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Fast Algorithm for Computing the Upper Endpoint of Sample Variance for Interval Data: Case of Sufficiently Accurate Measurements

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Abstract

When we have n results $x_1, \ldots, x_n$ of repeated measurement of the same quantity, the traditional statistical approach usually starts with computing their sample average E and their sample variance V. Often, due to the inevitable measurement uncertainty, we do not know the exact values of the quantities, we only know the intervals $\mathbf{x}_i$ of possible values of x_i. In such situations, for different possible values $x_i \in \mathbf{x}_i$, we get different values of the variance. We must therefore find the range $\mathbf{V}$ of possible values of V. It is known that in general, this problem is NP-hard. For the case when the measurements are sufficiently accurate (in some precise sense), it is known that we can compute the interval $\mathbf{V}$ in quadratic time $O(n^2)$. In this paper, we describe a new algorithm for computing $\mathbf{V}$ that requires time $O(n \cdot \log(n))$ (which is much faster than $O(n^2)$).

1 Introduction

Computing sample variance is important. When we have n results $x_1, \ldots, x_n$ of repeated measurement of the same quantity (at different points, or at different moments of time), the traditional statistical approach usually starts with computing their sample average

$$E = \frac{x_1 + \ldots + x_n}{n} \tag{1}$$

and their (sample) variance

$$V = \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^2 = \frac{x_1^2 + \ldots + x_n^2}{n} - E^2 \tag{2}$$

(or, equivalently, the sample standard deviation $\sigma = \sqrt{V}$); see, e.g., [14].

Problem: computing sample variance under interval uncertainty. Measurements are never 100% accurate, so in reality, the actual value x_i of the i-th measured quantity can differ from the measurement result $\widetilde{x}_i$. Often, we only know the intervals $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of possible values of x_i. In this case, for different possible values $x_i \in \mathbf{x}_i$, we get different values of E and V. In such situations, it is desirable to find the ranges of possible values of E and V.

Since both E and V are continuous functions of n variables $x_1 \in \mathbf{x}_1, \ldots, x_n \in \mathbf{x}_n$, the range of each of these functions on the box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$ is an interval. So, in such situations, our objective is to compute the intervals $\mathbf{E}$ and $\mathbf{V}$ of possible values of E and V:

$$\mathbf{E} = [\underline{E}, \overline{E}] \stackrel{\text{def}}{=} \left\{ \frac{x_1 + \ldots + x_n}{n} \,|\, x_1 \in \mathbf{x}_1 \,\&\, \ldots \,\&\, x_n \in \mathbf{x}_n \right\};$$

$$\mathbf{V} = [\underline{V}, \overline{V}] \stackrel{\text{def}}{=} \left\{ \frac{x_1^2 + \ldots + x_n^2}{n} - E^2 \,|\, x_1 \in \mathbf{x}_1 \,\&\, \ldots \,\&\, x_n \in \mathbf{x}_n \right\}.$$

The practical importance of the problem of computing sample variance under interval uncertainty was emphasized, e.g., in [7, 8] on the example of processing geophysical data and in [3] on the example of processing environmental data.

What is known. For E, the straightforward interval computations [9, 10, 11, 13] lead to the exact range:

$$\mathbf{E} = \frac{\mathbf{x}_1 + \ldots + \mathbf{x}_n}{n}, \text{ i.e., } \underline{E} = \frac{\underline{x}_1 + \ldots + \underline{x}_n}{n}, \text{ and } \overline{E} = \frac{\overline{x}_1 + \ldots + \overline{x}_n}{n}.$$

For V, straightforward interval computations lead to an excess width, and moreover, the problem of computing the range $\mathbf{V}$ is, in general, NP-hard [5] (this result originally appeared in [4]).

In [5], it was shown that we can compute the lower endpoint $\underline{V}$ of the desired range in quadratic time $O(n^2)$. For the upper bound $\overline{V}$ of the desired range, in [5], it was proven that we can compute it in quadratic time if the measurements are sufficiently accurate in the sense that different measurement results can still be distinguished from each other – i.e., when intervals $\mathbf{x}_i$ corresponding to different measurement do not intersect.

Moreover, it was proven that a quadratic time algorithm is possible not only when the original intervals $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ do not intersect, but also in a more general case when the "narrowed" intervals $[\widetilde{x}_i - \Delta_i/n, \widetilde{x}_i + \Delta_i/n]$ do not intersect. In fact, this quadratic time algorithm even works in the case when for some integer $c < n$, no sub-collection of greater than c narrowed intervals of $\mathbf{x}_i$ has a common intersection [5].

For large n, n^2 is still a lot of time; it is therefore desirable to speed up the computations.

Many applications fall into one of two cases: (i) measurement error is constant, either in absolute or as a fraction of x (e.g., if we the same physical instrument to get all the measurement results), (ii) the real line is partitioned into pre-assigned bins, and one learns the bin into which each observation falls (e.g., income brackets). In either of these cases, we can compute the range of variance in time $O(n \cdot \log(n))$ [1, 15, 16]. However, if we use different measuring instruments, then the measurement error is no longer constant, so we cannot directly apply the algorithms developed for these special cases.

In [6], it was shown that in the general case of "sufficiently accurate" measurements, we can compute $\underline{V}$ in time $O(n \cdot \log(n))$ – which is much faster than $O(n^2)$. A natural question is: can we similarly speed up the computation of $\overline{V}$?

What we are planning to do. In this paper, we describe a new algorithm for computing $\mathbf{V}$ that requires time $O(n \cdot \log(n))$ in the case when for some integer c, no sub-collection of more than c narrowed intervals of $\mathbf{x}_i$ has a common intersection.

2 Previously Known Quadratic-Time Algorithm: A Brief Reminder

The *input* to our problem is a finite list of intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$. There are two standard ways to represent an interval in the computer:

- first, by describing two real numbers $\underline{x}_i$ and $\overline{x}_i$;
- second, by describing the midpoint $\widetilde{x}_i \overset{\text{def}}{=} (\underline{x}_i + \overline{x}_i)/2$ and the half-width $\Delta_i \overset{\text{def}}{=} (\overline{x}_i - \underline{x}_i)/2$ of this interval.

Once we know the midpoint and the half-width, we can reconstruct the endpoints of the interval as $\underline{x}_i = \widetilde{x}_i - \Delta_i$ and $\overline{x}_i = \widetilde{x}_i + \Delta_i$.

We have already mentioned that we consider the case when for some given integer c, no sub-collection of more than c narrowed intervals $[\widetilde{x}_i - \Delta_i/n, \widetilde{x}_i + \Delta_i/n]$ has a common intersection.

For this situation, the following quadratic-time algorithm for computing $\overline{V}$ was described in [5]:

- First, we sort all $2n$ endpoints of the narrowed intervals $\widetilde{x}_i - \Delta_i/n$ and $\widetilde{x}_i + \Delta_i/n$ into a sequence $x_{(1)} \le x_{(2)} \le \ldots \le x_{(2n)}$. This enables us to divide the real line into $2n+1$ zones $[x_{(k)}, x_{(k+1)}]$, where we denote $x_{(0)} \overset{\text{def}}{=} -\infty$ and $x_{(2n+1)} \overset{\text{def}}{=} +\infty$.

- Second, we compute $\underline{E}$ and $\overline{E}$ and pick all zones $[x_{(k)}, x_{(k+1)}]$ that intersect with $[\underline{E}, \overline{E}]$.
- For each of remaining zones $[x_{(k)}, x_{(k+1)}]$, for each i from 1 to n, we pick the following value of x_i:
  - if $x_{(k+1)} \le \widetilde{x}_i - \Delta_i/n$, then we pick $x_i = \overline{x}_i$;
  - if $\widetilde{x}_i + \Delta_i/n \le x_{(k)}$, then we pick $x_i = \underline{x}_i$;
  - for all other i, we consider both possible values $x_i = \overline{x}_i$ and $x_i = \underline{x}_i$.
- As a result, we get one or several sequences of x_i. For each of these sequences, we check whether the average E of the selected values $x_1, \ldots, x_n$ is indeed within this zone, and if it is, compute the variance by using the formula (2).
- Finally, we return the largest of the computed variances as $\overline{V}$.

The proof that this algorithm requires only $O(n^2)$ time is based on the fact that for each zone, there are at most c indices i for which the i-th narrowed interval $[\widetilde{x}_i - \Delta_i/n, \widetilde{x}_i + \Delta_i/n]$ contains this zone and therefore, at most c indices for which we had to consider both choices $\underline{x}_i$ and $\overline{x}_i$. As a result, for each zone, there are at most 2^c corresponding sequences x_i.

3 New Algorithm

1°. Let us first sort the lower endpoints $\widetilde{x}_i - \Delta_i/n$ of the narrowed intervals into an increasing sequence. Without losing generality, we can therefore assume that these lower endpoints are ordered in increasing order:

$$\widetilde{x}_1 - \Delta_1/n \le \widetilde{x}_1 - \Delta_2/n \le \ldots$$

It is well known that sorting requires time $O(n \cdot \log(n))$; see, e.g., [2].

2°. Then, similar to the previously known algorithm, we sort *all* the endpoints of the narrowed intervals into a sequence $x_{(1)} \le x_{(2)} \le \ldots \le x_{(k)} \le \ldots \le x_{(2n)}$. Sorting means that for every i, we know which element $k^-(i)$ represents the lower endpoint of the i-th narrowed interval and which element $k^+(i)$ represents the upper endpoint of the i-th narrowed interval.

This sorting also requires $O(n \cdot \log(n))$ steps.

3°. On the third stage, we produce, for each of the resulting zones $[x_{(k)}, x_{(k+1)}]$, the set S_k of all the indices i for which the i-th narrowed interval

$$[\widetilde{x}_i - \Delta_i/n, \widetilde{x}_i + \Delta_i/n]$$

contains this zone.

As we have mentioned, for each i, we know the value $k = k^-(i)$ for which $\widetilde{x}_i - \Delta_i/n = x_{(k)}$. So, for each i, we place i into the set $S_{k^-(i)}$ corresponding to the zone $[x_{(k^-(i))}, x_{(k^-(i)+1)}]$, into the set corresponding to the next zone, etc., until we reach the zone for which the upper endpoint is exactly $\widetilde{x}_i + \Delta_i/n$.

Here, we need one computational step for each new entry of i into the set corresponding to a new zone. Therefore, filling in all these sets requires as many steps as there are items in all these sets. For each of $2n+1$ zones, as we have mentioned, there are no more than c items in the corresponding set; therefore, overall, all the sets contain no more than $c \cdot (2n+1) = O(n)$ steps. Thus, this stage requires $O(n)$ time.

4°. On the fourth stage, for all integers p from 0 to n, we compute the sums

$$E_p \stackrel{\text{def}}{=} \frac{1}{n} \cdot \sum_{i=1}^{p} \underline{x}_i + \frac{1}{n} \cdot \sum_{i=p+1}^{n} \overline{x}_i;$$

$$M_p \stackrel{\text{def}}{=} \frac{1}{n} \cdot \sum_{i=1}^{p} (\underline{x}_i)^2 + \frac{1}{n} \cdot \sum_{i=p+1}^{n} (\overline{x}_i)^2.$$

We compute these values sequentially. Once we know E_p and M_p, we can compute E_{p+1} and M_{p+1} as $E_{p+1} = E_p + \underline{x}_{p+1} - \overline{x}_{p+1}$ and $M_{p+1} = M_p + (\underline{x}_{p+1})^2 - (\overline{x}_{p+1})^2$.

Transition from E_p and M_p to E_{p+1} and M_{p+1} requires a constant number of computational steps; so overall, we need $O(n)$ steps to compute all the values E_p and M_p.

5°. Finally, for each zone k, we compute the corresponding values of the variance. For that, we first find the smallest index i for which $x_{(k+1)} \le \widetilde{x}_i - \Delta_i/n$. We will denote this value i by $p(k)$.

Since the values $\widetilde{x}_i - \Delta_i/n$ are sorted, we can find this i by using bisection [2]. It is known that bisection requires $O(\log(n))$ steps, so finding such $p(k)$ for all $2n+1$ zones requires $O(n \cdot \log(n))$ steps.

Once $i \ge p(k)$, then $\widetilde{x}_i - \Delta_i/n \ge \widetilde{x}_{p(k)} - \Delta_{p(k)}/n \ge x_{(k+1)}$. So, in accordance with the above justification for the quadratic-time algorithm, we should select $x_i = \overline{x}_i$, as in the sums $E_{p(k)}$ and $M_{p(k)}$.

In accordance with the same justification, the only values $i < p(k)$ for which we may also select $x_i = \overline{x}_i$ are the values for which the i-th narrowed intervals contains this zone. These values are listed in the set S_k of no more than c such intervals. So, to find all possible values of V, we can do the following.

We then consider all subsets $s \subseteq S_k$ of the set S_k; there are no more than 2^c such subsets. For each subset s, we replace, in $E_{p(k)}$ and $M_{p(k)}$, values $\underline{x}_i$ and $(\underline{x}_i)^2$ corresponding to all $i \in s$, with, correspondingly, $\overline{x}_i$ and $(\overline{x}_i)^2$.

Each replacement means subtracting no more than c terms and then adding no more than c terms, so each computation requires no more than $2c$ steps.

Once we have E and V corresponding to the subset s, we can check whether E belongs to the analyzed zone and, if yes, compute $V = M - E^2$.

For each subset, we need no more than $2c + 2$ computations, so for all no more than 2^c subsets, we need no more than $(2c + 2) \cdot 2^c$ computations. For a fixed c, this value does not depend on n; in other words, for each zone, we need $O(1)$ steps.

To perform this computation for all $2n + 1$ zones, we need $(2n + 1) \cdot O(1) = O(n)$ steps.

6°. Finally, we find the largest of the resulting values V – this will be the desired value $\overline{V}$.

Finding the largest of $O(n)$ values requires $O(n)$ steps.

Overall, we need

$$O(n \cdot \log(n)) + O(n \cdot \log(n)) + O(n) + O(n) + O(n \cdot \log(n)) + O(n) = O(n \cdot \log(n))$$

steps. Thus, we have proven that our algorithm computes $\overline{V}$ in $O(n \cdot \log(n))$ steps.

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References

[1] F. A. Cowell, "Grouping Bounds for Inequality Measures under Alternative Informational Assumptions," *Journal of Econometrics*, 1991, Vol. 48, pp. 1–14.

[2] Th. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, MIT Press, Cambridge, MA, 2001.

[3] S. Ferson, *RAMAS Risk Calc 4.0: Risk Assessment with Uncertain Numbers*, CRC Press, Boca Raton, Florida, 2002.

[4] S. Ferson, L. Ginzburg, V. Kreinovich, L. Longpré, and M. Aviles, "Computing Variance for Interval Data is NP-Hard", *ACM SIGACT News*, 2002, Vol. 33, pp. 108–118.

[5] S. Ferson, L. Ginzburg, V. Kreinovich, L. Longpré, and M. Aviles, "Exact Bounds on Finite Populations of Interval Data", *Reliable Computing*, 2005, Vol. 11, No. 3, pp. 207–233.

[6] L. Granvilliers, V. Kreinovich, and N. Müller, "Novel Approaches to Numerical Software with Result Verification", In: R. Alt, A. Frommer, R. B. Kearfott, and W. Luther (eds.), *Numerical Software with Result Verification*, Springer Lectures Notes in Computer Science, 2004, Vol. 2991, pp. 274–305.

[7] P. Nivlet, F. Fournier, and J. Royer, "A new methodology to account for uncertainties in 4-D seismic interpretation", *Proceedings of the 71st Annual International Meeting of the Society of Exploratory Geophysics SEG'2001*, San Antonio, Texas, September 9–14, 2001, pp. 1644–1647.

[8] P. Nivlet, F. Fournier, and J. Royer, "Propagating interval uncertainties in supervised pattern recognition for reservoir characterization", Proceedings of the 2001 Society of Petroleum Engineers Annual Conference SPE'2001, New Orleans, Louisiana, September 30–October 3, 2001, paper SPE-71327.

[9] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis: With Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer, London, 2001.

[10] R. B. Kearfott, *Rigorous Global Search: Continuous Problems*, Kluwer, Dordrecht, 1996.

[11] R. B. Kearfott and V. Kreinovich (eds.), *Applications of Interval Computations*, Kluwer, Dordrecht (1996)

[12] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1966.

[13] R. E. Moore, *Methods and Applications of Interval Analysis*, SIAM, Philadelphia, 1979.

[14] S. Rabinovich, *Measurement Errors: Theory and Practice*, American Institute of Physics, New York, 1993.

[15] S. A. Starks, V. Kreinovich, L. Longpré, M. Ceberio, G. Xiang, R. Araiza, J. Beck, R. Kandathi, A. Nayak, and R. Torres, "Towards combining probabilistic and interval uncertainty in engineering calculations", *Proceedings of the Workshop on Reliable Engineering Computing*, Savannah, Georgia, September 15–17, 2004, pp. 193–213.

[16] G. Xiang, S. A. Starks, V. Kreinovich, and L. Longpré, "New Algorithms for Statistical Analysis of Interval Data", *Proceedings of the Workshop on State-of-the-Art in Scientific Computing PARA'04*, Lyngby, Denmark, June 20–23, 2004, Vol. 1, pp. 123–129.