Beyond [0,1] to Intervals and Further: Do We Need All New Fuzzy Values?

Yeung Yam[1], Masao Mukaidono[2], and Vladik Kreinovich[3]

***Abstract*— In many practical applications of fuzzy methodology, it is desirable to go beyond the interval $[0,1]$ and to consider more general fuzzy values: e.g., intervals, or real numbers outside the interval $[0,1]$. When we increase the set of possible fuzzy values, we thus increase the number of bits necessary to store each degree, and therefore, increase the computation time which is needed to process these degrees. Since in many applications, it is crucial to get the result on time, it is therefore desirable to make the smallest possible increase. In this paper, we describe such smallest possible increases.**

I. Introduction

In classical (two-valued) logic, there are only two truth values: "true" (which, in the computer, is usually denoted by 1) and "false" (which is usually denoted by 0). To represent the uncertainty of human reasoning, L. Zadeh proposed, in his *fuzzy logic*, to use additional truth values, including truth values which are intermediate between "true" and "false", i.e., intermediate between 1 and 0 [12]. Due to this motivation, the most natural choice of the set of truth values is the interval $[0,1]$. This set is indeed used in most applications of fuzzy logic.

However, in some applications, it is desirable to go beyond the interval $[0,1]$. Let us give three examples of such situations:

- Traditional $[0,1]$-based fuzzy logic is based on the assumption that we can describe an expert's uncertainty by a single real number. However, if an expert is uncertain about something, he can as well be uncertain about his degree of belief as well, and it is quite possible that the expert will not be able to describe his degree of uncertainty exactly: e.g., a person can meaningfully distinguish between his degrees of belief 0.6 and 0.7, but hardly anyone will be able to describe his degree of belief as 0.6 and not 0.601. As a result, a more adequate representation of degree of belief is not by a *single* real number, but by an *interval* or even a more general *set* of possible real numbers; see, e.g., [1, 2, 5, 6, 7, 8, 10] and references therein.
- Another problem with traditional fuzzy logic is that the corresponding set of degrees of belief is linearly ordered, i.e., for every two degrees of belief a and b, we are either 100% sure that a indicates more belief than b, or that b indicates more belief than a. Again, since we are talking about the expert uncertainty, it is quite reasonable to imagine situations in which an expert is unable to definitely sort his degrees of belief. To describe such situations, it is desirable to allow the set of degrees of belief to be only *partially* ordered.
- The third problem stems from the fact that although originally, fuzzy logic was proposed to describe human reasoning, it turned out that fuzzy logic leads to a good description of general systems. This fact, in itself, should not be surprising, because fuzzy logic reflects human reasoning which is known to be good in controlling different systems. Indeed, it was shown that for some reasonable criteria, fuzzy logic indeed leads to an optimal approximation (see, e.g., [4]). When we consider fuzzy logic as a method of representing human uncertainty, then it is usually natural to restrict ourselves to values from the interval $[0,1]$ (usually, but not always: see, e.g., [9]). However, when we consider fuzzy logic as simply a convenient approximation tool (see, e.g., [3]), then this restriction is no longer justified. Moreover, it has been shown [11] that if we allow negative values and values greater than 1, we sometimes drastically decrease the number of rules which are necessary to describe the same input-output behavior with the same accuracy.

The authors are with the [1]Department of Mechanical & Automation Engineering The Chinese University of Hong Kong, Shatin, NT, Hong Kong, China, email yyam@mae.cuhk.edu.hk; [2]Department of Computer Science, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki-shi 214 Japan, email masao@cs.meiji.ac.jp; [3]Department of Computer Science, University of Texas at El Paso, El Paso, TX 79968, USA, email vladik@cs.utep.edu

These and other arguments show that we do need to go beyond the interval $[0,1]$. However, if we increase the set of possible values for representing uncertainty, we thus increase the number of bits needed to store possible values, and hence, increase the computation time necessary to process these values. In many applications (e.g., in real-time control) time is really crucial and therefore, we must keep it as small as possible. Hence, although we must increase the set of truth values, it is desirable to use the smallest possible increase. Of course, this increase should satisfy some reasonable conditions: e.g., we still want to have a logic (and an extension of fuzzy logic), i.e., we want to have this set of values closed under logical operations, hedges (like "very" and "sightly"), etc. Thus, we arrive at the following problem: *What is the smallest possible increase in the set of truth values which is still closed under all these operations?*

Crudely speaking, our resulting answers are as follows:

- we do need all intervals, and
- we do need all real numbers.

II. First result: We need all intervals

All standard fuzzy logic operations can be naturally extended to intervals: e.g., we define "very" for an interval as

$$[a^-,a^+]^2 = \{a^2 \mid a \in [a^-,a^+]\},$$

"slightly" as

$$\sqrt{[a^-,a^+]} = \{\sqrt{a} \mid a \in [a^-,a^+]\},$$

and we can define logical operations "and"

$$\min([a^-,a^+],[b^-,b^+]) =$$

$$\{\min(a,b) \mid a \in [a^-,a^+], b \in [b^-,b^+]\},$$

"or"

$$\max([a^-,a^+],[b^-,b^+]) =$$

$$\{\max(a,b) \mid a \in [a^-,a^+], b \in [b^-,b^+]\},$$

and negation

$$1-[a^-,a^+] = \{1-a \mid a \in [a^-,a^+]\}.$$

One can easily see that, due to monotonicity of these operations,

$$[a^-,a^+]^2 = [(a^-)^2,(a^+)^2],$$

$$\sqrt{[a^-,a^+]} = [\sqrt{a^-},\sqrt{a^+}],$$

$$\min([a^-,a^+],[b^-,b^+]) =$$

$$[\min(a^-,b^-),\min(a^+,b^+)],$$

$$\max([a^-,a^+],[b^-,b^+]) =$$

$$[\max(a^-,b^-),\max(a^+,b^+)],$$

and

$$1-[a^-,a^+] = [1-a^+,1-a^-].$$

Definition 2.1. *We say that a set S of intervals $[a,b] \subseteq [0,1]$ forms an interval logic if it contains all degenerate intervals $[a,a]$, at least one non-degenerate interval, and is closed under the logical operations $\min(a,b)$ ("and"), $\max(a,b)$ ("or"), $1-a$ ("not"), and hedges "very" $a \to a^2$, and "slightly" $a \to \sqrt{a}$.*

Theorem 2.1. *There exist four and only four interval logics:*

- *the set of all intervals $[a,b] \subseteq [0,1]$;*
- *the set of all intervals $[a,b]$ with $b<1$, together with an interval $[1,1]$;*
- *the set of all intervals $[a,b]$ with $a>0$, together with an interval $[0,0]$;*
- *the set of all intervals $[a,b]$ with $0<a$ and $b<1$, together with an intervals $[0,0]$ and $[1,1]$.*

In short, disregarding the degenerate cases $a=0$ and $b=1$, we need all the intervals.

Comment. For readers' convenience, the proof of Theorem 2.1 is placed in a special Proofs section at the end of the paper.

III. Second result: We need all real numbers

We are interested in the sets S of real numbers which are closed under logical operations, e.g., under "and" operation $a \cdot b$ and negation $\neg a = 1-a$ (i.e., if $a,b \in S$, then $a \cdot b \in S$ and $1-a \in S$). The following results characterize such sets:

Theorem 3.1. *If a set S of real numbers contains the interval $[0,1]$ and is closed under the "and"-operation $a \cdot b$ and under negation, then either $S=[0,1]$ or S coincides with the set R of all real numbers.*

Theorem 3.2. *If a closed set S of real numbers is contained in the interval $[0,1]$ and is closed under the "and"-operation $a \cdot b$ and under negation, then either $S=[0,1]$ or $S=\{0,1\}$.*

Thus, we only have three choices: we either have a classical logic $S=\{0,1\}$, or the standard fuzzy logic $S=[0,1]$, or a logic which uses all real numbers.

Comment. These results are proven in [9].

IV. Proof of Theorem 2.1

First, let us remark that since the interval logic contains all real numbers and is closed under min and max, it must therefore, contain, with every interval, all its subintervals. Indeed, if $[a^-, a^+] \in S$ and $[b^-, b^+] \subseteq [a^-, a^+]$, then we can conclude that:

- the degenerate intervals $[b^-, b^-]$ and $[b^+, b^+]$ belong to S;
- hence, the interval $\min([b^+, b^+], [a^-, a^+]) = [a^-, b^+]$ belongs to S;
- therefore, the interval $\max([b^-, b^-], [a^-, b^+]) = [b^-, b^+]$ also belongs to S.

In view of this remark, to prove our theorem it is sufficient to prove that for every $\delta > 0$, the interval logic S contains an interval which contains $[\delta, 1-\delta]$. Indeed, by definition of n interval logic, it contains a non-degenerate interval $[a^-, a^+]$. Since S is closed under "very", it also contains the intervals $[(a^-)^2, (a^+)^2]$, $[(a^-)^4, (a^+)^4]$, and, in general, $[(a^-)^{2^k}, (a^+)^{2^k}]$ for every natural number k.

Since S is also closed under negation, it must contain the interval $[z, Z]$, where $z^- = 1 - (a^+)^{2^k}$ and $z^+ = 1 - (a^-)^{2^k}$. (We will choose the value k later.)

Since the interval logic is closed under the hedge "slightly", it must also contain, for every natural number l, the interval $[z^{1/2^l}, Z^{1/2^l}]$. We want to choose k and l in such a way that this interval contains $[\delta, 1-\delta]$, i.e., that $z^{1/2^l} \le \delta$ and $Z^{1/2^l} \ge 1-\delta$.

As $l \to \infty$, we have $Z^{1/2^l} \to 1$; therefore, for some l, we can get this value to be greater than or equal to $1-\delta$. Let us take the first l for which $Z^{1/2^l} \ge 1-\delta$. Then, for the previous l, this inequality is no longer true, i.e., $(Z^{1/2^l})^2 < 1-\delta$. Therefore, $Z^{1/2^l} < \sqrt{1-\delta}$, i.e.,

$$1-\delta \le Z^{1/2^l} < \sqrt{1-\delta}.$$

By turning to logarithms, we conclude that

$$\ln(1-\delta) \le 2^{-l} \cdot \ln(Z) =$$

$$2^{-l} \cdot \ln(1-(a^-)^{2^k} < (1/2) \cdot \ln(1-\delta).$$

It is sufficient to prove the result only for small δ. Asymptotically, neglecting higher order terms in δ, we have $\ln(1-\delta) \approx -\delta$ and therefore, the above inequality takes the asymptotic form

$$-\delta \le -2^{-l} \cdot (a^-)^{2^k} \le -\frac{\delta}{2},$$

i.e.,

$$\frac{\delta}{2} \le 2^{-l} \cdot (a^-)^{2^k} \le \delta.$$

Therefore, the value 2^{-l} is, within a multiplicative constant, equal to

$$2^{-l} \sim \frac{\delta}{2} \cdot \left(\frac{1}{a^-}\right)^{2^k}.$$

We want to choose k for which $z^{1/2^l} \le \delta$, i.e., for which $2^{-l} \cdot \ln(1-(a^+)^{2^k}) \le \ln(\delta)$. Asymptotically, this inequality takes the form $-2^{-l} \cdot (a^+)^{2^k} \le \ln(\delta)$, or, equivalently, $2^{-l} \cdot (a^+)^{2^k} \ge -\ln(\delta)$. Substituting the above expression for 2^{-l} into the desired inequality, we reformulate it in the form

$$\left(\frac{\delta}{2} \cdot \left(\frac{1}{a^-}\right)^{2^k}\right) \cdot (a^+)^{2^k} \ge -\ln(\delta),$$

or, equivalently,

$$\left(\frac{a^+}{a^-}\right)^{2^k} \ge \frac{|\ln(\delta)|}{\delta/2}.$$

The left-hand side of this inequality tends to ∞ as $k \to \infty$ and therefore, for sufficiently large k, this inequality is indeed true. For such sufficiently large k and for the corresponding l, the interval $[z^{1/2^l}, Z^{1/2^l}] \in S$ contains $[\delta, 1-\delta]$. The theorem is proven.

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