3-1-2001

An Even More Realistic (Non-Associative) Interval Logic and its Relation to Psychology of Human Reasoning

I. R. Goodman
Raul Trejo
Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu
Jesus Martinez
Reginaldo Gonzalez

Follow this and additional works at: http://digitalcommons.utep.edu/cs_techrep

Part of the Computer Engineering Commons

Comments:
UTEP-CS-00-32b.

Recommended Citation
http://digitalcommons.utep.edu/cs_techrep/494

This Article is brought to you for free and open access by the Department of Computer Science at DigitalCommons@UTEP. It has been accepted for inclusion in Departmental Technical Reports (CS) by an authorized administrator of DigitalCommons@UTEP. For more information, please contact lweber@utep.edu.
An Even More Realistic (Non-Associative) Logic
And Its Relation to Psychology of Human Reasoning

I. R. Goodman
Code D42215, Space & Naval Warfare Systems Center
San Diego, CA 92152, USA
goodman@spawar.navy.mil

Vladik Kreinovich, Raúl Trejo,
Jesus Martinez, and Reginaldo Gonzalez
Computer Science, Univ. of Texas at El Paso
El Paso, TX 79968, USA
vladik@cs.utep.edu

Abstract

If we know the degrees of certainty (subjective probabilities) \( p(S_1) \) and \( p(S_2) \) in two statements \( S_1 \) and \( S_2 \), then possible values of \( p(S_1 & S_2) \) form an interval \( p = [\max(p_1 + p_2 - 1, 0), \min(p_1, p_2)] \). As a numerical estimate, it is natural to use a midpoint \( p \) of this interval; this midpoint is a mathematical expectation of \( p(S_1 & S_2) \) over a uniform (second order) distribution on all possible probability distributions.

This midpoint operation \& is not associative. We show that the upper bound on the difference \( 3 \) \( 45 \) \( 768 \) \( 3 \) \( 59 : \) \( 76 \) is 1/9, so if the size of the corresponding granules is \( \leq 10 \), we will not notice this associativity. This may explain the famous \( ?A@CB \) law, according to which we use no more than 9 granules.

1. In Expert Systems, We Need Estimates for the Degree of Certainty of \( S_1 & S_2 \) and \( S_1 \vee S_2 \)

In many areas (medicine, geophysics, military decision-making, etc.), top quality experts make good decisions, but they cannot handle all situations. It is therefore desirable to incorporate their knowledge into a decision-making computer system.

Experts describe their knowledge by statements \( S_1, \ldots, S_n \) (e.g., by if-then rules). Experts are often not 100% sure about these statements \( S_i \); this uncertainty is described by the subjective probabilities \( p_i \) (degrees of belief, etc.) which experts assign to their statements. The conclusion \( C \) of an expert system normally depends on several statements \( S_i \). For example, if we can deduce \( C \) either from \( S_2 \) and \( S_3 \), or from \( S_4 \), then the validity of \( C \) is equivalent to the validity of a Boolean combination \( (S_2 \& S_3) \vee S_4 \). So, to estimate the reliability \( p(C) \) of the conclusion, we must estimate the probability of Boolean combinations. In this paper, we consider the simplest possible Boolean combinations are \( S_1 \& S_2 \) and \( S_1 \vee S_2 \).

In general, the probability \( p(S_1 & S_2) \) of a Boolean combination can take different values depending on whether \( S_1 \) and \( S_2 \) are independent or correlated. So, to get the precise estimates of probabilities of all possible conclusions, we must know not only the probabilities \( p(S_i) \) of individual statements, but also the probabilities of all possible Boolean combinations. To get all such probabilities, it is sufficient to describe \( 2^n \) probabilities of the combinations \( E_1^+ \& \cdots \& E_n^+ \), where \( \varepsilon_i \in \{+, -, \} \), \( E^+ \) means \( E \), and \( E^- \) means \( \neg E \). The only condition on these probabilities is that their sum should add up to 1, so we need to describe \( 2^n - 1 \) different values. A typical knowledge base may contain hundreds of statements; in this case, the value \( 2^n - 1 \) is astronomically large. We cannot ask experts about all \( 2^n \) such combinations, so in many cases, we must estimate \( p(S_1 & S_2) \) or \( p(S_1 \vee S_2) \) based only on the values \( p_1 = p(S_1) \) and \( p_2 = p(S_2) \).

2. Interval Estimates Are Possible, But Sometimes, Numerical Estimates Are Needed

It is known that for given \( p_1 = p(S_1) \) and \( p_2 = p(S_2) \):

- possible values of \( p(S_1 \& S_2) \) form an interval \( p = [p^-, p^+] \), where \( p^- = \max(p_1 + p_2 - 1, 0) \) and
\[ p^+ = \min(p_1, p_2); \]

- possible values of \( p(S_1 \lor S_2) \) form an interval \( p = [p^-, p^+] \), where \( p^- = \max(p_1, p_2) \) and \( p^+ = \min(p_1 + p_2, 1) \).

(see, e.g., a survey [12] and references therein).

So, in principle, we can use such interval estimates and get an interval \( p(C) \) of possible values of \( p(C) \). Sometimes, this idea leads to meaningful estimates, but often, it leads to a useless \( p(C) = [0, 1] \) (see, e.g., [12, 13]). In such situations, it is reasonable, instead of using the entire interval \( p \), to select a point within this interval as a reasonable estimate for \( p(S_1 \lor S_2) \) (or, correspondingly, for \( p(S_1 \lor S_2) \)).

### 3. Natural Idea: Selecting a Midpoint as the Desired Estimate

Since the only information we have, say, about the unknown probability \( p(S_1 \land S_2) \) is that it belongs to the interval \( [p^-, p^+] \), it is natural to select a midpoint of this interval as the desired estimate. In other words, if we know the probabilities \( p_1 \) and \( p_2 \) of the statements \( S_1 \) and \( S_2 \), then, as estimates for \( p(S_1 \land S_2) \) and \( p(S_1 \lor S_2) \), we can take the values \( p_1 \land p_2 \) and \( p_1 \lor p_2 \), where

\[
\begin{align*}
p_1 \land p_2 & = \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2); \\
p_1 \lor p_2 & = \frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1). 
\end{align*}
\]

This midpoint selection is not only natural from a common sense viewpoint; it also has a deeper justification. Namely, in accordance of our above discussion, for \( n = 2 \) statements \( S_1 \) and \( S_2 \), to describe the probabilities of all possible Boolean combinations, we need to describe \( 2^2 = 4 \) probabilities \( x_1 = p(S_1 \land S_2), x_2 = p(S_1 \land \neg S_2), x_3 = p(\neg S_1 \land S_2), \) and \( x_4 = p(\neg S_1 \land \neg S_2) \); these probabilities should add up to 1: \( x_1 + x_2 + x_3 + x_4 = 1 \). Thus, each probability distribution can be represented as a point \((x_1, \ldots, x_4)\) in a 3-D simplex

\[
\mathcal{S} = \{(x_1, x_2, x_3, x_4) \mid x_1 \geq 0 \land x_1 + \ldots + x_4 = 1\}.
\]

We know the values of \( p_1 = p(S_1) = x_1 + x_2 \) and \( p_2 = p(S_2) = x_1 + x_3 \), and we are interested in the values of \( p(S_1 \land S_2) = x_1 \) and \( p(S_1 \lor S_2) = x_1 + x_2 + x_3 \). It is natural to assume that \( a \) priori, all probability distributions (i.e., all points in a simplex \( \mathcal{S} \)) are “equally possible”, i.e., that there is a uniform distribution (“second-order probability”) on this set of probability distributions. Then, as a natural estimate for the probability \( p(S_1 \land S_2) \) of \( S_1 \land S_2 \), we can take the conditional mathematical expectation of this probability under the condition that the values \( p(S_1) = p_1 \) and \( p(S_2) = p_2 \):

\[
E(p(S_1 \land S_2) \mid p(S_1) = p_1 \land p(S_2) = p_2) = \\
\frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1).
\]

### 4. Problem: Midpoint Operations Are Not Associative

Any “and” operation \( p_1 \land p_2 \) enables us to produce an estimate for \( P(S_1 \land S_2) \) provided that we know estimates \( p_1 \) for \( p(S_1) \) and \( p_2 \) for \( p(S_2) \). If we are interested in estimating the degree of belief in a conjunction of three statements \( S_1 \land S_2 \land S_3 \), then we can use the same operation twice:

- first, we apply the “and” operation to \( p_1 \) and \( p_2 \) and get an estimate \( p_1 \land p_2 \) for the probability of \( S_1 \land S_2 \);
- then, we apply the “and” operation to this estimate \( p_1 \land p_2 \) and \( p_3 \), and get an estimate \( (p_1 \land p_2) \land p_3 \) for the probability of \( (S_1 \land S_2) \land S_3 \).

Alternatively, we can get start by combining \( S_2 \) and \( S_3 \), and get an estimate \( p_1 \land (p_2 \land p_3) \) for the same probability \( p(S_1 \land S_2 \land S_3) \). Intuitively, we would expect these two estimates to coincide: \( (p_1 \land p_2) \land p_3 = p_1 \land (p_2 \land p_3) \), i.e., in algebraic terms, we expect the operation & to be associative. Unfortunately, midpoint operations are not associative [2]: e.g., \((0.4 \land 0.6) \land 0.8 = 0.2 \land 0.8 = 0.1\), while \(0.4 \land (0.6 \land 0.8) = 0.4 \land 0.5 = 0.2 \neq 0.1\).

By itself, a small non-associativity may not be so bad:
• associativity comes from the requirement that our reasoning be rational, while
• it is well known that our actual handling of uncertainty is not exactly following rationality requirements; see, e.g., [14].

So, it is desirable to find out how non-associative can these operations be.

5. How Non-Associative Are Natural (Midpoint) Operations? Main Results and Their Psychological Interpretation

We know that the midpoint operations are non-associative, i.e., that sometimes, \((a \& b) \& c \neq a \& (b \& c)\). We want to know how big can the difference \((a \& b) \& c - a \& (b \& c)\) be.

Theorem 1. \(\max_{a,b,c} |(a \& b) \& c - a \& (b \& c)| = 1/9.\)

Theorem 2. \(\max_{a,b,c} |(a \lor b) \lor c - a \lor (b \lor c)| = 1/9.\)

(The proof of these two theorems use interval computations; for readers’ convenience, the proofs are placed in the special proofs section.)

Human experts do not use all the numbers from the interval [0, 1] to describe their possible degrees of belief; they use a few words like “very probable”, “mildly probable”, etc. Each of words is a “granule” covering the entire sub-interval of values. Since the largest possible non-associativity degree \(|(a \& b) \& c - a \& (b \& c)|\) is equal to 1/9, this non-associativity is negligible if the corresponding realistic “granular” degree of belief have granules of width \(\geq 1/9\). One can fit no more than 9 granules of such width in the interval [0, 1]. This may explain why humans are most comfortable with \(\leq 9\) items to choose from – the famous “7 plus minus 2” law; see, e.g., [9, 10].

This general psychological law has also been confirmed in our specific area of formalizing expert knowledge: namely, in [3], it was shown that this law explains why in intelligent control, experts normally use \(\leq 9\) different degrees (such as “small”, “medium”, etc.) to describe the value of each characteristic.

6. Interval Computations: A Tool Used in Our Proof

To prove our results, we use a technique called interval computations (see, e.g., [6, 7, 15]). Namely, our goal is to find the maximum of the function \(|(a \& b) \& c - a \& (b \& c)|\) when \(a \in [0, 1]\), \(b \in [0, 1]\), and \(c \in [0, 1]\).

We know that the minimum of this function is 0: it is attained, e.g., if \(a = b = c = 0\). Thus, what we are looking for is the range of the above function of three real variables.

Interval computations is a technique which allows us, given a function \(y = f(x_1, \ldots, x_n)\) of several real variables and a “box” \(B = x_1 \times \cdots \times x_n\), where \(x_i = [x^-_i, x^+_i]\), to compute either the range of the given function on the given box:

\[y = f(x_1, \ldots, x_n) = \{f(x_1, \ldots, x_n) \mid x_1 \in [x^-_1, x^+_1], \ldots, x_n \in [x^-_n, x^+_n]\},\]

or an interval \(Y\) which is guaranteed to contain the desired range, i.e., for which \(Y \subseteq y\). (We cannot always compute the exact range because computing this exact range is intractable even for quadratic functions \(f(x_1, \ldots, x_n)\); see, e.g., [8].)

This technique is based on the fact that in the computer, the computation of a function \(f\) consists of several elementary steps. For example, a compiler will translate the computation of the midpoint “and” operation

\[f(p_1, p_2) = \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2)\]

into the following sequence of elementary steps (\(r_1, r_2, \ldots, r_6\) denote the preliminary computation results):

- first, we compute \(r_1 := p_1 + p_2;\)
- then, we compute \(r_2 := r_1 - 1;\)
- compute \(r_3 := \min(r_2, 0);\)
- compute \(r_4 := (1/2) \cdot r_3;\)
- compute \(r_5 := \min(p_1, p_2);\)
- compute \(r_6 := (1/2) \cdot r_5;\)
- finally, compute the result as \(y := r_4 + r_6.\)

In this example, we have two input variables \(x_1 = p_1\) and \(x_2 = p_2.\) In general, for each input variable \(x_i,\) we know the interval \(x_i = [x^-_i, x^+_i]\) of possible values. For each elementary step \(h(a, b),\) if we know the intervals \(a = [a^-, a^+]\) and \(b = [b^-, b^+]\) of possible values for each of the input, then we can compute the interval \(h(a, b)\) of possible values of the results:

\[\begin{align*}
[a^-, a^+] + [b^-, b^+] &= [a^- + b^-, a^+ + b^+]; \\
[a^-, a^+] - [b^-, b^+] &= [a^- - b^+, a^+ - b^-]; \\
[a^-, a^+] \cdot [b^-, b^+] &= [c^-, c^+], & \text{where } c^- = \min(a^- \cdot b^-, a^- \cdot b^+, a^+ \cdot b^-, a^+ \cdot b^+), \\
&\quad \text{and } c^+ = \max(a^- \cdot b^-, a^- \cdot b^+, a^+ \cdot b^-, a^+ \cdot b^+);
\end{align*}\]
• $\min([a^-, a^+], [b^-, b^+]) = [\min(a^-, b^-), \min(a^+, b^+)]$;
• $\max([a^-, a^+], [b^-, b^+]) = [\max(a^-, b^-), \max(a^+, b^+)]$.

These formulas are called formulas of interval arithmetic.

So, to find an interval that contains the desired range, we follow the original algorithm step-by-step, on each step replacing the original elementary operation with the corresponding operation of interval arithmetic.

In particular, if we want to know the range of the values of the function $f(p_1, p_2) = p_1 \& p_2$ when $p_1 \in p_1$ and $p_2 \in p_2$, we do the following:

• first, we compute $r_1 := p_1 + p_2$;
• then, we compute $r_2 := r_1 - [1, 1]$;
• compute $r_3 := \min (r_2, [0, 0])$;
• compute $r_4 := [0.5, 0.5] \cdot r_3$;
• compute $r_5 := \min (p_1, p_2)$;
• compute $r_6 := [0.5, 0.5] \cdot r_5$;
• finally, compute the result as $Y := r_4 + r_6$.

It is easy to prove (by induction) that at any given moment of time, the result of this procedure is guaranteed to contain the result of the interval of possible values of the corresponding quantity.

It is also easy to show that this “naive” interval computation procedure sometimes overestimates. For example, for a function $f(x_1) = x_1 \cdot (1 - x_1)$ on the interval $[0, 1]$, the computational procedure consists of the following two steps:

• $r_1 := 1 - x_1$;
• $y := x_1 \cdot r_1$,

so we get the following estimate:

• $r_1 := [1, 1] - x_1 = [1, 1] - [0, 1] = [1 - 1, 1 - 0] = [0, 1]$;
• $Y := x_1 \cdot r_1 = [0, 1] \cdot [0, 1] = [0, 1]$.

While the actual range is $y = [0, 0.25] \subset Y = [0, 1]$.

To decrease the overestimation, we can use the following methodology of interval computations: we divide each interval $x_i$ into several sub-intervals, thus dividing the original box into many sub-boxes; then, we estimate the range of the function over each of the subintervals, and then take the union of the resulting ranges as an estimate for the range over the whole original box.

If we are interested not only in the actual value of the maximum, but if we also want to know where exactly this maximum is attained, then we can use this sub-boxes as follows: if we have two sub-boxes $B_1$ and $B_2$ with range estimates $[m_1, M_1]$ and $[m_2, M_2]$, and $M_1 < m_2$, then we are guaranteed that an arbitrary value $f(x_1, \ldots, x_n)$ for $(x_1, \ldots, x_n)$ from the first sub-box is smaller than every value from the second sub-box. Thus, we can safely claim that the (global) maximum of the given function cannot be attained in the first sub-box – hence, this first sub-box can be safely removed from the list of possible location of the global maximum.

7. Proof Itself

One can easily see that the operation $\lor$ is dual to the operation $\land$ in the sense that $a \lor b = 1 - (1 - a) \land (1 - b)$. Because of this duality, we can easily deduce Theorem 2 from Theorem 1. Thus, it is sufficient to prove Theorem 1.

Every triple can be sorted: $a \leq b \leq c$. For these sorted real numbers, we want to know the relation between $t_a \equiv a \land (b \lor c)$, $t_b \equiv b \lor (a \land c)$, and $t_c \equiv c \land (a \lor b)$. The formulas for $a \land b, a \lor c$, and $b \lor c$ depend on the relation between $a + b, a + c, b + c$, and $1$. Since $a \leq b \leq c$, we have $a + b \leq a + c \leq b + c$. Thus, there are exactly 4 possible locations of number 1 in relation to these three sums:

I. $a + b \leq a + c \leq b + c \leq 1$;
II. $a + b \leq a + c \leq 1 < b + c$;
III. $a + b \leq 1 < a + c \leq b + c$;
IV. $1 < a + b \leq a + c \leq b + c$.

We prove by considering these cases one by one. In each case, we get expressions for $a \land b, a \lor c$, and $b \lor c$ which do not contain min and max.

We can subdivide each of these cases into subcases depending on which of the maximized and minimized terms in the expressions for $a \land (b \lor c), b \lor (a \land c)$, and $(a \lor b)$ are larger.

For example, in case IV, all three sums $a + b, a + c$, and $b + c$ are greater than 1, so $a \land b = a + 0.5 \cdot b - 0.5$, $a \lor c = a + 0.5 \cdot c - 0.5$, and $b \lor c = b + 0.5 \cdot c - 0.5$.

• The value of $t_a = (b \lor c) \land a$ depends on whether $(b \lor c) + a \leq 1$, i.e., whether $b + 0.5 \cdot c - 0.5 + a \leq 1$.
  If we move terms which do not contain $a$, $b$, or $c$
to the right hand-side, and rearrange terms which do contain $a$, $b$, or $c$, in alphabetic order, we get an equivalent inequality $a + b + 0.5 \cdot c \leq 1.5$.

- Similarly, the value of $t_b = (a \& c) \& b$ depends on whether $(a \& c) + b \leq 1$, i.e., whether $a + 0.5 \cdot c - (1 - b) + b \leq 1$, which is also equivalent to the same inequality $a + b + 0.5 \cdot c \leq 1.5$.

- Finally, the value of $t_c = (a \& b) \& c$ depends on whether $(a \& b) + c \leq 1$, i.e., whether $a + 0.5 \cdot b - 0.5 + c \leq 1$, which is equivalent to the inequality $a + 0.5 \cdot b + c \leq 1.5$.

So, to find the expressions for $t_a$, $t_b$, and $t_c$, we must know where 1.5 stands in comparison with $a + b + 0.5 \cdot c$ and $a + 0.5 \cdot b + c$. Since $b \leq c$, we have $0.5 \cdot b \leq 0.5 \cdot c$, hence

$$a + b + 0.5 \cdot c = (a + b + c) - 0.5 \cdot c \leq (a + b + c) - 0.5 \cdot b = a + 0.5 \cdot b + c.$$  

Due to this inequality, we have exactly three possibilities:

- A. the number 1.5 can be larger than the largest of the above two expressions; in this case, both expressions are $\leq 1.5$, i.e.,

  $$a + b + 0.5 \cdot c \leq a + 0.5 \cdot b + c \leq 1.5;$$

- B. the number 1.5 is in between the above two expressions; in this case,

  $$a + b + 0.5 \cdot c \leq 1.5 < a + 0.5 \cdot b + c;$$

- C. the number 1.5 is smaller than the smallest of the above two expressions; in this case, both expressions are $\geq 1.5$, i.e.,

  $$1.5 < a + b + 0.5 \cdot c \leq a + 0.5 \cdot b + c.$$  

These subcases can be further subdivided, etc. For each of the resulting final subcases, all three combinations $t_a$, $t_b$, and $t_c$ are described by linear expressions.

There are many such subcases, so the proof is possible but very lengthy. It turns out that interval computations can reduce this length.

Indeed, we want to find the maximum of the expression

$$|[(a \& b) \& c - a \& (b \& c)]|$$

when $a, b, c \in [0, 1]$. To help with the proof, we divided each interval $[0, 1]$ into 100 subintervals of length 0.01, thus generating $100^3 = 10^6$ sub-boxes. We use the above interval arithmetic (with additional operations for min and max). For each subbox, we applied the “naive” interval computations technique to get the estimate $[m_i, M_i]$ for the range of the desired function on this subbox. Then, we eliminated all subboxes for which $M_i < 1/9$.

As a result, out of the original $10^6$ boxes, we have only 80 possible locations of the global maximum. For these 80 boxes, $b \in [0.54, 0.58]$, and:

- either $a \in [0.43, 0.46]$ and $c \in [0.75, 0.79]$;
- or $a \in [0.75, 0.79]$ and $c \in [0.43, 0.46]$.

When we sort $a$, $b$, and $c$, we get $a \in [0.43, 0.46]$, $b \in [0.54, 0.58]$, and $c \in [0.75, 0.79]$. Hence, $a + c > 1$ and we only need proofs for half of the cases: Cases III and IV.

Some subcases of Case IV were also eliminated. Indeed, within the above interval bounds for $a$, $b$, and $c$, the upper bound for $a + b + 0.5 \cdot c$ is equal to $0.46 + 0.58 + 0.5 \cdot 0.79 = 1.435 < 1.5$. Thus, to check that the value of the desired function cannot exceed 1/9, we only need to consider cases when $a + b + 0.5 \cdot c < 1.5$. Hence, we can dismiss Subcase C when this inequality is not satisfied, and only consider Subcases A and B in our proof.

For each final subcase, the difference

$$(a \& b) \& c - a \& (b \& c)$$

is a linear function, and the constraints describing this subcase are linear inequalities. Thus, for each subcase, we have a linear programming problem with rational coefficients. We can analytically solve each of these problems by computing the vertices of the corresponding polytope, and finding the vertex on which the objective function attains the largest value. As a result, we get the desired proof.

8. Auxiliary Results: Alternatives to Midpoint

Instead of selecting a midpoint, we can make a more general selection of a value in the interval $p$.

By a choice function, we mean a function $s$ that maps every interval $u = [u^-, u^+]$ into a point $s(u) \in u$ so that for every $c$ and $\lambda > 0$:

- $s([u^- + c, u^+ + c]) = s([u^-, u^+]) + c$ (shift-invariance);
- $s([\lambda \cdot u^-, \lambda \cdot u^+]) = \lambda \cdot s([u^-, u^+])$ (unit-invariance).
Proposition. [11] Every choice function has the form
\( s[u^-, u^+] = \alpha \cdot u^- + (1 - \alpha) \cdot u^+ \) for some \( \alpha \in [0, 1] \).

The combination \( p = \alpha \cdot p^- + (1 - \alpha) \cdot p^+ \) (first proposed by Hurwicz [5]) has been successfully used in areas ranging from submarine detection to petroleum engineering [11]; in [16], this approach is applied to second-order probabilities.

With this approach, we get the following formulas which generalize the above definitions:

\[
p_1 \& p_2 \overset{\text{def}}{=} \alpha \cdot \min(p_1 + p_2 - 1, 0) + (1 - \alpha) \cdot \min(p_1, p_2);
\]

\[
p_1 \lor p_2 \overset{\text{def}}{=} \alpha \cdot \max(p_1, p_2) + (1 - \alpha) \cdot \min(p_1 + p_2, 1).
\]

Theorem 3.

\[
\max_{a, b, c} |(a \& b) \& c - a \& (b \& c)| = \frac{\alpha \cdot (1 - \alpha)}{2 + \alpha \cdot (1 - \alpha)}.
\]

\[
\max_{a, b, c} |(a \lor b) \lor c - a \lor (b \lor c)| = \frac{\alpha \cdot (1 - \alpha)}{2 + \alpha \cdot (1 - \alpha)}.
\]

Comment. This non-associativity degree is the smallest (= 0) when \( \alpha = 0 \) or \( \alpha = 1 \), and the largest (= 1/9) for midpoint operations (\( \alpha = 0.5 \)).

In our proof, it was useful to first show that the new operations have some properties of associativity: namely, it turns out that for every \( \alpha \), both operations are semi-associative in the sense that \( a \leq b \leq c \) implies that \( a \ast (b \ast c) \geq b \ast (a \ast c) \geq c \ast (a \ast b) \).

Acknowledgments

This work was supported in part by NASA under cooperative agreement NCC5-209, by Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-00-1-0365, and by Grant No. W-00016 from the U.S.-Czech Science and Technology Joint Fund.

The authors are very thankful to H.T. Nguyen and T. Whalen for helpful discussions.

References


