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Optimal Prices in the Presence of Discounts: A New Economic Application of Choquet Integrals

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Abstract

In many real-life situations, there are "package deals", when customers who buy two or more different items get a special discount. For example, when planning a trip, a deal including airfare and hotel costs less than the airfare and hotel on their own; there are often also special deals when we combine airfare with car rental, and deals that cover all three items plus tickets to local tourist attractions, etc.

Usually, there are several such deals with different combinations and different discounts. What if we plan to buy several different copies of each item: e.g., we plan a group tour in which some tourists want to rent car, some want to visit certain tourist attractions, etc. What is the best way to use all available discounts? We show that, under reasonable assumptions, the optimal price is provided by the Choquet integral.

1 Practical Problem and Its Reformulation in Precise Terms

Practical situation: discounts. In many real-life situations, there is a "package deal": customers who buy two or more different items get a special discount. For example:

- when planning a trip, a deal including airfare and hotel costs less than the airfare and hotel on their own;
- there are often also special deals when we combine airfare with car rental, and deals that cover all three items plus tickets to local attractions, etc.;
- when a customer buys a house, there are often package deals which enable the customer to also buy the original furniture and/or car at a special discount;

- in fast food restaurants, such group discounts are a norm.

Usually, there are several such deals with different combinations and different discounts.

How to describe discounts in precise terms. Let us denote, by n, the total number of different items, and let us denote these items by $x_1, \ldots, x_n$, so that the set of all the items is $X = \{x_1, \ldots, x_n\}$. Let $v(x_i)$ denote the price of each individual item.

A discount means that for some sets of items $S \subseteq X$, the overall price $v(S)$ of buying all the items from this set is smaller than the total price $\sum\limits_{x_i \in S} v(x_i)$ that we would have to pay if we bough them separately. For example, if x_1 is the airfare and x_2 is the hotel, then a package deal means that $v(\{x_1, x_2\}) < v(x_1) + v(x_2)$.

Not all combinations lead to package deals. For some combinations S, the cheapest price $v(S)$ is exactly the sum $\sum\limits_{x_i \in S} v(x_i)$ of individual prices.

We assume that we know all the discounts. We assume that for every subset S, we know the cheapest price $v(S)$ that we have to pay if we want to buy all the items from this set S.

Natural assumption: the larger the group, the better the discount. A natural assumption is that the larger the group, the better the discount. So, if we want to buy items from two sets S and S', then, instead of buying them separately and paying the sum $v(S) + v(S')$, it is cheaper (or at least not more expensive) to buy the whole group $S \cup S'$ – and then buy additionally all the duplicate items, from the intersection $S \cap S'$. In other words, we require that for all sets S and S', we have $v(S) + v(S') \geq v(S \cup S') + v(S \cap S')$.

What if we want to buy several items of each type: formulation of the practical problem. What if we want to buy several items of each type, i.e., we want to buy d_1 items of type x_1, d_2 items of type x_2, etc. For example, we want to plan a group tour in which some tourists want to rent car, some want to visit certain attractions, etc.

The problem is: What is the best way to use all available discounts?

Mathematical comment. We assume that we know the prices $v(S)$ corresponding to all possible sets S, i.e., situations in which for each item i, we either buy it ($d_i = 1$) or not ($d_i = 0$). Now, we want to extend the price function to situations in which each d_i can take any natural value 0, 1, 2, ... In mathematics, such combinations in which we may have several copies of each item are called *bags* (also known as *multisets*)– because of their relation to grocery bags, for which, by the way, discounts are also frequent; see, e.g., [2, 4]. Thus, from the mathematical viewpoint, the problem is how to extend a given function $v(S)$ from sets to bags.

Towards the formulation of our main problem in precise terms. Each way to use the discounts consists of selecting discounts – i.e., sets $S_1, \ldots, S_m$ – and selecting how many times $t_1, \ldots, t_m$

we use each of the discounts so that totally, we get exactly d_1 objects of type x_1, d_2 objects of type x_2, etc.

The desired amounts can be described by a tuple $d = (d_1, \ldots, d_n)$. Each set $S_i \subseteq X$ can be identified with its characteristic function, for which $\chi_{S_j}(x_i) = 1$ if the item x_i is in the set S_j and $\chi_{S_j}(x_i) = 0$ otherwise. For each selection of sets S_i and times t_i, the overall price is equal to $t_1 \cdot v(S_1) + \ldots + t_m \cdot v(S_m)$. Thus, we arrive at the following precise optimization problem.

2 Precise Optimization Formulation of the Practical Problem

Definition 1. *Let $X = \{x_1, \ldots, x_n\}$ be a finite set. By a* discount function*, we mean a function $v(S)$ that maps every subset of X into a non-negative real number in such a way that for every two sets S and S', we have*

$$v(S) + v(S') \geq v(S \cup S') + v(S \cap S'). \tag{1}$$

Definition 2. *Let $X = \{x_1, \ldots, x_n\}$ be a finite set.*

- *By a* task*, we mean a tuple $d = (d_1, \ldots, d_n)$ in which d_i are natural numbers (i.e., non-negative integers).*
- *By a* purchasing plan*, we mean a pair $P = \langle (S_1, \ldots, S_m), (t_1, \ldots, t_m) \rangle$, in which S_j are subsets of the set X and t_j are non-negative integers.*
- *We say that a plan $P = \langle (S_1, \ldots, S_m), (t_1, \ldots, t_m) \rangle$* satisfies *the task d if for every element x_i, we have $d_i = \sum\limits_{j=1}^{m} t_j \cdot \chi_{S_j}(x_i)$.*
- *By a* price $v(P)$ *of the purchasing plan $P = \langle (S_1, \ldots, S_m), (t_1, \ldots, t_m) \rangle$, we mean the value*

$$v(P) \stackrel{\text{def}}{=} t_1 \cdot v(S_1) + \ldots + t_m \cdot v(S_m). \tag{2}$$

Problem:

GIVEN: a discount function v and a task d,

AMONG: all the purchasing plans which are consistent with the task,

FIND: the purchasing plan with the smallest price $v(P)$.

The price of this cheapest purchasing plan will be called the *price of the task* and denoted by $v(d)$.

3 Main result

Theorem. *For every discount function v and for every task d, the price of this task can be determined as follows. If we order the values d_i in the increasing order, into the sequence*

$$d_{(1)} \leq d_{(2)} \leq \ldots \leq d_{(n)} \tag{3}$$

and order the items accordingly, then

$$v(d) = d_{(1)} \cdot v(\{x_{(1)}, x_{(2)} \ldots, x_{(n)}\}) + (d_{(2)} - d_{(1)}) \cdot v(\{x_{(2)}, \ldots, x_{(n)}\}) + \ldots +$$

$$(d_{(i)} - d_{(i-1)}) \cdot v(\{x_{(i)}, x_{(i+1)}, \ldots, x_{(n)}\}) + \ldots + (d_{(n)} - d_{(n-1)}) \cdot v(\{x_{(n)}\}). \tag{4}$$

Comment. The expression (4) is exactly (discrete) Choquet integral; see, e.g., [1, 5]. Thus, under reasonable assumptions, the optimal price is indeed provided by the Choquet integral.

Historical comment. For the case of $n = 2$, this interpretation of the Choquet integral was first outlined in [3]. For other economic applications of Choquet integral – to decision making – see, e.g., [6] and references therein.

Proof. The above value is attained when we take the following purchasing plan P_0: $d_{(1)}$ copies of the set $\{x_{(1)}, x_{(2)} \ldots, x_{(n)}\}$, $d_{(2)} - d_{(1)}$ copies of the set $\{x_{(2)}, \ldots, x_{(n)}\}$, ..., $d_{(i)} - d_{(i-1)}$ copies of the set $\{x_{(i)}, x_{(i+1)}, \ldots, x_{(n)}\}$, ..., and $d_{(n)} - d_{(n-1)}$ copies of the set $\{x_{(n)}\}$. So, to prove the theorem, it is sufficient to prove that for every other purchasing plan $P = \langle (S_1, \ldots, S_m), (t_1, \ldots, t_m) \rangle$ which is consistent with the task d, we have $v(P) \geq v(P_0)$. Indeed, let P be such a plan.

Without losing generality, we can ignore the items which are not included in the task d, i.e., for which $d_i = 0$. As a result, we can assume that the list of items X includes only the items x_i for which $d_i > 0$. Without losing generality, we can also assume that all the sets S_i are different – otherwise, if some of them are equal, we can simply add up the corresponding times t_i that this bargain is used.

Let us denote, by $t_{(1)}$, the smallest possible value t_i: $t_{(1)} \stackrel{\text{def}}{=} \min(t_1, \ldots, t_m)$. Due to the property (1), if we replace S_1 and S_2 with $S_1 \cup S_2$ and $S_1 \cap S_2$, the sum of the prices either decreases or stays the same:

$$v(S_1) + v(S_2) \geq v(S_1 \cup S_2) + v(S_1 \cap S_2). \tag{5}$$

By adding $v(S_3)$ to both sides, we get

$$v(S_1) + v(S_2) + v(S_3) \geq v(S_1 \cup S_2) + v(S_1 \cap S_2) + v(S_3). \tag{6}$$

Similarly, if we replace $S_1 \cup S_2$ and S_3 with $S_1 \cup S_2 \cup S_3$ and $(S_1 \cup S_2) \cap S_3$, the sum of prices does not decrease:

$$v(S_1 \cup S_2) + v(S_3) \geq v(S_1 \cup S_2 \cup S_3) + v((S_1 \cup S_2) \cap S_3). \tag{7}$$

By adding $v(S_1 \cap S_2)$ to both sides, we get

$$v(S_1 \cup S_2) + v(S_1 \cap S_2) + v(S_3) \geq v(S_1 \cup S_2 \cup S_3) + v((S_1 \cup S_2) \cap S_3) + v(S_1 \cap S_2), \quad (8)$$

and hence, due to (6), we get

$$v(S_1) + v(S_2) + v(S_3) \geq v(S_1 \cup S_2 \cup S_3) + v((S_1 \cup S_2) \cap S_3) + v(S_1 \cap S_2). \quad (9)$$

By adding $v(S_4)$ to both sides, we get

$$v(S_1) + v(S_2) + v(S_3) + v(S_4) \geq v(S_1 \cup S_2 \cup S_3) + v((S_1 \cup S_2) \cap S_3) + v(S_1 \cap S_2) + v(S_4). \quad (10)$$

If we replace $S_1 \cup S_2 \cup S_3$ and S_4 with $S_1 \cup S_2 \cup S_3 \cup S_4$ and $(S_1 \cup S_2 \cup S_3) \cap S_4$, the sum of prices does not increase:

$$v(S_1 \cup S_2 \cup S_3) + v(S_4) \geq v(S_1 \cup S_2 \cup S_3 \cup S_4) + v((S_1 \cup S_2 \cup S_3) \cap S_4), \quad (11)$$

and thus,

$$v(S_1) + v(S_2) + v(S_3) + v(S_4) \geq$$
$$v(S_1 \cup S_2 \cup S_3 \cup S_4) + v((S_1 \cup S_2 \cup S_3) \cap S_4) + v((S_1 \cup S_2) \cap S_3) + v(S_1 \cap S_2). \quad (12)$$

After repeating this argument for all n items, we conclude that

$$v(S_1) + \ldots + v(S_n) \geq$$

$$v(S_1 \cup \ldots \cup S_n) + v((S_1 \cup \ldots \cup S_{n-1}) \cap S_n) + \ldots + v((S_1 \cup \ldots \cup S_{i-1}) \cap S_i) + \ldots v(S_1 \cap S_2). \quad (13)$$

Thus, in the purchasing plan P, we can replace $S_1, \ldots, S_n$ with a new sequence of sets in which one of them is the union $S'_1 = S_1 \cup \ldots \cup S_m$ of the original sets – and do not increase the total price. We can do this with all $t_{(1)}$ copies of all the sets $S_1, \ldots, S_m$.

All the remaining sets in the resulting purchasing plan are then proper subsets of the union S'_1.

To the remaining sets, we can apply the same procedure, and get a new collection of sets in which one of the sets S'_2 is a union of the remaining sets (and hence $S'_2 \subset S'_1$) and others are proper subsets of S'_2. Repeating this procedure for all the remaining sets, we get a collection with a new set $S'_3 \subset S'_2$, etc. After we repeat this procedure m times, we get a new purchasing plan, with sets $S'_1, S'_2, \ldots, S'_m$ repeated certain number of times $t'_1, \ldots, t'_m$. By construction, we have

$$S'_1 \supset S'_2 \supset \ldots \supset S'_m. \quad (14)$$

Since on each step of this construction, the price of the purchasing plan does not increase, we conclude that

$$v(P) = t_1 \cdot v(S_1) + \ldots + t_m \cdot v(S_m) \geq t'_1 \cdot v(S'_1) + \ldots + t'_m \cdot v(S'_m). \quad (15)$$

Let us show that the new purchasing plan $P' = \langle (S'_1, \ldots, S'_m), (t'_1, \ldots, t'_m) \rangle$ is exactly the plan P_0. Then, (15) would mean that $v(P) \geq v(P_0)$.

Indeed, at each step, we did not change the total number of objects of each type, so the new purchasing plan still satisfies the original task d. Thus, for every element x_i, d_i is equal to $\sum_{j=1}^{m} t'_j \cdot \chi_{S'_j}(x_i)$. For values x_i that belong to S'_m, due to (14), we have $d_i = t'_1 + \ldots + t'_m$. For values x_i that belong to S'_{m-1} but do not belong to S'_m, i.e., that belong to the set difference $S'_{m-1} \setminus S'_m$, we get $d_i = t'_1 + \ldots + t'_{m-1}$. In general, for values x_i that belong to $S'_j \setminus S'_{j+1}$, we have

$$d_i = t'_1 + \ldots + t'_j. \tag{16}$$

Thus, all elements from each set $S'_j \setminus S'_{j+1}$ have the exact same value d_i, which we will denote by $d(j)$. By (16), the larger j, the larger the corresponding values $d(j)$, i.e., $d(1) \leq d(2) \leq \ldots \leq d(m)$. In other words, the values $d(j)$ are the values d_i sorted in increasing order – i.e., in effect, the values $d_{(j)}$.

The set S'_1 consists of all the items. The set S'_2 consists of all the items except for the items from the set difference $S'_1 \setminus S'_2$. Thus, to get S'_2, we exclude the elements with the smallest possible value $d(1)$, and only include the elements x_i for which $d_i \geq d(2)$. Similarly, the set S'_j consists of all the elements x_i for which $d_i \geq d(j)$. Thus, if we sort the elements x_i in the increasing order of their values d_i, we conclude that $S'_j = \{x_{(j)}, x_{(j+1)}, \ldots, x_{(n)}\}$, i.e., we get the same sets as in the purchasing plan P_0.

By comparing the values $d_{(i)} = t'_1 + \ldots + t'_{i-1} + t'_i$ and $d_{(i-1)} = t'_1 + \ldots + t'_{i-1}$ computed according to the formula (16), we conclude that $t'_i = d_{(i)} - d_{(i-1)}$, i.e., that not only the sets in the purchasing plan P' are the same as in P_0, but the numbers of times are the same. Thus, indeed, $P' = P_0$, and the theorem is proven.

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