space, this means that for every $a_- < a < a_+$, there exists a K for which, for all $k \geq K$, we have $a_- < a_{n_k} < a_+$. Let us show that $a_n \to a$, i.e., that for every a_- and a_+, there exists an N for which, for all $n \geq N$, we have $a_- < a_n < a_+$. Indeed, let K be the value corresponding to these a_- and a_+, and let us take $N = n_K$. In this case, $a_N = a_{n_K} < a_+$.

When $n \geq N$, then, due to the fact that the sequence is $\leq$-decreasing, we have $a_n \geq a_N$, so due to $a_N < a_+$, we have $a_n < a_+$.

Since $n_k \to \infty$, there exists a value $k_0 \geq K$ for which $n_{k_0} \geq n$ and hence, $a_{n_{k_0}} \leq a_n$. Thus, from $a_- < a_{n_{k_0}}$ and $a_{n_{k_0}} \leq a_n$, we conclude that $a_- < a_n$. Convergence is proven, and so it the proposition.

3.6. Proof of the Theorem

The proof of this result uses the following natural auxiliary notions:

Definition 12. *For every element $e \in X$, let S_e denote the set of all $\le$-monotonically decreasing sequences $a = \{a_n\}$ for which $e \le a_n$ for all n and $\bigcap\limits_{n=1}^{\infty} [e, a_n] = [e, e]$. On this set of sequences, we can define a new pre-ordering $a \ge b \Leftrightarrow \forall n\, \exists m (a_n \ge b_m)$.*

Our proof is based on the following three lemmas:

Lemma 3. *For every complete separable normal kinematic space, if $a \in S_e$, then $a_n \to e$.*

Lemma 4. *For every complete separable normal kinematic space, if a sequence $a \in S_e$ is $<$-decreasing, then it is a largest element in S_e, i.e., $a \ge b$ for all $b \in S_e$.*

Lemma 5. *For every complete separable normal kinematic space, $a > b$ if and only if there exists a $\le$-decreasing sequence $\{s_n\}$ with $s_1 = a$ and a limit $e \ge b$ which is the largest element in the set S_e.*

Comment. Lemma 5 describes $<$ in terms of $\le$. Thus, $\le = \le'$ indeed implies $< = <'$.

Proof of Lemma 3. If $a \in S_e$, then a_n is a $\le$-decreasing sequence which is bounded by e. Since the kinematic space is complete, this sequence has a limit. Let us denote this limit by b.

From $a_n \ge e$ and $a_n \to b$, in the limit, we get $b \ge e$; see Proposition 2. From the fact that $a_N \le a_n$ for all $N \ge n$, in the limit, we get $b \le a_n$ for all n. Thus, $e \le b \le a_n$ for all n, i.e., b belongs to all the closed intervals $[e, a_n]$ and so, b belongs to the intersection $[e, e]$ of all these closed intervals.

The fact that $b \in [e, e]$ means that $b \le e$ and $e \le b$. Now, for every element x, if $x < b$ then from $x < b$ and $b \le e$, we conclude that $x < e$. Vice versa, if $x < e$, then from $x < e$ and $e \le b$, we conclude that $x < b$. Thus, $x < b$ if and only if $x < e$. Similarly, for every element x, we have $b < x$ if and only if $e < x$. So, in terms of the open relation $<$, the elements e and b are interchangeable. Since the limit is defined in terms of the open relation $<$, the fact that $a_n \to b$ implies that $a_n \to e$. The lemma is proven.

Proof of Lemma 4. Let us show that if a is an $<$-decreasing element of S_e and $b \in S_e$, then $a \geq b$, i.e., that for every n, there exist an m for which $a_n \geq b_m$. Indeed, by Lemma 3, $b \in S_e$ implies that $b_n \to e$. By definition of convergence, this means that for every $a_- < e < a_+$, there exists an m_0 for which, for all $m \geq m_0$, we have $a_- < b_m < a_+$.

By definition of a kinematic space, there exists an element $a_- < e$. Since $a_n > a_{n-1}$ and $a_{n-1} \geq e$, we conclude, by Proposition 2, that $a_n > e$. So, we can take $a_+ = a_n$. Then, there exists an m for which $b_m < a_+ = a_n$ and thus, $b_m \leq a_n$. The Lemma is proven.

Proof of Lemma 5. Let $a > b$. Then, similarly to the proof of Lemma 1, we can construct a $<$-decreasing sequence s_n for which $s_1 = a$, $s_n > b$, and $s_n \to b$: the only difference is that we select $s_1 = a$ instead of $s_1 = b_+$. One can easily show that this sequence belongs to the set S_b. Indeed, if $b \leq x \leq s_n$ for all n, then in the limit $s_n \to b$, we conclude that $b \leq x \leq b$, i.e., that $x \in [b, b]$. Then, due to Lemma 4, we conclude that s is the largest element in the set S_b.

Vice versa, let us assume that $a = s_1$ for some sequence s which is the largest in S_e for some $e \geq b$. This means that for every other sequence $s' \in S_e$, we have $s \geq s'$. In particular, as s', we can take a $<$-decreasing sequence s'_n for which $s'_n \to e$. For this sequence, $s'_n > e$ for all n. From $s \geq s'$, we conclude, in particular, that there exists an m for which $a = s_1 \geq s'_m$. From $a \geq s'_m > e \geq b$, we now conclude – via Proposition 1 – that $a > b$.

Proposition is proven.

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