6-1-2010

How to Describe Spatial Resolution: An Approach Similar to the Central Limit Theorem

Omar Ochoa
University of Texas at El Paso, omar@miners.utep.edu

Martine Ceberio
University of Texas at El Paso, mceberio@utep.edu

Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu

Comments:
Technical Report: UTEP-CS-10-13
Published in *Applied Mathematical Sciences*, 2010, Vol. 4, No. 63, pp. 3153-3160.

How to Describe Spatial Resolution: An Approach Similar to the Central Limit Theorem

Omar Ochoa, Martine Ceberio, and Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
omar@miners.utep.edu, mceberio@utep.edu, vladik@utep.edu

Abstract

For spatially distributed quantities $v(x)$, there are two main reasons why the measured value is different from the actual value. First, the sensors are imprecise, so the measured value is slightly different from the actual one. Second, sensors have a finite *spatial resolution*: they do not simply measure the value at a single point, they are "blurred", i.e., affected by the values of the nearby points as well. It is known that uncertainty can be often described by the Gaussian distribution. This possibility comes from the Central Limit Theorem, according to which the sum of many independent small measurement errors has an approximately Gaussian distribution. In this paper, we show how a similar technique can be applied to spatial resolution: a combination of several independent small blurrings can be described by a Gaussian blurring function.

Uncertainty and spatial resolution. For spatially distributed quantities $v(x)$, there are two main reasons why the measured value is different from the actual value (see, e.g., [3]):

- First, the sensors are imprecise. As a result, even for the case when there is no spatial variation, i.e., when the value of the measured quantity is the same at all spatial locations x ($v(x) = v_0$), the measured value $\widetilde{v}$ is, in general, slightly different from the actual one: the measurement error $\Delta v \stackrel{\text{def}}{=} \widetilde{v} - v$ is different from 0.
- Second, sensors have a finite *spatial resolution* in the following sense: the signal generated by a sensor located at a point x does not depend only on the value of $v(x)$ at the sensor's location x, it is also depending on the values $v(y)$ at nearby spatial locations y.

Thus, to understand the measurement accuracy, we need to understand both uncertainty and spatial resolution of the corresponding measurement.

How uncertainty is usually described. To describe uncertainty, we need to know the probabilities of different values of the measurement error Δv. In other words, for each interval $[\underline{v}, \overline{v}]$, we need to know the probability $p([\underline{v}, \overline{v}])$ that Δv belongs to this interval. For narrow intervals $[\underline{v}, \overline{v}]$, with a small width $\delta v \stackrel{\text{def}}{=} \overline{v} - \underline{v}$, this probability $p([\underline{v}, \overline{v}])$ is proportional to the width: $p([\underline{v}, \overline{v}]) \approx c(\Delta v) \cdot \delta$. The corresponding coefficient $c(x) \approx \dfrac{p([\underline{v}, \overline{v}])}{\overline{v} - \underline{v}}$ is called the *probability density* and denoted by $\rho(\Delta v)$.

Thus, to describe measurement uncertainty, we need to describe the corresponding probability density $\rho(\Delta v)$.

Mathematical comment. Formally, the probability density is defined as a *limit*

$$\rho(\Delta v) \stackrel{\text{def}}{=} \lim_{\overline{v}-\underline{v}} \frac{p([\underline{v}, \overline{v}])}{\overline{v} - \underline{v}},$$

where the limit is taken over the intervals $[\underline{v}, \overline{v}]$ that contain the value Δv.

Since probabilities are non-negative, we have $\rho(x) \geq 0$. The sum of all probabilities is equal to 1, so we have $\int \rho(x)\, dx = 1$.

Physical comment. In general, a measuring instrument may have different accuracy on different parts of its scale. For example, usually, when we measure values which are close to the limits of this measuring instrument, we get much lower accuracy. As a result, we can have different probabilities at different parts (sub-scales) of the original scale. To simplify our description, we limit ourselves to one of these sub-scales – and thus, we can safely assume that the measurement accuracy is the same for all the values v.

Central Limit Theorem and Gaussian distribution. It is known that uncertainty can be often described by the Gaussian (= normal) distribution, with the probability density

$$\rho(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right). \tag{1}$$

This possibility comes from the Central Limit Theorem, according to which the sum $x = \sum\limits_{i=1}^{N} x_i$ of a large number N of independent small random variables x_i has an approximately Gaussian distribution. (To be more precise, the theorem says that in the limit $N \to \infty$, the distribution of the sum tends to the Gaussian distribution.)

In practice, often, the measurement error is caused by a joint effect of a large number of small independent factors, so it makes sense to conclude that the distribution is approximately Gaussian. This theoretical conclusion has been experimentally confirmed on the example of many actual measuring instruments; see, e.g., [1, 2].

Comment. The above result is about the 1-D distribution: of a random number. For the multi-D case – of a random vector $x = (x_1, \ldots, x_n)$ – a similar result also leads to multi-D Gaussian distribution, with an expression

$$\rho(x) = \text{const} \cdot \exp\left(- \sum_{i,j=1}^{n} c_{ij} \cdot (x_i - a_i) \cdot (x_j - a_j) \right). \tag{2}$$

How spatial resolution is usually described. The value $\widetilde{v}(x)$ measured by a sensor place at a point x is affected not only by the value $v(x)$ at this location, but also by the values $v(y)$ at nearby locations y.

In general, the dependence of the measured quantity $\widetilde{v}(x)$ on the values $v(x)$ and $v(y)$ can be non-linear: $\widetilde{v}(x) = F(v(x), v(y), \ldots)$.

In some cases, the signal that we are measuring is quite strong. For example, in geophysics, the seismic waves coming from strong nearby earthquakes are strong. In such cases, the signal-to-noise ratio is high, and the effects of uncertainty and spatial resolution are small.

The problem of detecting and processing uncertainty and spatial resolution becomes important only in the cases when the signal is relatively small, so small that it needs special data processing to detect. Such cases are abundant in data processing. For example, in geophysics, in the gravity measurements, we try to detect the values of density at different locations and at different depths based on slight deviations from the standard free fall acceleration $g = 9.81$ m/sec^2, deviations which require very accurate measuring instruments to detect. Similarly, seismic measurements are based on measuring the seismic waves against the noise background, measurements which also require highly sensitive equipment. For example, to measure seismic waves from relatively weak distant earthquakes, it is necessary to place the sensors below ground, in special noise-protected chambers.

Since the effect of all values $v(x)$ and $v(y)$ is usually small, the terms which are quadratic and or higher order in terms of these small signals are extremely small and can be therefore safely ignored. It is thus reasonable to expand the dependence $\widetilde{v}(x) = F(v(x), v(y), \ldots)$ in Taylor series and keep only linear terms in this expansion. As a result, we conclude that the dependence of the measured value $\widetilde{v}(x)$ on the values $v(x)$ and $v(y)$ is linear, i.e., in the discrete approximation, in which we only consider points y on a grid, $\widetilde{v}(x) \approx \sum\limits_{y} w(x, y) \cdot v(y)$, for some weights $w(x, y)$. The denser the grid, the more values $v(y)$ we take into account. In the limit, the sum turns into an integral

$$\widetilde{v}(x) = \int w(x, y) \cdot v(y)\, dy.$$

The weight $w(x, y)$ depends on how close the locations x and y are, i.e., in general, it depends on the difference $x - y$ between the two locations: $w(x, y) = w(x - y)$. To be more precise, similarly to the case of uncertainty, we may have different weights $w(x - y)$ in different subregions. To simplify our description,

we limit ourselves to one of these subregions – and thus, we can safely assume that the weight function $w(x-y)$ is the same for all the measured values $\widetilde{v}(x)$.

Under this assumption, we conclude that the measured value $\widetilde{v}(x)$ is related to the actual values $v(y)$ by the formula

$$\widetilde{v}(x) = \int w(x-y) \cdot v(y)\, dy, \tag{3}$$

for some weight function $w(x)$.

Terminological comment. In mathematical terms, the function $\widetilde{v}(x)$ as described by the formula (3) is called the *convolution* of the functions $w(x)$ and $v(x)$.

Comment 1: related to measurement theory. Usually, sensors are *calibrated* – e.g., by testing them on the case when the signal is constant $v(x) = v_0$ for some constant v_0. In this case, ideally, the measurement results should coincide with this constant value v_0.

In reality, according to (3), the actual measured value is equal to $\widetilde{v}_0 = v_0 \cdot \int w(x-y)\, dy$. By introducing an auxiliary variable $z = x-y$, we can conclude that $\widetilde{v}_0 = v_0 \cdot \int w(z)\, dz$. If this value is different from v_0, then the sensor is re-scaled (re-calibrated) so that the original signal $\widetilde{v}_0$ would now correspond to the signal v_0: e.g., by introducing an additional step of multiplying the original measurement result by the ratio $v_0/\widetilde{v}_0$. For the re-calibrated measuring instrument, we have $\widetilde{v}_0 = v_0$ and hence, $\widetilde{v}_0 = v_0 \cdot \int w(z)\, dz = v_0$; thus,

$$\int w(z)\, dz. \tag{4}$$

Because of this, in the following text, we can, without losing generality, assume that the weight function satisfies the condition (4).

Comment 2: related to physics. From the purely mathematical viewpoint, it is possible that some of the weights $w(y)$ can be negative. However, in most physical measurements, the nearby values $v(y)$ affect the sensor in the same way (although with somewhat smaller strength) as the value $v(x)$ at the sensor's location x. For example, the measured value of the gravity at a given location x is affected not only by the density at this location x, but also by the density y at nearby points y. Thus, from the physical viewpoint, we can assume that all the weights are non-negative: $w(z) \geq 0$.

This assumption makes perfect sense. The whole point of spatial resolution is that we are unable to distinguish between the contributions of the value $v(x)$ at location x and the contribution of the values $v(y)$ at nearby locations y. If we had positive weights for x and negative weights for nearby locations y, then we would, in effect, measure the (weighted) difference between the signals at x and signals at nearby locations – which would enable us to effectively separate the values at these locations.

Thus, in the following text, we assume that the weight function $w(x)$ is everywhere non-negative:

$$w(z) \geq 0. \tag{5}$$

Problem: it is desirable to limit the class of weight functions corresponding to spatial resolution. In principle, we can have different weight functions $w(z)$ satisfying conditions (4) and (5) – just like for uncertainty, we could have different possible probability density functions $\rho(x)$.

For the case of uncertainty, if we take into account that the actual measurement error comes from many different independent sources, then (as we have mentioned earlier) we can conclude that the distribution is Gaussian. Thus, we can limit ourselves to a 2-parametric family (1) of probability density functions.

It is desirable to come up with a similar few-parametric class of weight functions $w(z)$ for describing spatial resolution.

Main idea. Our main idea is that, similar to uncertainty, the "blurring" (3) of the original signal $v(x)$ is caused by a large number N of different independent factors, each of them relatively small.

Main idea: towards a precise formulation. In precise terms, we assume that the original signals goes through a sequence of a large number N of "small" blurrings. Let $v_i(x)$ denote the signal at the i-th stage of this sequential procedure. We start with the original signal $v_0(x) = v(x)$, and on each stage, apply the corresponding small blurring to get the next-stage signal:

$$v_i(x) = \int w_i(x-y) \cdot v_{i-1}(y)\, dy, \tag{6}$$

where the weight function $w_i(z)$ is concentrated in a small vicinity of 0. The observed signal $\widetilde{v}(x)$ corresponds to the result of going through all N stages: $\widetilde{v}(x) = v_N(x)$.

It is desirable to find out which transformations (3) can be thus represented for large N – and in the limit when $N \to \infty$.

Analysis of the problem. It is known that when we apply two blurrings corresponding to weight functions $w(x)$ and $w'(x)$ one after another, the result is equivalent to a single blurring with a weight functions $w''(x)$ which is equal to the convolution of the functions $w(x)$ and $w'(x)$:

$$\widetilde{w}''(x) = \int w'(x-t) \cdot w(t)\, dt. \tag{7}$$

Indeed, suppose that we start with a function $v(x)$, and then apply the two blurrings one after another:

$$v_1(x) = \int w(x-y) \cdot v(y)\, dy, \tag{8}$$

and then

$$\widetilde{v}(x) = \int w'(x-y) \cdot v_1(y)\, dy. \tag{9}$$

Substituting the formula (8) into the expression (9), we conclude that

$$\widetilde{v}(x) = \int\int w'(x-y) \cdot w(y-z) \cdot v(z)\, dydz. \tag{10}$$

On the other hand, the equation

$$\widetilde{v}(x) = \int w''(x-z) \cdot v(z)\, dz, \tag{11}$$

where $w''(x)$ is described by the formula (7), has the form

$$\widetilde{v}(x) = \int\int w'(x-z-t) \cdot w(t) \cdot v(z)\, dtdz, \tag{12}$$

By introducing new variable $y = z+t$, for which $t = y-z$ and $x-z-t = x-y$, we get the desired formula (10) he statement is proven.

Thus, the original question can be reformulated as follows: how can we describe the results of convolution of many different weight functions $w_i(x)$ corresponding to small changes (i.e., for which $w_i(x) > 0$ only in a small vicinity of 0). To answer this question, we would like to use the case of uncertainty, where we had a similar result about the sum of several independent random variables.

To use this result, let us recall how the sum of two independent random variables can be described. As we have mentioned, each random variable x_i can be described by its probability density function $\rho_i(x_i)$. For each value x_1, the probability density that the first variable takes the value x_1 is equal to $\rho_1(x_1)$, and for each value x_2, the probability density that the second variable takes the value x_2 is equal to $\rho_2(x_2)$. Since the random variables are independent, the probability that the first variable takes the value x_1 *and* the second variable takes the value x_2 is equal to the product: $\rho_1(x_1) \cdot \rho_2(x_2)$. In this case, the sum of the values of the two random variables is equal to $x = x_1 + x_2$. So, if we want to find the probability density $\rho(x)$ that the sum of the two random variables is equal to x, we must add up the probabilities $\rho_1(x_1) \cdot \rho_2(x_2)$ (with $x_2 = x - x_1$) corresponding to all possible values of x_1. Thus, we have the formula

$$\rho(x) = \int \rho_1(x_1) \cdot \rho_2(x - x_1)\, dx_1. \tag{13}$$

In other words, the probability density function $\rho(x)$ corresponding to the sum of two random variables is equal to the convolution of the probability density functions $\rho_i(x_i)$ corresponding to individual variables.

Due to the conditions (4) and (5), the weight functions $w_i(x)$ can be (formally) interpreted as probability density functions of some random variables x_i, and their convolution $w(x)$ can be therefore interpreted as a probability density function for the sum $\sum\limits_{i=1}^{N} x_i$ of these random variables.

Since we assumed that $w_i(x) > 0$ only for values x close to 0, the corresponding random variables x_i are small. Thus, due to the central limit theorem, the distribution of their sum is close to Gaussian. Therefore, the probability density function $w(x)$ for this sum is close to the probability density function (2) of the multi-D Gaussian distribution. In the limit $N \to \infty$, we have the exact Gaussian function. Thus, we arrive at the following conclusion.

Conclusion. If we take into account that the actual blurring (3) is usually a combination of several independent small blurrings, then we conclude that the blurring can be described by a Gaussian weight function

$$w(x) = \text{const} \cdot \exp\left(-\sum_{i,j=1}^{n} c_{ij} \cdot (x_i - a_i) \cdot (x_j - a_j)\right). \tag{14}$$

Thus, similarly to the case of uncertainty, we have a reasonable finite-parametric family to describe spatial resolution.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and DUE-0926721, by Grant 1 T36 GM078000-01 from the National Institutes of Health, by Grant MSM 6198898701 from MŠMT of Czech Republic, and by Grant 5015 "Application of fuzzy logic with operators in the knowledge based systems" from the Science and Technology Centre in Ukraine (STCU), funded by European Union.

References

[1] P. V. Novitskii and I. A. Zograph, *Estimating the Measurement Errors*, Energoatomizdat, Leningrad, 1991 (in Russian).

[2] A. I. Orlov, "How often are the observations normal?", *Industrial Laboratory*, 1991, Vol. 57. No. 7, pp. 770–772.

[3] S. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer Verlag, New York, 2005.

[4] D. G. Sheskin, *Handbook of Parametric and Nonparametric Statistical Procedures*, Chapman & Hall/CRC Press, Boca Raton, Florida, 2007.