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OPEN-ENDED CONFIGURATIONS OF RADIO TELESCOPES: A GEOMETRICAL ANALYSIS

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Abstract. *The quality of radio astronomical images drastically depends on where we place the radio telescopes. During the design of the Very Large Array, it was empirically shown that the power law design, in which n-th antenna is placed at a distance n^α from the center, leads to the best image quality. In this paper, we provide a theoretical justification for this empirical fact.*

Why radio telescopes. According to modern physics, most elementary particles are photons, i.e., quanta of electromagnetic field. Not surprisingly, the main information about the extra-terrestrial objects comes from observing electromagnetic waves on different wavelengths. The Earth's atmosphere absorbs most of these waves, so there are only a few windows of observability.

The most well known window corresponds to visible light. The corresponding optical telescopes indeed bring a lot of astronomical information. However, this information is often not sufficient: many celestial objects are not bright in visible light. To complement this information, astronomers use radio telescopes, devices that use the second observability window of radio waves.

Why configurations of radio telescopes. According to optics, when we use a telescope of diameter d to make observations on wavelength λ, we can determine the location of the radiation sources with an error $\approx \lambda/d$. To increase the observation quality, we must decrease this error, and thus, we must increase the diameter d. For radio telescopes, from the technical viewpoint, the largest possible diameter is ≈ 100 m. Thus, if we want to further

increase the diameter d, we cannot simply design a *single* telescope of larger diameter. Instead, we must build a *configuration* of radio telescopes.

Why open-ended configurations of radio telescopes. In principle, the more telescopes we add, the more the noise decreases and therefore, the better the quality of the resulting images. However, telescopes are very costly devices, and these financial considerations severely limit our design abilities.

Sometimes, when a configuration is built, it turns out that for some observations, adding one or several appropriately placed radio telescopes would drastically increase the amount of astrophysical information that can be extracted from the resulting images. In this case, it makes sense to add a few telescopes to the existing configuration. In view of this possibility, many configurations are designed as *open-ended*, when it is always possible to add one or several telescopes.

We need optimal configurations. The image quality drastically depends on where exactly we place the telescopes. Depending on where we place them, we can get almost an order of magnitude improvement or decrease in image quality. We want to extract as much information from our investment in a radio telescope configuration as possible. Since telescopes are expensive, it makes sense to spend as much computational time and resources as necessary and find the truly optimal design.

Empirical analysis and the Very Large Array. The problem of optimally designing a configuration of radio telescopes was first handled during the design of the Very Large Array [Chow 1972], [Napier et al. 1983], F [Thompson et al. 1980], [Thompson et al. 2001]. First, experimental and theoretical analysis showed that in the optimal open-ended design, radio telescopes are placed along several semi-lines with a common origin. If we select n lines, then each line should form an angle of $2\pi/n$ with the neighboring one. For example, if we select 3 lines, they form a Y-shape configuration; if we select $n = 4$, we get a cross-shaped configuration, etc.

For each such configuration, it is important to describe where exactly the antennas should be placed on each line. When we have a large number of telescopes, then we can describe the desired placement by describing, for each n, the distance r_n between n-th telescope and the center.

Empirical comparison of several possible placement functions showed that for several different criteria, a power law $r_n = C \cdot n^\alpha$ leads to the best image quality [Napier et al. 1983], [Thompson et al. 1980], [Thompson

et al. 2001]. Because of this analysis, this placement was selected for the design of VLA [Napier et al. 1983], [Thompson et al. 1980], [Thompson et al. 2001].

For some criteria, it was even possible to theoretically prove that this placement is optimal [Chow 1972] – but, alas, not for the value α used in the actual VLA design. In this paper, we provide a theoretical proof that the power law placement is indeed optimal under any optimality criterion that satisfies certain reasonable properties.

Towards mathematical formulation of the problem: general idea. We want to find an "optimal" configuration r_n.

It is difficult to formulate exactly what "optimal" means because possible numerical criteria like quality of the observed images depend on what exactly source we observe. So, instead of trying to come up with an exact formalization of what "optimal" means, we will try to find geometric constraints that an optimal configuration should satisfy, and show that these constraints lead to the desired power law.

Scale-invariance. Our first comment about the geometry of optimal configurations comes from the fact that the equations that describe observation by radiotelescopes – i.e., equations of optics, and more generally, Maxwell equations that describe electromagnetic fields – are scale-invariant, i.e., they do not change when we change the unit for measuring length (and change related units accordingly).

Thus, if a configuration r_n is optimal, then for every scaling factor $c > 0$, the scaled version $r'_n = C \cdot r_n$ should also be optimal. So, instead of a *single* "optimal" configurations r_n, we should be looking for a *family* $\{C \cdot r_n\}_C$ of optimal configurations.

Open-endedness. The second comment is that we are looking for an *open-ended* configuration. This means that we should be able to add extra antennas to the original configuration, and still keep it optimal. In particular, this means, e.g., that it should be possible to built an additional antenna between every two consequent antennas of the original configuration, and still get the optimal configuration. It should also be possible, for every integer $k > 0$, to build k extra antennas between each two consequent antennas of the original configuration, and still get the optimal configuration.

How can we describe this requirement in formal terms? If we insert a new antenna between every two consequent antennas of the original configuration, then the antenna that was No. 1 becomes No. 2, the antenna that was

No. 2 becomes No. 4, etc., and in general, the antenna No. n in the original configuration becomes antenna No. $2n$ in the new configuration. In general, if we insert k new antennas between every two consequent antennas of the original configuration, then the antennas that was No. n in the original configuration becomes antenna No. $(k+1) \cdot n$ in the new configuration.

Let r_n be the optimal configuration. After inserting new antennas, the configuration must remain optimal. Since all the optimal configurations have the form $C \cdot r_n$, the new configuration must be of the type

$$r'_n = C \cdot r_n, \tag{1}$$

where r'_n is the distance of the n-th antenna in the new configuration from the center, and C is a constant that does not depend on n (it can only depend on k). Since the n-th antenna from the old configuration becomes antenna No. $(k+1) \cdot n$ in the new configuration, the corresponding distances r_n and $r'_{(k+1) \cdot n}$ must coincide: $r'_{(k+1) \cdot n} = r_n$. Substituting the expression (1) for r'_n and explicitly mentioning the possible dependence of C on k, we thus conclude that

$$C_k \cdot r_{(k+1) \cdot n} = r_n \tag{2}$$

for all n and k.

Now, we are ready for the main result:

Theorem. *If an increasing sequence $0 < r_1 < r_2 < \ldots < r_n < \ldots$ satisfies the equation (2) for all n and k, then $r_n = C \cdot n^\alpha$ for some real numbers C and α.*

In other words, the above conditions of scale-invariance and open-endedness imply that the optimal configuration should be of the desired type $r_n = C \cdot n^\alpha$. Thus, out theorem provides a theoretical justification for the empirical discovery that underlies the VLA design.

Proof. Let us simplify the equation (2). First, if we divide both sides by the coefficient C_k, we conclude that

$$r_{(k+1) \cdot n} = c_k \cdot r_k, \tag{3}$$

where we denoted $c_k \stackrel{\text{def}}{=} 1/C_k$.

To simplify this equation even further, we denote $k + 1$ by m. In terms of m, the coefficient c_k becomes c_{m-1}. For simplicity, we will denote $p_m \stackrel{\text{def}}{=} c_{m-1}$. In these new terms, the equation (2) takes the following form:

$$r_{m \cdot n} = p_m \cdot r_n. \tag{3}$$

Here, p_m is a ratio of two positive numbers and thus, is itself positive.

Substituting $n = 1$ into the formula (3), we get

$$r_m = p_m \cdot r_1. \tag{4}$$

Thus, if we know p_n, we can determine r_n as $\mathrm{const} \cdot p_n$. Since the sequence r_n is increasing, this means that the sequence $p_n = r_n/r_1$ is increasing too.

Substituting the expression (4) into the equation (3), we conclude that

$$p_{m \cdot n} \cdot r_1 = p_m \cdot p_n \cdot r_1, \tag{5}$$

i.e., that

$$p_{m \cdot n} = p_m \cdot p_n. \tag{6}$$

Since the values p_m are positive for all m, we can take logarithms of both sides and conclude that

$$P_{m \cdot n} = P_m + P_n, \tag{7}$$

where we denoted $P_n \stackrel{\text{def}}{=} \ln(p_n)$. Since the values p_n are increasing, their logarithms P_n also form an increasing sequence.

Functions satisfying equation (7) are called *totally additive number theoretic* functions; see, e.g., [Aczel et al. 1991]. It is known (see, e.g., [Aczel et al. 1991], [Erdős 1946]) that every monotonic totally additive number theoretic function has the form $P_n = \alpha \cdot \ln(n)$. Thus, $P_n = \alpha \cdot \ln(n)$. Since $P_n = \ln(p_n)$, we conclude that

$$p_n = \exp(P_n) = \exp(\alpha \cdot \ln(n)) = n^\alpha. \tag{8}$$

Using equality (4), we can now conclude that $r_n = C \cdot n^\alpha$, with $C = r_1$. The theorem is proven.

Open problem. In the above text, we used geometrical analysis to explain the empricial formula for the distance r_n between n-th antenna and the center of the configuration. It is desirable to be able to explain not only the distances, but also the angles: specifically, to explain why placing all antennas along three central rays turned out to be an optimal configuration.

Alternatively, maybe some other geometric configuration will turn out to be optimal?

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