1-1-2005

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Comments:
UTEP-CS-05-03.
Published in *Proceedings of the 12th International Conference Sensor'2005*, Nuremberg, Germany, May 10-12, 2005, pp. 317-322.

A Universal Sensor Model

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1. Introduction

Due to the continuing progress in science and engineering, we are able to measure more and more different quantities with a better and better accuracy. New ideas and techniques are used to develop new sensors and new measuring instruments.

Designing each new measuring instrument, each innovative measuring system is a difficult and time-consuming task. It is desirable to utilize our experience of designing such systems and come up with a general methodology that would help in designing individual measuring systems.

The required generalization is easier for some parts of the measuring system but much more difficult for other parts. In a nutshell, a measuring system consists of:

- sensors that transform the measured quantity into an electric signal;
- analog-to-digital transformers (ADT) that transform the electric signal into the binary code, and
- processors that convert the (raw) binary code into the desired measurement result.

No matter what physical quantity we measure, the ADTs and processors perform the same function. The designs may differ. For example, an ADT design for a highly dynamic system is mainly concerned with the ability to process many readings per second, while an ADT for a very accurate measurement of a stable signal is more about achieving cutting-edge accuracy. However, because of the similarity in function, these designs also have similar features, so it is possible to have a general methodology for designing ADTs and specialized processors.

In contrast to ADTs and processors, sensors vary drastically. Different sensors measure different physical quantities, they may be based on completely different physical and engineering principles; as a result, they have few things in common except for the fact that all the sensors transform the value of the measured quantity into an electric signal. This variety of sensors is what prevents us from developing general methodologies for designing measuring instruments. So, to develop such methodologies, we just have a general way of describing sensors.

Such a description is easy for simplified sensors, like the ones described in the Introduction to Measurement classes. These simplified sensors transform a static input x (e.g., temperature) into an electric signal y, and within a certain range, the dependence of y on x can be safely assumed to be linear: $y = a_0 + a_1 \cdot x$. Within this simplified model, to completely describe every sensor (at least within a given range), all we need to do is to describe the two coefficients a_0 and a_1 of this linear dependence. In other words, for simplified sensors, this linear dependence serves as a desired universal sensor model.

The simplified sensor model is a reasonable first approximation to actual sensors, an approximation that works reasonably well if we consider measurements of low and medium accuracy. However, the main purpose of the newly designed measurement systems is to go beyond this situation, to aim either for high accuracy measurements, or for measuring highly dynamical inputs. As a result, for the cutting-edge measuring systems, the dependence of the resulting electric signal y on the signal x differs from the above simplified model:

- First, while the dependence of y on x is often close to linear, it is not exactly linear. So, to accurately describe the sensor, we need to take this non-linearity into consideration.
- Second, the simplified sensor model assumes that the measured quantity is the only factor that affects the resulting sensor's signal. In reality, different characteristics of the sensor, of the object, and of the environment all affect each other.

From the purely theoretical viewpoint, it is possible to take all these deviations from the simplified sensor model into consideration.

For example, to describe a general non-linear dependence $y = f(x)$ of y on x, we can use polynomials (corresponding, e.g., to the Taylor expansion of $f(x)$). It is known that an arbitrary continuous dependence can be approximated by polynomials with arbitrary accuracy – the higher the accuracy, the more terms we need in the approximation. From the practical viewpoint, however, such descriptions require too many parameters and are, as a result, not very practically useful.

It is desirable to come up with a universal sensor model that would require only a few parameters and at the same time provide a good description of the existing sensors. In this paper, we provide a brief description of such a model. Specifically, in the following sections, we will show how our model takes into account all the aspects of sensor complexity: non-linearity and the interaction between the sensor, the object, and the environment.

2. How to take sensor nonlinearity $y(x)$ into consideration

Many advanced sensors do not directly transform the quantity into the electric current; instead, they perform a sequence of transformations that starts with the original quantity (signal) and end up with the electric signal. On each step of this transformation, we may have deviations from non-linearity. The resulting non-linear transformation is thus a composition of several nonlinear transformation functions.

Our objective is to approximate such dependences by a family of non-linear functions that can be described by a small number of parameters. Since the sensor's nonlinear transformation is often a composition of two such nonlinear functions, it is reasonable to select this family in such a way that a composition of two functions from this family also belongs to this family. Since some transformations are reversible, it is also reasonable to require that the inverse transformation also belong to the same family. In mathematical terms, this means that these transformations form a *group*. It can be shown that there is only one group that contains all linear transformations – the group of all fractional linear transformations. Thus, we arrive at the idea to describe the sensor's nonlinearity by a fractional-linear transformation [1,4]. Such transformations indeed provide a reasonably good description of close-to-linear transformations.

To improve the accuracy, measuring instruments often combine several sensors and return, e.g., the average of their measured results. For example, the standard of time is based on several super-precise maser clocks whose readings are then combined to improved the accuracy. In real-life situations, different sensors are, in general, non-linear, and moreover, different sensors are, in general, described by different fractional-linear functions. As a result, the non-linear transformation function of such a combined sensor can no longer be described by a single fractional linear function; instead, it is best described by a linear combination of fractional linear functions – i.e., equivalently, by a sum of fractional-linear functions.

It turns out that such sums indeed provide a good description of sensors' non-linearity. For example, for the platinum thermo-resistor – a sensor that is used to implement the international practical temperature scale – the dependence of the resistance $R(t)$ at the temperature t°C to the resistance $R(0)$ at zero temperature can be described by the following formula:

$$\frac{R(t)}{R(0)} = \frac{-2.244 \cdot 10^4}{t + 2.768 \cdot 10^3} + \frac{2.925}{t + 280.063} + \frac{-7.227 \cdot 10^4}{t - 7.947 \cdot 10^3}.$$

This formula holds for all the temperatures from -259°C to $+660^\circ$C with a mean square error 0.7°C, i.e., 0.08% – accurate enough for most practical applications.

For a platinum-Rhodium thermo-pair 30% Rh – 6% Rh, the dependence of the thermo-pair characteristic E on the temperature t can be described as

$$E = \frac{-4.514 \cdot 10^6 + 5.287 \cdot 10^7 \cdot \mathrm{i}}{t - 186.827 - 2.807 \cdot 10^3 \cdot \mathrm{i}} - \frac{4.514 \cdot 10^6 + 5.287 \cdot 10^7 \cdot \mathrm{i}}{t - 186.827 + 2.807 \cdot 10^3 \cdot \mathrm{i}} - \frac{4.86 \cdot 10^8}{t - 1.302 \cdot 10^4}$$

with a mean square accuracy accuracy 0.3% for temperatures from 0°C to $1,600^\circ$C.

In both cases, we use very few parameters and get a very accurate description of non-linearity.

How can we get a general approximation of this type? For that, we can start, e.g., by approximating the desired nonlinear dependence $y(x)$ with a rational (fractional-polynomial) function $y(x) = \dfrac{P(x)}{Q(x)}$, where $P(x) = \sum\limits_{i=0}^{m} a_i \cdot x^i$ and $Q(x) = \sum\limits_{i=0}^{n} b_i \cdot x^i$. Once we know the outputs $y_1, \ldots, y_N$ corresponding to different inputs $x_1, \ldots, x_N$, we can form equations $P(x_k) \approx y_k \cdot Q(x_k)$, i.e., $\sum\limits_{i=0}^{m} a_i \cdot x_k^i \approx \sum\limits_{i=0}^{n} b_i \cdot y_k \cdot x_k^i$. These equations are linear in a_i and b_i; thus, by using, e.g., the Least Square Method, we can find the values a_i and b_i for which the corresponding rational function is the best fit. To be able to represent the original dependence as a sum of fractional linear functions, we should use $m \le n + 1$.

In the general case, all n roots $\alpha_1, \ldots, \alpha_n$ of the n-th order polynomial $Q(x)$ are different (these roots are, in general, complex numbers). It is known that for such polynomials, the rational function $\dfrac{P(x)}{Q(x)}$ can be represented as a sum of a linear term and fractional linear terms $\sum\limits_{i=1}^{n} \dfrac{A_i}{x - \alpha_i}$, where $A_i \stackrel{\text{def}}{=} \dfrac{P(\alpha_i)}{Q'(\alpha_i)}$.

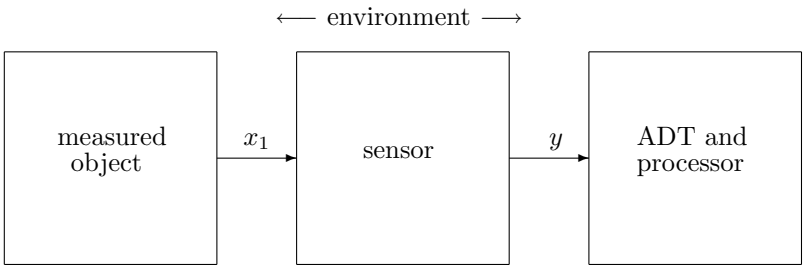
3. How to describe deviations of the sensor signal y from its "average" (non-linear) value $y(x)$

The non-linear calibration function $y(x)$ describes the dependence of the "average" sensor signal y on the value x of the measured quantity. The actual signal y deviates from this "average" value $y(x)$. This deviation contributes to the inaccuracy of the resulting measuring instrument. Therefore, when we design a measuring instrument, it is important to know how different characteristics of this deviation change with design.

These characteristics include characteristics that describe both systematic and random deviations. Let us consider one of such characteristics z_y.

In the ideal case, the sensor's signal depends only on the value of the measured quantity. Realistically, not only the sensor is influenced by the measured object, but the measured object itself is influenced by the sensor, and, as a result, the value of the measured quantity may be slightly changed during the measurement process. In addition, the environment affects (and is affected) both by the sensor and by the measured object.

Let x_i be the parameters that characterize the sensor, the measured object, and the environment:



Let z_{xi}, $1 \le i \le n$, be the characteristics of the sensor, the object, and the environment that describe their deviations – both systematic and random – from their ideal behavior. For example, this list of characteristics may include the standard deviation of the environment noise, the difference between the actual and the nominal resistances of the sensor, etc.

For each sensor design, we know how these characteristics z_{xi} affect the value z_y. In some situations, we have an explicit formula for this dependence; in more complex situations, we have a numerical algorithm that, given z_{xi}, computes z_y. In all these cases, we know how to compute this dependence, i.e., we know the (explicit or implicit) function $z_y = z_y(z_{x1}, \ldots, z_{xn})$ that describes this dependence.

Do we really need to know the values of all the characteristics $z_{x1}, \ldots, z_{xn}$ to find the desired value z_y? In the simplest case, when the entire system (consisting of the sensor, the object, and the environment) is linear, random and systematic errors do not influence each other, so we can treat them separately. In this simple case, if we are interested in knowing the characteristic of a systematic error

z_y, then it is sufficient to only consider characteristics z_{xi} that describe systematic deviations, and vice versa, if we are interested in knowing the characteristic of a random error z_y, then it is sufficient to only consider characteristics z_{xi} that describe random deviations.

Realistically, however, as we have mentioned, to get a an adequate description of a high-accuracy measurement, we must take non-linearity into consideration. Because of the non-linearity, we must take random characteristics z_{xi} into consideration when computing a systematic characteristic z_y, and vice versa, we must take systematic characteristics z_{xi} into consideration when computing a random characteristic z_y.

Let us explain this on the simplest possible case of non-linearity. Indeed, the simplest possible non-linear terms are quadratic terms, where $y(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2$. For a linear sensor for which $y(x) = a_0 + a_1 \cdot x$, the average value $E[y]$ of the output signal is equal to $E[y] = a_0 + a_1 \cdot E[x]$, i.e., it depends only by the average value $E[x]$ of the measured quantity and does not depend on the level of random fluctuations of this quantity. Thus, if the signal fluctuation is the only reason why the sensor signal y deviates from $y(x)$, then for the largest possible values Δ_x and Δ_y of such deviations, we get the dependence $\Delta y = |a_1| \cdot \Delta_x$. However, for a quadratic sensor, we have

$$E[y] = a_0 + a_1 \cdot E[x] + a_2 \cdot E[x^2] = a_0 + a_1 \cdot E[x] + a_2 \cdot (E[x]^2 + V[x]).$$

Thus, the output signal $E[y]$ depends not only on the average value $E[x]$ of the measured quantity, but also on the variance $V[x] = \sigma_x^2$ of this quantity – i.e., on the level of its random fluctuations. In this case, $\Delta_y = |a_1| \cdot \Delta_x + |a_2| \cdot (\Delta_x^2 + \sigma_x^2)$.

In the sensor design, we are interested in how the changes in the characteristics z_{xi} would affect the value of the desired characteristic z_y. In other word, for each of the n characteristics z_{xi}, we are interested in knowing the derivative $\dfrac{dz_y}{dz_{xi}}$. In the simplest case when all the values z_{xi} are independent – in the sense that they do not affect each other – we can find this value by simply differentiating the dependence $z_y = z_y(z_{x1}, \ldots, z_{xn})$ with respect to z_{xi}: $\dfrac{dz_y}{dz_{xi}} = \dfrac{\partial z_y}{\partial z_{xi}}$. In real life, however, as we have mentioned, the sensor, the object, and the environment are inter-related; as a result, all the n parameters are inter-related, and the change in one of them causes others to change as well. As a result, instead of the above simple formula, we have to use a general chain rule formula to describe the desired derivative:

$$\frac{dz_y}{dz_{xi}} = \sum_{j=1}^{n} \frac{\partial z_y}{\partial z_{xj}} \cdot \frac{\partial z_{xj}}{\partial z_{xi}}.$$

In sensor design, it is desirable to know not only the overall value of this derivative, but also which part of it comes from what exactly interaction. In other words, we need to know the values $a_{ij} \stackrel{\text{def}}{=} \dfrac{\partial z_y}{\partial z_{xj}} \cdot \dfrac{\partial z_{xj}}{\partial z_{xi}}$. We will call this tensor (matrix) an *affinor*.

An affinor describes the relation between the different input variables z_{xi} and the output variable z_y. Specifically, the value a_{ij} describes how the change in i-th input quantity z_{xi} affects z_y via its influence on z_{xj}.

We have mentioned that in the linearized case, the characteristics that describe systematic and random deviations do not influence each other, i.e., the corresponding values a_{ij} are 0. We can use this feature of the matrix a_{ij} to simplify computations related to this matrix if we order the characteristics in such a way that systematic characteristics come first and random characteristics come next. In this ordering, the matrix a_{ij} takes the following form:

1	2
3	4

Here, submatrix 1 reflects systematic-systematic interactions, submtarix 4 reflects random-random interactions, and submatrices 2 and 3 reflect random-systematic interaction.

In the linear case, both submtarix 2 and submtarix 3 are 0s. In a more accurate model, there is some dependence between these characteristics, but still, in general, the values a_{ij} that correspond to the interaction between systematic and random characteristics are much smaller than the values a_{ij} that correspond to systematic-systematic and random-random interaction, so submatrices 2 and 3 are still much smaller than submatrices 1 and 4.

The actual values of a_{ij} depend on which characteristic we use. For example, if we are interested in the (absolute) difference $z_y = y - y^{(0)}$ between the actual value y of the output signal and the "ideal" value $y^{(0)} = f(x^{(0)})$ determined the the nominal value $x^{(0)}$ of the input and by the non-linear calibration function $f(x)$, and we use the absolute differences $z_{xi} = x_i - x_i^{(0)}$ as the input parameters, then $a_{ij} = \dfrac{\partial y}{\partial x_j} \cdot \dfrac{\partial x_j}{\partial x_i}$. If we want to describe how the relative deviation of the output $z_y = \dfrac{y - y^{(0)}}{y^{(0)}}$ depends on the absolute deviations of the input, then we get $a_{ij} = \dfrac{1}{y} \cdot \dfrac{\partial y}{\partial x_j} \cdot \dfrac{\partial x_j}{\partial x_i}$. Finally, if we want to know how the relative change in the output depends on the relative changes in the input $z_{xi} = \dfrac{x_i - x_i^{(0)}}{x_i^{(0)}}$, then we get $a_{ij} = \dfrac{x_i}{y} \cdot \dfrac{\partial y}{\partial x_j} \cdot \dfrac{\partial x_j}{\partial x_i}$. For example, if x_j is one of the sensor's parameters, then:

- a small value $a_{jj} < 1$ means that changes in x_j do not lead to strong changes in the output, while
- $a_{jj} > 1$ means that this parameter strongly affects the sensor output, so when designing a sensor, we must impose high requirements on the stability of this parameter x_j.

For random-random interactions, formulas become more complex; e.g. (see [3] for details):

$$a_{ij} = \frac{\dfrac{1}{|y(\underline{x}_i)| \cdot \rho(\underline{x}_i)} \cdot \dfrac{\partial y}{\partial x_j} \cdot \dfrac{\partial x_j}{\partial \underline{x}_i} + \dfrac{1}{|y(\overline{x}_i)| \cdot \rho(\overline{x}_i)} \cdot \dfrac{\partial y}{\partial x_j} \cdot \dfrac{\partial x_j}{\partial \overline{x}_i}}{\dfrac{1}{|\underline{x}_i| \cdot \rho(\underline{x}_i)} + \dfrac{1}{|\overline{x}_i| \cdot \rho(\overline{x}_i)}},$$

where $\underline{x}_i$ and $\overline{x}_i$ are the lower and upper endpoint of the confidence interval for x_i, and $\rho(x_i)$ is the probability density for x_i.

Another example is a dynamic sensor, for which the dependence on the systematic change in frequency takes the form $a_{ii} = \omega \cdot \left(\dfrac{1}{S(\omega)} \cdot \dfrac{\partial S(\omega)}{\partial \omega} + j \cdot \dfrac{\partial \varphi_y}{\partial \omega} \right)$, where $j = \sqrt{-1}$, $S(\omega)$ is the sensor's amplitude-frequency characteristic, and φ_y is the phase of the output signal. For a random change in ω, we similarly have [3]:

$$a_{ii} = \frac{\dfrac{1}{S(\omega)} \cdot \left(\dfrac{1}{\rho(\underline{\omega})} \cdot \left| \dfrac{\partial S(\omega)}{\partial \underline{\omega}} \right| + \dfrac{1}{\rho(\overline{\omega})} \cdot \left| \dfrac{\partial S(\omega)}{\partial \overline{\omega}} \right| \right) + j \cdot \left(\dfrac{1}{\rho(\underline{\omega})} \cdot \left| \dfrac{\partial \varphi_y}{\partial \underline{\omega}} \right| + \dfrac{1}{\rho(\overline{\omega})} \cdot \left| \dfrac{\partial \varphi_y}{\partial \overline{\omega}} \right| \right)}{\dfrac{1}{|\underline{\omega}| \cdot \rho(\underline{\omega})} + \dfrac{1}{\overline{\omega} \cdot \rho(\overline{\omega})}}.$$

4. Combining quantities: towards metric spaces

Sometimes, the sensor directly measures the quantity y in which we are interested. However, often, it is difficult (or even impossible) to directly measure the desired quantity. In such situations, we can measure y *indirectly*: namely, we measure the quantities $y_1, \ldots, y_m$ that are related to y by a known relation $y = f(y_1, \ldots, y_m)$, and then use the measured values $\widetilde{y}_1, \ldots, \widetilde{y}_m$ of the measured quantities to provide an estimate $\widetilde{y} = f(\widetilde{y}_1, \ldots, \widetilde{y}_m)$ for y.

Systematic and random errors of the direct measurements lead to the errors in y. It is therefore desirable to estimate the systematic and random characteristics of the error $\Delta y \stackrel{\text{def}}{=} \widetilde{y} - y$ of the indirect measurement. Thus, when we design a universal sensor model, it is desirable that this model describe not only individual measurements by this sensor, but also how the resulting measurement errors combine with errors from different sensors.

As we have mentioned, we are interested in the analysis and design of cutting-edge high accuracy sensors. For such sensors, the outputs $\widetilde{y}_i$ are very close to the (correspondingly rescaled) actual values y_i of the measured quantities: $\widetilde{y}_i \approx y_i$, so $\Delta y_i \stackrel{\text{def}}{=} \widetilde{y}_i - y_i \ll y_i$. Since the values Δy_i are small, we can expand the dependence $\Delta y = f(\ldots, \widetilde{y}_i, \ldots) - f(\ldots, \widetilde{y}_i - \Delta y_i, \ldots)$ into Taylor series in Δy_i and safely ignore quadratic and higher order terms in this expansion. As a result, we arrive at the linearized formula $\Delta y = \sum_{i=1}^{n} c_i \cdot \Delta y_i$, where $c_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial y_i}$. Since the values Δy_i are determined modulo rescaling

anyway, we can simplify the problem by assuming that each input y_i in already rescaled in such a way that $c_i = 1$, so that $\Delta y = \sum_{i=1}^{m} \Delta y_i$. In this formulation, the question is: we know the metrological characteristics of n signals Δy_i, we must find the corresponding characteristic of their sum Δy.

For example, for the standard deviations σ_i and σ, if all the measurement errors are independent, we get a standard formula $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$. This formula has a natural geometric interpretation: we can describe each possible collection of m corresponding sensors by a vector $(\sigma_1, \ldots, \sigma_m)$, then σ is the Euclidean distance between the starting point 0 of the coordinate system (the point corresponds to ideal perfect sensors) and the point corresponding to the actual sensors.

If we take correlations r_{ij} into account, then $\sigma^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} r_{ij} \cdot \sigma_i \cdot \sigma_j$. This formula can also be interpreted as a distance in Euclidean space – but in non-orthogonal (affine) coordinates. The coefficients r_{ij} in the formula for a metric form a *metric tensor*.

For actual sensors, we may have different values of the correlation on different ranges of the measured quantities. So, we may have different metric tensors at different points – a metric tensor *field*; see, e.g., [3]. In geometry and space-time physics, such spaces are called *Riemann spaces*, so we need Rimann spaces to describe sensor interaction.

In some cases, we even end up with non-quadratic formulas for the metric; in geometric terms, these formula correspond to *Finsler spaces*, a generalization of Riemann spaces.

5. Conclusion

The main objective for our universal sensor model was to provide a general description that would help in designing sensors and sensor-based measuring systems. In this model, to fully describe a sensor, we must describe both the corresponding non-linear calibration function $y(x)$ and the affinor a_{ij} that describes how the deviation of the sensor's output signal y from the "average" output $y(x)$ depends on the parameters of the sensor, of the measured object, and of the environment.

For the existing sensors, the calibration characteristics $y(x)$ is the most important part of this description. In most cases, to provide a metrological analysis of a measuring system that uses such a sensor, it is sufficient to know this characteristics and the overall metrological characteristics that limit possible systematic and random deviations of the actual output signal from $y(x)$. For such systems, it is sufficient to know the overall values of these deviations, and there is no need to know the values a_{ij} that describe which part of this deviation is caused by which factor.

However, in some cases, we need to use a sensor under the conditions which are different from those described in the sensor's documentation. In such cases, to predict the sensor's accuracy under these new conditions, we must use the complete model including the values a_{ij}.

We implemented our methodology in MathCAD, and we tested it on the examples of an acceleration sensor and a sensor for detecting electromagnetic field. For each of the selected sensors, there exists a detailed and reasonably accurate computer-based simulation model. So, to test our methodology, we compared the results of our computations with the simulation model. Our results turned out to be in good accordance with this simulation model.

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