A New Look at Fuzzy Theory Via Chu Spaces

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Abstract

We propose to use Chu categories as a general framework for uncertainty analysis, with a special attention to fuzzy theory. We emphasize the fact that by viewing fuzzy concepts as Chu spaces, we can discover new aggregation operators, and model interactions and relationship between fuzzy data; these possibilities are due, in essence, to the category structure of Chu spaces, and especially to their morphisms. This paper is a tutorial introduction to the subject.

1 Introduction

1.1 A General Mathematical Framework is Needed for Data Mining

Most existing data mining techniques are *semi-heuristic* in the sense that they include parameters whose values must be experimentally adjusted to the data.

For simple data relationships, it is possible, by trying different values of these parameters, to "tune" the method and successfully uncover these relationships. To uncover more complicated data relationships, however, we must tune a larger number of parameters; as a result, for complex relationship, empirical tuning often becomes practically difficult, and the existing data mining techniques do not work that well.

To uncover complicated relationships between the data, we must, therefore, develop a general mathematical framework for the description of such relationships. The need for such a framework was emphasized in a recent paper [1].

1.2 This General Formalism Must Include a General Description of Combination Operations

Complexity of data relationships reflects the complexity of the systems described by this data; this system complexity, in turn, reflects the complex nature of interaction between the corresponding subsystems.

Therefore, to describe data relationship, we must be able to describe different *combination* operations which combine simpler subsystems into a more complex system.

1.3 Chu Spaces as a Natural Description of Combination Operations

Chu spaces can be viewed as a general formalism for describing different combination operations.

We first got acquainted with the notion of Chu spaces through the paper [6] presented at the First International Workshop on Current Trends and Developments in Fuzzy Logic (Thessaloniki, Greece, October 1998). This notion was originally introduced as a purely mathematical notion, by Po-Hsiang Chu [2]. Since then, Chu spaces were used to model concurrency in computer science (see, e.g., [7]), and more recently, to model information flow in distributed systems [3].

1.4 It Is Desirable to Use Chu Spaces to Describe Uncertainty

In the above applications, Chu spaces have a clear advantage over the previously proposed techniques; namely, the previous techniques usually started with a simple model which, due to its simplicity, is often not general enough. To make this model more general, the previous techniques add more and more features to the simple model; as a result, we get the desired generality, but the theory stops being simple, elegant, and consistent. The theory of Chu spaces, on the other hand, is both sufficiently *generality* (it has already been successfully applied to different situations) and reasonably *simple* and *elegant*.

Traditionally, Chu spaces have been mostly used to describe complex *deterministic* systems such as parallel computations. In view of the important features of Chu spaces, it is natural to explore the possibility of using this general algebraic framework for modeling and analysis of *general uncertainty*, both in the context of *probabilistic* uncertainty and non-probabilistic uncertainty as described by *fuzzy* theory.

2 What is a Chu Space?

2.1 Definition and Basic Interpretations

Let K be a set. A *K-Chu space* is a triple (X, r, A), where X and A are sets, and r is a map from $X \times A$ to K. Thus, a K-Chu space is a K-valued *relation* between elements of two sets X and A.

This relation can also be interpreted as a *classification scheme*: X is the set of things to be classified, A is the collection of properties used to classify elements of X, and r specifies the degree to which an element of X satisfies a property from A.

Let us describe some basic examples of Chu spaces.

2.2 Combination Function: First Example

A *combination function* c, developed in [1] to describe the general combination, can be viewed as an example of a $[0,1]$-Chu space.

Here, $X = 2^{S\mathcal{L}}$, where $S\mathcal{L}$ is the set of all *instantiated predicates*, i.e., the set of all formulas of the type $P(t)$, and $A = W$ is the set of all *support functions*, i.e., mappings $w : S\mathcal{L} \to [0,1]$ which describe to what extent a statement $P(t) \in S\mathcal{L}$ is supported by the available knowledge.

The value $r(x,a) = c(x,a)$ of a combination function $c : X \times A \to [0,1]$ describes the degree of support for a set $x \in X = 2^{S\mathcal{L}}$ provided that the support for each statement $P(t) \in x$ is described by the support function a.

2.3 Fuzzy Subset: Second Example

A fuzzy subset A of a set X can be viewed as a $[0,1]$-Chu space $(X, r, \{A\})$, where $r(x, A) = A(x)$ is the membership function of the fuzzy set A.

2.4 Fuzzy Partition: Third Example

A (finite) *fuzzy partition* $A_1, \ldots, A_m$ of a space X is usually defined as a collection of fuzzy subsets for which, for every i, there is an $x \in X$ such that $A_i(x) = 1$, and for any $y \in X$, $\sum_j A_j(y) = 1$.

Such a partition can be interpreted as a $[0,1]$-Chu space (X, r, A), where $A = \{A_1, A_2, \ldots, A_m\}$, and $r(X, A_i) = A_i(x)$.

2.5 Fuzzy Objects and Fuzzy Properties: Fourth Example

A $[0,1]$-Chu space $(X, r, [0,1]^Y)$ models a relationship between objects from X and fuzzy properties on Y.

Here, X and Y are (crisp) sets, $[0,1]^Y$ denotes the class of all fuzzy subsets of Y, and $r(x, B)$ is a degree to which x satisfies the fuzzy concept B.

If all the concepts B are crisp, then we replace $[0,1]^Y$ by 2^Y; the corresponding function r can be determined either by experts (i.e., using fuzzy measures as in Orlowski's model of decision-making), or by using covering functions of random sets.

2.6 Conditional Probabilities: Fifth Example

According to the (Bayesian) viewpoint, *conditional* probabilities capture the main aspects of information flow. The corresponding formalism of Bayesian networks can be described as a Chu space.

Namely, a conditional probability Chu space is of the form $(\mathcal{A}, P(\cdot|\cdot), \mathcal{B})$, where $\mathcal{A}$ is a σ-field of subsets of some set Ω, P is a probability measure on $(\Omega, \mathcal{A})$, and $P(\cdot|\cdot)$ is the conditional probability operator $P(\cdot|\cdot) : \mathcal{A} \times \mathcal{B} \to [0,1]$, defined by the formula $P(a|b) = P(ab)/P(b)$.

3 Why Chu Spaces?

3.1 Morphisms: The Main Reason for Using Chu Spaces

Chu spaces were originally defined as objects of *Chu categories*. Therefore, relationships between Chu spaces can be formulated in terms of *morphisms* of these categories. In our view, Chu morphisms are the main reason why we should translate different concepts into the language of Chu spaces.

3.2 Definition of a Morphism

Let us describe these morphisms formally.

Let $Ob(K)$ denote the set of all K-Chu spaces (for some fixed set K). If $\mathbf{A} = (X, r, A)$ and $\mathbf{B} = (Y, s, B)$ are objects from $Ob(K)$, then a *morphism* from $\mathbf{A}$ to $\mathbf{B}$ is defined as a pair of maps $\varphi = (f, g)$, where $f : X \to Y$, $g : B \to A$ such that the following adjointness condition holds:

$$\forall x \in X, \forall b \in B, r(x, g(b)) = s(f(x), b).$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
X \times B \arrow[r, "f \times 1"] \arrow[d, "1 \times g"'] & Y \times B \arrow[d, "s"] \\
X \times A \arrow[r, "r"'] & K
\end{tikzcd}$$

3.3 Morphisms Lead to New Combination Operations

By looking at this diagram, we see that a new K-Chu space appears, namely, $\mathbf{C} = (X, t, B)$, where the function $t : X \times B \to K$ is determined, by the adjointness condition, either from r and g, or from f and s:

$$t(x, b) = r(x, g(b)) = s(f(x), b).$$

In other words, two Chu spaces and a morphism give rise to a new Chu object.

This observation leads to a new way of "combining" objects relative to a morphism, in the same way as in [1], where "generalized fuzzy conditional rules" relative to a combination function are defined. We can thus define a general operator, a "cross product" $\mathbf{C} = \mathbf{A} \otimes_\varphi \mathbf{B}$; this general operator enlarges the list of combination operators such as fuzzy connectives and aggregation operators like OWA operators and fuzzy integrals.

We intend to use this operator (as well as many other categorical combination operators defined within the context of Category Theory) to search for more "fusion" operators.

3.4 Morphisms Are a Natural Analogue of Implication

From a logical inference viewpoint, this reminds us of an *implication* operator, when an if-then combination of the rules can be viewed as a new rule.

Similarly,
implications between "conditional events" are again conditional events, etc.

3.5 Morphisms Describe Data Interaction and Data Dependence

In [3], Chu morphisms are called *infomorphisms* and are interpreted in terms of information flow; in this interpretation, morphisms describe the interaction and relationship between different sources of *information*.

When we view Chu objects as *data*, then morphisms correspond to interaction or dependence between *data*. Therefore, we expect that morphisms can provide general models for data interaction and data dependence.

3.6 Chu Spaces are a Natural Generalization of Existing Techniques

An additional advantage of Chu space approach is that since this approach is very general, more traditional set-theoretic models can be viewed as a particular case of this approach and therefore, this approach can be viewed as a natural generalization of these more traditional ones.

Let us give one example of this possibility. By $\mathcal{C}([0,1])$, we well denote the class of all $[0,1]$-Chu spaces of the form (X, r, A), where A is an arbitrary set, $A = [0,1]^X = \{a : X \to [0,1]\}$, and $r(a, x) = a(x)$. What are the morphisms between these Chu objects?

If $\varphi = (f, g)$ is a morphism from $\mathbf{A} = (X, r, A) \in \mathcal{C}([0,1])$ to $\mathbf{B} = (Y, s, B) \in \mathcal{C}([0,1])$, then, $\forall x \in X, \forall b \in B$, we have

$$r(x, g(b)) = g(b)(x) = s(f(x), b) = b(f(x)).$$

Thus, it suffices to specify $f : X \to Y$, since then $g : B \to A$ can be uniquely defined by the formula $g(b) = b \circ f$ (where $\circ$ denotes composition of maps). Hence, the above category $\mathcal{C}([0,1])$ of "evaluation fuzzy spaces" is indeed in 1-1 correspondence with the standard category *Set* of sets and functions. So, traditional set categories are indeed a *particular* case of Chu spaces.

Now that we provided the arguments of why Chu spaces and their morphisms are important, let us describe the class of all Chu spaces and corresponding morphisms, i.e., *Chu categories*.

4 Chu Categories

4.1 Definition of a Chu Category

We have already defined the notion of a Chu space, and the notion of a morphism between the Chu spaces. To complete the definition of a Chu category, we must define the *composition* of morphisms.

This composition is defined as follows. Let (f, g) and (u, v) be morphisms, correspondingly, from $\mathbf{A} = (X, r, A)$ to $\mathbf{B} = (Y, s, B)$ and from $\mathbf{B}$ to $\mathbf{C} = (Z, t, C)$. Then, one can easily show that a pair $(u \circ f, g \circ v)$ is a morphism from $\mathbf{A}$ to $\mathbf{C}$. Indeed, for any $x \in X$ and $c \in C$, we have

$$r(x, g \circ v(c)) = r(x, g(v(c))) = s(f(x), v(c)) =$$

$$t(u(f(x)), c) = t(u \circ f(x), c).$$

Thus, we can define the composition $*$ of morphisms as

$$(u, v) * (f, g) = (u \circ f, g \circ v).$$

4.2 Illustration

Let us illustrate this definition by showing how it can be used to formalize the above result that "evaluation fuzzy spaces" are in 1-1 correspondence with sets and functions. Namely, we will show that there is a covariant function F from the category *Set* of sets and functions into the category $\mathcal{C}([0,1])$.

> It is also easy to show that this functor is an onto mapping of the objects, and that the inverse functor F^{-1} is a covariant functor $F^{-1} : \mathcal{C}([0,1]) \to$ *Set*; thus, the categories *Set* and $\mathcal{C}([0,1])$ are isomorphic.

A functor maps objects into objects, and morphisms into morphisms. Objects of the set category *Set* are sets. For every set $X \in Set$, we define $F(X) = (X, r, A) \in \mathcal{C}([0,1])$. Morphisms of the set category *Set* are functions. For every function $f : X \to Y$, we define $F(f)$ as the following morphism between the objects $F(X) = (X, r, A)$ and $F(Y) = (Y, s, B)$: $F(f) = (f, \Psi_f)$, where the function $\Psi_f : B \to A$ is defined by the formula $\Psi_f(b)(x) = b \circ f(x)$, for all $x \in X$. It can be checked that $F(f)$ is indeed a morphism in the category $\mathcal{C}([0,1])$, and that the above-defined mapping F is indeed a functor, i.e., it preserves composition: if $f : X \to Y$ and $g : Y \to Z$, then the corresponding morphisms $F(f) : F(Z) \to F(y)$ and $F(g) : F(Y) \to F(Z) = (Z, t, C)$ satisfy the condition

$$F(g \circ f) = F(g) * F(f).$$

Indeed, by definition of the function F, we have

$$F(g \circ f) = (g \circ f, \Psi_{g \circ f}),$$

and

$$\Psi_{g \circ f}(c)(x) = c \circ (g \circ f)(x) =$$

$$(c \circ g)(f(x)) = \Psi_g(c) \circ f(x) = \Psi_f(\Psi_g(c))(x),$$

i.e.,

$$\Psi_{g \circ f} = \Psi_f \circ \Psi_g.$$

On the other hand,

$$F(g) * F(f) = (g, \Psi_g) * (f, \Psi_f) = (g \circ f, \Psi_f \circ \Psi_g).$$

5 Further Interpretations and Applications

5.1 Data Fusion is Very Important

One of the most important problems in data processing is the problem of *data fusion*: how can we combine data coming from different sources? In particular, for uncertain data (e.g., for expert knowledge), we have a problem of combining uncertainty of different data sources.

The fact that this problem is of extreme importance can be illustrated by the fact that the very first paper introducing the ideas of fuzzy logic also introduced the basic operations for combining different fuzzy data: namely, the logical operations "or" and "and" which generalize the logical operations of traditional logic.

5.2 Traditional Logical Approach to Data Fusion in Fuzzy Theory: Successes and Drawbacks

These logic-induced fusion operations work well in many applications, especially in the cases when we do not know the relation between different data. However, when we do have some information about the relation between the data, the choice between a few logical fusion operators is too narrow.

We would like to have the whole spectrum of possible fusion operations which describe different degree of dependence and correlation between different data.

5.3 Chu Spaces: a New Sources of Combination Operations

Chu spaces provide us with such a possibility: namely, we naturally have the continuum of combination operations $\mathbf{A} \otimes_\varphi \mathbf{B}$ which correspond to different morphisms φ and which describe different correlation between the Chu spaces $\mathbf{A}$ and $\mathbf{B}$.

Acknowledgments

This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by the United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grant No. MDA904-98-1-0564.

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