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ARITHMETIC OF COMPLEX SETS: NICKEL’S CLASSICAL PAPER REVISITED FROM A GEOMETRIC VIEWPOINT

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Abstract. Due to measurement uncertainty, after measuring a value of a physical quantity (or quantities), we do not get its exact value, we only get a set of possible values of this quantity (quantities). In case of 1-D quantities, we get an interval of possible values. It is known that the family of all real intervals is closed under point-wise arithmetic operations (+, −, ×) (i.e., this family forms an arithmetic). This closeness is efficiently used to estimate the set of possible values for $y = f(x_1, \ldots, x_n)$ from the known sets of possible values for $x_i$.

In some practical problems, physical quantities are complex-valued; it is therefore desirable to find a similar closed family (arithmetic) of complex sets. We follow K. Nickel’s 1980 paper to show that, in contrast to 1-D interval case, there is no finite-dimensional arithmetic.

We prove this result by reformulating it as a geometric problem of finding a finite-dimensional family of planar sets which is closed under Minkowski addition, rotation, and dilation.
Data processing: a practical problem which leads to arithmetic of complex sets. In many real-life situations, we are interested in the value of some physical quantity \( y \) which is difficult (or even impossible) to measure directly. To estimate \( y \), we measure directly measurable quantities \( x_1, \ldots, x_n \) which have a known relationship with \( y \), and then reconstruct \( y \) from the results \( \tilde{x}_1, \ldots, \tilde{x}_n \) of these measurements by using this known relation: 
\[
\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n),
\]
where \( f \) is a known algorithm.

Measurements are never 100% accurate; as a result, the actual value \( x_i \) of each measured quantity may differ from the measured value \( \tilde{x}_i \). If we know the upper bound \( \Delta_i \) for the measurement error \( \Delta x_i = |\tilde{x}_i - x_i| \), then after we get the measurement result \( \tilde{x}_i \), we can conclude that the actual value \( x_i \) of the measured quantity belongs to the interval \( x_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i] \). A natural question is: when \( x_i \in x_i \), what is the resulting interval \( y = f(x_1, \ldots, x_n) = \{f(x_1, \ldots, x_n) | x_i \in x_i\} \) of possible values of \( y \)?

Computing the exact bounds for the range interval is, in general, computationally difficult (see e.g., [Kreinovich et al. 1997]). However, there are efficient methods of computing an enclosure \( Y \supseteq y \) for this range; these methods are called methods of interval computations (see, e.g., [Hammer et al. 1993], [Hansen 1992], [Kearfott 1996], [Kearfott et al. 1996], [Moore 1979]). For example, we can use “naive interval computations”: describe the algorithm \( f \) as a sequence of elementary arithmetic operations (+, -, \cdot, /), and on each step, replace each operation \( \circ \) with numbers by the corresponding operation with intervals:

\[
x \circ y = \{x \circ y | x \in x, y \in y\}.
\]

For intervals, we have explicit formulas for these arithmetic operations: e.g., \([a, \bar{a}] + [b, \bar{b}] = [a + b, \bar{a} + \bar{b}]\), etc.

For example, to estimate the range of the function \( f(x_1) = x_1 \cdot (1 - x_1) \), we describe the algorithm \( f \) as a sequence of two arithmetic operations:

- computing the intermediate value \( r_1 := 1 - x_1 \), and
- computing the product \( f := x_1 \cdot r_1 \).

So, to estimate the range \( f([0,1]) \), we compute \( r_1 := 1 - [0,1] = \)
[0, 1], and then get the final enclosure $Y := x_1 \cdot r_1 = [0, 1] \cdot [0, 1] = [0, 1]$ (this is, of course, a superset of the actual range $[0, 0.25]$).

Similar range estimation problems appear when the physical quantities are described by complex numbers. It is therefore desirable to find a similar technique for complex numbers. The methodology of naive interval computations is based on the fact that the set of all intervals (including degenerate intervals – real numbers) is closed under point-wise arithmetic operations (1) (except, of course, division by an interval $y$ containing 0). In other words, arithmetic operations are well defined on the family of all intervals, so we can talk about the arithmetic of intervals. Hence, it is desirable to look for families of subsets of complex numbers which are also closed under arithmetic operations, i.e., to look for an arithmetic of complex sets.

We want these subsets to be representable in a computer, where we can only store finitely many parameters and therefore, we want these sets to form a finite-dimensional (finite-parametric) family.

Also, we want to take into consideration that real numbers are an important practical case of complex numbers; therefore, real-line intervals (corresponding to imprecisely known real numbers) should be a particular case of this more general family of complex sets.

Reasonable families of complex sets do not form a complex arithmetic: the empirical fact and the resulting question. There are several natural complex analogues of real-line intervals:

- boxes, i.e., rectangular parallel to real axis;
- ellipses (including real-line intervals as degenerate ellipses), etc.

None of these families is closed under point-wise arithmetic operations (1). Moreover, they are not even closed under a limited set of arithmetic operations which includes addition and multiplication by complex numbers. A natural question is: Is there a finite-dimensional family of complex sets which is closed under these operations? To answer this question, let us reformulate it in geometric terms.
Reformulating the question in geometric terms. In geometric terms, a complex plane is simply a plane, so we are looking for families of planar sets. The sum (1) of two planar sets is simply their Minkowski sum.

In geometric terms, if we multiply a complex number $t$ by another complex number $z = \rho \cdot \exp(i\varphi)$, this means that we first rotate $t$ by an angle $\varphi$ around the origin $O = (0, 0)$ of the coordinate system, and then dilate the rotated point $\rho$ times. Thus, the pointwise product $z \cdot T$ of a complex number $z$ and a set $T$ means that we first rotate the set $T$, and then dilate the result of this rotation.

Hence, we arrive at the following definition:

**Definition.** Let $\mathbb{R}^2$ be a plane. By an arithmetic of complex sets, we mean a family $\mathcal{F}$ of planar sets which satisfies the following three properties:

- $\mathcal{F}$ contains all sub-intervals of the $x$-axis $\mathbb{R} \times \{0\}$;
- $\mathcal{F}$ is closed under Minkowski addition, and
- $\mathcal{F}$ is closed under rotations and dilations around $O = (0, 0)$.

A finite-dimensional family can be defined in a standard topological way: if we restrict ourselves to bounded and closed (hence, compact) sets, we can use Hausdorff distance between sets to define a topology; once the family is a topological space, we can use standard topological definitions to define its dimension.

The question is: does there exist a finite-dimensional arithmetic of complex sets?

Nickel’s answer, and why it is not final. In his paper [Nickel 1980], K. Nickel proves that “finite-dimensional” arithmetics of complex sets do not exist. However, in his formulation, he only considers sets with piece-wise smooth boundaries, and he uses a non-standard (and non-topological) definition of dimension.

To be more precise, he calls a family “at least $m$-dimensional” if this family contains at least one set with $m$ “corner” (non-smooth) points, and he proves that every arithmetic of complex sets is “infinite-dimensional” in this sense by proving that it contains a $m$-cornered set $B_m$ for each $m$. From the topological viewpoint, all
these sets \( B_m \) form a family of dimension 0, and therefore, Nickel’s proof does not answer our question.

**Final answer.** We will show that a minor modification of Nickel’s construction does lead to the final answer:

**Proposition.** There exists no finite-dimensional arithmetic of complex sets.

**Proof.** We will show that every arithmetic of complex sets \( F \) contains, for every \( n \), an \( n \)-dimensional subfamily. Indeed, by definition of an arithmetic of complex sets, the family \( F \) contains a horizontal (real-line) interval \( I_0 = [0, 1] \times \{0\} \), and also the results \( I_1, \ldots, I_n \) of its rotation by angles \( \varphi_0, 2\varphi_0, \ldots, n \cdot \varphi_0 = \pi/2 \), where \( \varphi_0 = \pi/(2n) \). Since \( F \) is closed under dilations, for every \( n + 1 \) positive real numbers \( \rho_0, \ldots, \rho_n \), this family contains the dilated sets \( J_i = \rho_i \cdot I_i, 0 \leq i \leq n \). Since \( F \) is closed under Minkowski addition, the family \( F \) also contains their Minkowski sum \( J_0 + \ldots + J_n \). One can easily see that this Minkowski sum is a polygon, and if we count its sides starting from the horizontal side, we get sides of lengths \( \rho_0, \ldots, \rho_n \) which make angles of \( 0, \varphi_0, 2\varphi_0, \ldots, n \cdot \varphi_0 = \pi/2 \) with the horizontal axes. Thus, different values of \( n \) parameters \( \rho_i \) lead to different sets from \( F \). Hence, the family \( F \) contains a \((n + 1)\)-dimensional subfamily. The proposition is proven.

**Open problem.** This result prompts the following open problem: what if, in our Definition, we do not require that a family \( F \) contain real-line intervals? What finite-dimensional families we will then have? For one, we will have a 1-D family of all circles with a center in \( O = (0, 0) \), a 3-D family of all circles. We will also have several other families of rotation-invariant sets (e.g., circles + circles with a narrow circular gap + circles with a concentric circular holes in them, etc.) Is there any finite-dimensional rotation- and dilation-invariant family of compact sets which is closed under Minkowski addition and whose sets are not rotation-invariant?

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