Chu Spaces - A New Approach to Diagnostic Information Fusion

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Abstract *This paper is rather theoretical. Its aim is to describe a general algebraic framework, known as Chu spaces, in which different type of information can be transformed into the same form, so that fusion procedures can be investigated in a single general framework.*

Keywords: Chu spaces, data fusion, fuzziness, probability

1 A Motivating Example

Data fusion means that we combine ("fuse") several pieces of information (measurement results, expert estimates) about one or several objects. To describe our new approach to formalizing data fusion, we will start with a physically meaningful (and mathematically simple) example.

In order to find the location of distant radio sources, we measure the signals from these sources received on different radiotelescopes, and then fuse the measurement results. The larger the telescope, the more accurate the measurements. Therefore, to achieve maximum accuracy, antennas forming a radiotelescope are placed as far away from each other as possible: ideally, on different continents. The resulting Very Long Baseline Interferometry method (VLBI, for short) works as follows: whenever a pair of antennas is oriented towards a radio source (e.g., a quasar), we record the signals $s_1(t)$ and $s_2(t)$ on these two antennas, and compare the records. From trigonometry, one can easily deduce that the difference between the lengths of the paths from the source to the two antennas is equal to $\tau = \vec{B} \cdot \vec{s}$, where $\vec{B}$ is a *baseline* (i.e., a vector from the first to the second antenna), and $\vec{s}$ is a unit vector in the direction of the radio source. This difference in paths leads to the corresponding difference $\Delta t = \tau / c$ between the times when the same signal reaches the two antennas (here, c is the speed of light, with which the radio signal travels). Thus, the signal $s_1(t)$ recorded by one of the antennas is *delayed* by Δt from the signal recorded by the second one. Hence, by comparing the signals $s_1(t)$ and $s_2(t)$, we can determine the delay Δt and therefore, the value $\tau = c \cdot \Delta t = \vec{B} \cdot \vec{s}$.

Our goal is to determine the source location (i.e., the vector $\vec{s}$). If we knew the baselines *exactly*, then we would get a system of linear equations for finding $\vec{s}$. In real life, we only know the *approximate* values of $\vec{B}$, and the exact values of the baselines must be determined by the same measurements. In other words, for different baselines $\vec{B}_i$ and for different sources $\vec{s}_j$, we measure the values $\tau_{ij} = \vec{B}^{(i)} \cdot \vec{s}^{(j)}$; we would like to extract, from the exact measurement results, the exact values of the source locations $\vec{s}^{(j)}$.

The corresponding problem has two aspects:

- First, a *theoretical* (*fundamental*) aspect: If we make a sufficient number of measurements, *can we*, in principle, uniquely reconstruct all the locations? If we cannot reconstruct all the locations uniquely, then what exactly information about the source locations can be determined?
- Second, a *practical* (*computational*) aspect: *how* can we actually extract the locations $\vec{s}^{(j)}$ (or whatever information we can) from the measurement results τ_{ij}?

2 Chu Spaces and Automorphisms

2.1 General Description of Data Fusion: Chu Spaces

We have described the data fusion problem on one specific example. In general:

- We have a set of objects of interest which we will denote by X; in the above example, each object of interest $x \in X$ is characterized by a unit vector $\vec{s}$, so we can say that X is the set of all possible unit vectors.
- We also have a set of measuring instruments (or estimators) which will be denoted by A; in the above example, measuring instruments are pairs of antennas; each pair is characterized by its baseline vector $\vec{B}$, so we can say that A is the set of all (3-D) vectors.
- We assume that the construction of measuring instruments is known, and therefore, if we know the exact parameters of the object $x \in X$ and the exact parameters of the measuring instrument $a \in A$, then we can uniquely predict the measurement result; this measurement *r*esult will be denoted by $r(x, a)$, and the set of all possible measurement results will be denoted by K. In mathematical terms, we have a *map* r from $X \times A$ to K. In the above example, K is the set $\mathbb{R}$ of all real numbers, and $r(\vec{B}, \vec{s}) = \vec{B} \cdot \vec{s}$.

In mathematical terms, a general data fusion situation can be thus described as a *triple* (X, r, A), where X and A are arbitrary sets, and r is a map $r : X \times A \to K$ into the set K. Such triples are called K-*Chu spaces*, or simply *Chu spaces* [1] (when the choice of K is clear). Chu spaces have been successfully used to describe parallelism [5], information flow in distributed systems [2], etc.

2.2 General Formulation of a Fundamental Problem of Data Fusion: Chu Automorphisms

In the above terms, the fundamental problem of data fusion can be reformulated as follows: in the ideal situation, when we know the results of all the measurements, can we uniquely reconstruct all the objects? In other words, if we know the values $r(x, a)$ for all $x \in X$ and all $a \in A$, will we be able to reconstruct all x, or it is possible to mis-interpret every object x as a different object $f(x)$, so that under a certain associated mis-interpretation $a \to h(a)$ of the measuring instruments, the results are still the same:

$$r(x, a) = r(f(x), h(a)) \qquad (1)$$

In other words, the unique reconstruction is possible if and only if there are no non-trivial pairs (f, h) with a property (1), and if there are such pairs, then we can only reconstruct x uniquely modulo transformations $x \to f(x)$.

For mathematical reasons, it is sometimes convenient to consider the *inverse* transformation $g(a) = h^{-1}(a)$. In terms of the inverse transformation, the condition (1) takes the form

$$r(x, g(a)) = r(f(x), a). \qquad (2)$$

A pair of functions which satisfies this property is called an *automorphism* of a Chu space (X, r, A). Thus, the data fusion problem has a unique solution if and only if the corresponding Chu space does not have any non-trivial automorphisms, and if its has, then we only have uniqueness modulo these automorphisms.

For example, for VLBI radioastrometry, there is no uniqueness, because we can apply a *rotation* $f(x)$ and a similar rotation $h(a)$, and the resulting scalar (dot) product will not change. One can prove, however, that this is the only possible non-uniqueness, i.e., that the only pair of transformations (f, h) which satisfies the property (1) is a pair of identical rotations. Thus, from VLBI measurements, we can reconstruct the locations of all radiosources modulo rotation: e.g., we can reconstruct the *arcs* between the sources.

From the physical viewpoint, the fact that we cannot uniquely reconstruct the coordinates of all the sources makes perfect sense: the axes of the coordinate system are determined only by convention, so this non-uniqueness simply means that we can select an arbitrary Cartesian coordinate system.

2.3 From Theoretical Analysis to Practical Data Fusion

We have just shown that Chu spaces allow us to answer a *theoretical* question about data fusion. Let us now show that we can also get a *practical* data fusion algorithm out of this analysis.

In our example, both sets X and A are represented as *manifolds*, i.e., each element $x \in X$ can be characterized by several numerical characteristics ("coordinates") $x_1, \ldots, x_n$, and each element $a \in A$ can be characterized by several numerical characteristics $a_1, \ldots, a_m$ (in this example, $n = 2$ and $m = 3$). In general, when X and A are manifolds, a uniqueness theoretical result leads to a *practical* algorithm. Namely, we know:

- the measurement results $r_{ij} = r(x^{(i)}, a^{(j)})$,
- the approximate values $\widetilde{x}^{(i)}$ of the parameters $x^{(i)}$ which characterize the objects, and
- the approximate values $\widetilde{a}^{(j)}$ of the parameters $a^{(j)}$ which characterize the measuring instruments.

To find the exact values $x^{(i)}$ and $a^{(j)}$ of these parameters, it is sufficient to find the differences $\Delta x^{(i)} = x^{(i)} - \widetilde{x}^{(i)}$ and $\Delta a^{(j)} = a^{(j)} - \widetilde{a}^{(j)}$. In terms of these unknown differences, we have $x^{(i)} = \widetilde{x}^{(i)} + \Delta x^{(i)}$ and $a^{(j)} = \widetilde{a}^{(j)} + \Delta a^{(j)}$, and the above expression for r_{ij} takes the form

$$r_{ij} = r(\widetilde{x}^{(i)} + \Delta x^{(i)}, \widetilde{a}^{(j)} + \Delta a^{(j)}). \quad (3)$$

The approximate values are usually reasonably good, so these differences are small, and we can therefore expand the right hand side of the equation (3) into Taylor series and ignore quadratic and higher order terms in this expansion. As a result, we get the following system of linear equations for determining the unknown differences:

$$\sum_{\alpha=1}^{n} A_{ij\alpha} \cdot \Delta x_\alpha^{(i)} + \sum_{\beta=1}^{m} B_{ij\beta} \cdot \Delta a_\beta^{(j)} = \Delta r_{ij}, \quad (4)$$

where:

$$A_{ij\alpha} = \frac{\partial r(x^{(i)}, a^{(j)})}{\partial x_\alpha^{(i)}}_{|x^{(i)} = \widetilde{x}^{(i)}, a^{(j)} = \widetilde{a}^{(j)}},$$

$$B_{ij\beta} = \frac{\partial r(x^{(i)}, a^{(j)})}{\partial a_\beta^{(j)}}_{|x^{(i)} = \widetilde{x}^{(i)}, a^{(j)} = \widetilde{a}^{(j)}},$$

$$\Delta r_{ij} = r_{ij} - r(\widetilde{x}^{(i)}, \widetilde{a}^{(j)}).$$

Solving a system of linear equations is easy.

For a detailed description of our example – and for a more realistic description of VLBI astrometry which takes into consideration the inaccuracy of the clocks – see, e.g., [3, 4].

3 Other Examples of Data Fusion

In the previous section, we showed that Chu spaces can be used to formalize a general class of data fusion problems. Data fusion is a very general concept which includes situations more general than the ones described above. In this section, we enumerate such situations; in the following section, we will argue that (at least some of) these more general situations can also be naturally described in terms of Chu spaces.

3.1 Classical Statistics

In the above example, we assumed that the measurement result is uniquely determined if we know the object x and the measuring instrument a. In real life, there are a lot of random factors (noise), as a result of which, repeated measurements of the same object leads, as a rule, to slightly different results. So, instead of the *exact* value of $r(x,a)$, we have a *probability distribution* on the set of all measurement results. The measurement results may vary, but the *probability distribution* is uniquely determined by the measurement situation (i.e., by the pair of an object and of a measuring instrument).

Let X denote the set of all measurement results x, Θ be the set of all possible measurement situations θ, and let

$$\mathcal{F} = \{f(x,\theta) \,|\, x \in X, \theta \in \Theta\}$$

denote the class of all corresponding probability density functions $f(x,\theta)$. As a result of repeated measurements, we observe a random sample $x_1,\ldots,x_n$ from X. Based on this sample, we want to estimate either the value θ (i.e., the probability distribution itself), or some characteristic $\varphi(\theta)$ of this distribution (e.g., the standard deviation). Each of the measurement results x_i provides some estimate for $\varphi(\theta)$; to get a better estimate, we must "fuse" these estimates into a single estimate depending on all the measurement results $x_1,\ldots,x_n$. Usually, we seek some "good" estimator $T(x_1,\ldots,x_n)$, in fact, the *best* one, e.g., in the sense that it will maximize (or minimize) some performance characteristic (e.g., the expected squared deviation of our estimate from the true value of $\varphi(\theta)$).

The same is true in general: *we look for fusion operator which optimizes a given performance characteristic.*

3.2 Coalitional Games

Coalitional games, i.e., situations where several participants have different interests but are willing to cooperate, are non-measurement examples of data fusion.

Let us denote the set of players (participants) by Ω. In a coalitional game, every subset $A \subseteq \Omega$ can form a *coalition*, i.e., act together as a group against all the others. For each possible coalition (i.e., for each subset $A \subseteq \Omega$), we thus get a zero-sum (antagonistic) game, and we can use known techniques to determine the *payoff* $G(A)$ of this game. Thus, a coalitional game can be described as a *set-function* $G : 2^\Omega \to I\!R$. This function is *monotone* in the sense that increasing the coalition increases its payoff (if $A \subset B$, then $G(A) \le G(B)$). The main objective of coalitional game theory is to avoid the time-consuming coalition forming and dissolving process, and to come up with a solution which is fair to all the participants. In other words, we must "fuse" (combine) the payoffs $G(A)$ corresponding to different coalitions into a single solution.

As a desired performance characteristic, we can take, e.g., *fairness* (in situations describing distribution of goods), *productivity* (in situations describing the production of goods), etc.

In mathematics, the most well-known example of a function $2^\Omega \to R$ is *measure* – an additive function μ from the set 2^Ω of subsets of Ω to the set of real numbers $I\!R$. The most natural operation which maps a measure μ to a number is a (Lebesgue) *integral* $\int f\,d\mu$. Payoff functions are not necessarily additive, so, to describe the corresponding fusion, we can, e.g., use *Choquet integrals* – a generalization of Lebesgue integrals to monotone (not necessarily additive) set-functions. This indeed leads to reasonable solutions.

3.3 Expert Systems

A typical problem for which an expert system is useful is to predict, based on the known symptoms, whether or not an individual with these symptoms has a certain disease. To solve this diagnostic problem, we solicit the knowledge of an expert. An expert usually formulates his or her knowledge in terms of different *rules*; these rules form what is called a *rule base.* For a given patient, different rules lead to different degree of confidence that the patient has (or does not have) the disease in question. The main goal of the expert system is to combine ("fuse") these (sometimes conflicting) degrees of confidence into a single result.

3.4 Probabilistic Inference

Similar to the previous example, we consider the problem of diagnosing a certain type of disease. Let $X = (X_t, t \in T)$ be the set of all variable which describe a patient: i.e., the variables which characterize the degree of the disease, the directly measurable variables (like body temperature, blood pressure, etc.), which are used in describing the symptoms, and the variables which are not directly measurable but which are used in the expert's arguments about the disease. Usually, the set T of these variables has some *neighborhood* structure in the sense that some pairs of variables t, t' are closely related ("close", t' belongs to the neighborhood N_t of t) while other pars are not directly related ("not close"). For example, we may be able to place all these variables on a plane so that "close" variables are the ones for which the distance is smaller than a certain threshold. The notion of a neighborhood structure is naturally formalized by a condition $P(X_t \,|\, X_s, s \neq t) = P(X_t \,|\, X_s, s \in N_t\}$ which describes a *Markov random field.*

From the experience, we can collect the conditional probabilities $P(X_t = x \,|\, X_s = y)$ which describe our degree of confidence in a rule "if $X_t = x$ then $X_s = y$". The main objective of data fusion is to combine these probabilities into a single symptom-determined probability of the given disease.

3.5 Randomness and Fuzziness

In the above fusion problems, all pieces of information had *the same* type of uncertainty. Here is a situation where *different* types of uncertainty can coexist in data.

In his pioneering work on random elements in metric spaces, Fréchet pointed out that besides standard random objects (such as points, vectors, functions), nature, science, and technology offer other random elements which, he claimed, "cannot be described mathematically". For example, for a randomly chosen group of people, we may be interested in their "morality" or "spirit"; for a randomly chosen town, its "beauty" of "shape" may be of interest, etc. Nowadays, these "fuzzy" concepts are described mathematically as *fuzzy sets.* Thus, examples of Fréchet are *random fuzzy sets.*

The existence of the two types uncertainty – randomness and fuzziness – requires new fusion procedures.

4 Chu Spaces and Morphisms As A Description of General Data Fusion Problems

4.1 Chu Morphisms

As we have already argued, each measurement procedure, each type of uncertainty, can be characterized by a Chu space. In some real-life situations, we must combine *different* types of uncertainty (e.g., random and fuzzy), so, we must consider relations between *different* Chu spaces.

It's possible to combine, e.g., probabilistic and fuzzy approaches: a fuzzy set can be described as a random set and thus, combined with probabilities. However, these combinations are complicated and hardly practical.

In general, for each type of uncertainty, we have a list of objects X and a list of properties A. Ideally, we would like to know exactly which object has which property; due to uncertainty, however, we only have the "degree" (probability, degree of certainty, etc.) $r(x,a)$ to which an object x has the property a. So, a general piece of uncertain knowledge can be described as a K-Chu space (X, r, A), where K is the set of all possible degrees (usually, $K = [0,1]$).

Often, to check whether an object has a certain property, we design a similar object (e.g., a scaled version), find its properties, and then make conclusions about the properties of the original system. In other words, we have a transformation $f : X \to Y$ which maps each object into a new one, and a transformation $g : B \to A$ which transforms the properties of the new object into properties of the old one in such a way that if the object $f(x)$ has a property b, then the original object x has the corresponding property $g(b)$, i.e., that

$$s(f(x), b) = r(x, g(b)). \quad (5)$$

A pair (f, g) is called a *morphism* between the Chu spaces (X, r, A) and (Y, s, B).

4.2 Categories of Chu Spaces

For every K-Chu space, a pair of identical maps is an (auto)morphism. If (f, g) is a morphism between K-Chu spaces (X, r, A) and (Y, s, B), and (Z, t, C) is another K-Chu space with (u, v) being a morphism from (Y, s, B) to it, then there is a morphism from (X, r, A) to (Z, t, C) given by

$$(f, g) * (u, v) = (u \circ f, g \circ v). \quad (6)$$

In the terminology of Category Theory, this means that K-Chu spaces and morphisms form a *category* in which a morphism composition is defined by the formula (6). This category will be denoted by $\mathcal{CHU}(\mathcal{K})$.

4.3 Fuzzy Sets as Chu Spaces

In fuzzy set theory, for a given set of objects X, properties are described as *fuzzy subsets*, i.e., $A = [0,1]^X = \{a : X \to [0,1]\}$, and the degree $r_X(x, a)$ to which an object x satisfies the property a is described as $r_X(x, a) = a(x)$.

Let us denote the $[0,1]$-Chu category of the corresponding Chu spaces $F(X) = (X, r_X, [0,1]^X)$ by $\mathcal{FUZZ}$. The morphisms of this category are easy to describe: if $f : X \to Y$ is a function from X to Y, then the pair $F(f) = (f, \varphi_f)$, where $\varphi_f : [0,1]^Y \to [0,1]^X$ is defined by a formula $(\varphi_f(b))(x) = b(f(x))$, is a morphism $F(f) : F(X) \to F(Y)$. By choosing an arbitrary function $f : X \to Y$, we can conclude that there exists a morphism between every two objects of the category $\mathcal{FUZZ}$.

It is easy to check that F preserves composition, i.e., $F(h \circ f) = F(h) * F(f)$, and therefore, that F is a *covariant functor* from the category $\mathcal{SET}$ of sets and functions to $\mathcal{FUZZ}$.

4.4 Chu Category of Conditional Probabilities

In a probabilistic approach to diagnosis, the basic pieces of information (which are combined in data fusion) consist of conditional probabilities $P(a|b)$ for different events a and b. So here, X and A are both sets of events, and $r(x, a) = P(x|a)$. Let us describe the corresponding Chu space in precise terms.

A *probability (measure) space* is usually defined as a triple $\Omega = (\Omega, P, \mathcal{A})$, where $\mathcal{A}$ is a σ-field over a set Ω, and $P : \mathcal{A} \to [0,1]$ is a probability measure on $\mathcal{A}$. For each probability space Ω, we define the corresponding Chu space as a triple $P(\Omega) = (\mathcal{A}, r_P, \mathcal{A})$, where $r_P(a, b) = P(a|b)(= P(a \cap b)/P(b))$ if $P(b) > 0$, and $r_P(a, b) = 0$ if $P(b) = 0$ (i.e, if the above formula for conditional probability cannot be directly applied).

How can we describe morphisms between these Chu spaces? Let $\Omega = (\Omega, P, \mathcal{A})$ and $\Sigma = (\Sigma, Q, \mathcal{B})$ be probability spaces. A mapping $\varphi : \Sigma \to \Phi$ is called *measurability preserving* if it is one-to-one, $\varphi(\Omega) = \Sigma$, and both φ and φ^{-1} are measurable transformations. A measurability preserving transformation is called *measure preserving* if $P(\varphi^{-1}(b)) = Q(b)$ for every $b \in \mathcal{B}$, and *isomorphic* if both φ and φ^{-1} are measure preserving. We say that a pair (φ, ψ) of measurability preserving maps is *mutually measure preserving* if $P(a \cap \varphi^{-1}(b)) = Q(\psi^{-1}(a) \cap b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. One can prove that a composition of mutually measure preserving maps is measure preserving:

Proposition 1. *Let $\Omega = (\Omega, P, \mathcal{A})$, $\Sigma = (\Sigma, Q, \mathcal{B})$, and $\Gamma = (\Gamma, R, \mathcal{C})$ be probability spaces, and let $\varphi : \Omega \to \Sigma$, $\psi : \Sigma \to \Omega$, $\theta : \Sigma \to \Gamma$, and $\lambda : \Gamma \to \Sigma$ be measurability preserving maps. If (φ, ψ) and (θ, λ) are mutually measure preserving, then $(\theta\varphi, \psi\lambda)$ is also mutually measure preserving.*

One can prove that if a pair is mutually measure preserving, then the corresponding mapping are also measure preserving; thus, they preserve conditional probabilities and define a Chu morphism:

Proposition 2. *If (φ, ψ) is mutually measure preserving, then both φ and ψ are measure preserving.*

Proposition 3. *Let $\Omega = (\Omega, P, \mathcal{A})$, $\Sigma = (\Sigma, Q, \mathcal{B})$ be probability spaces, and let $\varphi : \Omega \to \Sigma$ and $\psi : \Sigma \to \Omega$ be measurability preserving maps. Then, the pair (φ, ψ) is mutually measure preserving if and only if the mapping $(\psi^{-1}, \varphi^{-1})$ is a Chu morphism $P(\Omega) \to P(\Sigma)$.*

An example of mutually measure preserving transformation is given by the following proposition:

Proposition 4. *If both φ and φ^{-1} are measure preserving, then the pair (φ, φ^{-1}) is mutually measure preserving.*

5 Cross Product Of Chu Spaces As A Data Fusion Operation

5.1 Motivating Example

In traditional probability theory, conditional probability $P(a|b)$ is defined for events a and b from the same σ-field of events. However, from the practical viewpoint, we start with *two* different sets of properties and, correspondingly, two different σ-fields: a σ-field $\mathcal{A}$ of events related to disease and a σ-field of events $\mathcal{B}$ related to symptoms; the only reasons why we have to combine these events is because otherwise, we will not be able to use the probability formalism.

How can we describe this "combination"? To even *describe* the conditional probability $P(a|b)$ of a given disease under given symptoms, we must represent the symptoms and diseases within the same probability space. We can achieve it in two ways:

- We can describe the symptoms in the disease space. For that, we need a transformation $g : \mathcal{B} \to \mathcal{A}$ which reformulates each disease-related property b into diseases-related terms: e.g., "sneezing" would translate into "cold or allergy". In this case, the desired conditional probability of a disease a under the symptoms b can be formalized as $P(a|g(b))$.
- We can also describe the diseases in terms of symptoms. For that, we need a transformation $f : \mathcal{A} \to \mathcal{B}$ which reformulates each symptom-related property a into symptom-related form. In this case, the desired conditional probability of a disease a under the symptoms b can be formalized as $P(f(a)|b)$.

The resulting conditional probability should not depend on how exactly we define it, and therefore, the corresponding two expressions must coincide:

$$P(a|g(b)) = P(f(a)|b). \qquad (7)$$

5.2 Reformulation in Terms of Chu Spaces

Let us re-describe the above construction in terms of Chu spaces. If we take into consideration that for probability Chu spaces, $P(a|b) = r(a, b)$, then the formula (7) turns into the formula (2), which defines a Chu morphism.

Thus, in terms of Chu spaces, we have the following situation:

- Originally, we had two Chu spaces $P(\Omega = (\mathcal{A}, r_A, \mathcal{A})$ and $P(\Sigma) = (\mathcal{B}, r_B, \mathcal{B}))$, and a Chu morphism $(f, g) : P(\Omega) \to P(\Sigma)$.
- Based on this information, we design a *new* Chu space $(\mathcal{A}, r_{\text{new}}, \mathcal{B})$ for which $r_{new}(a, b) = r_A(a, g(b)) = r_B(f(a), b)$.

This constructed can be repeated for an arbitrary morphism between two Chu spaces:

- We start with two Chu spaces $\mathcal{X} = (X, r, A)$ and $\mathcal{Y} = (Y, s, B)$ and a Chu morphism $F = (f, g) : (X, r, A) \to (Y, s, B)$.
- Based on this information, we design a new Chu space (X, t, B), with $t(a, b) = r(a, g(b)) = s(f(a), b)$.

This new Chu space is called a *cross-product* of two original Chu spaces with respect to the morphism (f, g) and denoted by $\mathcal{X} \otimes_F \mathcal{Y}$.

5.3 One More Possible Application of Chu Cross Product to Data Fusion: Fuzzy Logic

In traditional fuzzy approach, fuzzy logic operations ("and", "or") are used to combine fuzzy data. This combination lacks the ability to describe relationship between the fused data. The notion of a Chu cross-product gives us a general way of describing such a relationship. So, we get the following new method of fusing two pieces of fuzzy data:

- first, we find the Chu morphism which best describes the relationship between these two pieces of data, and
- then, we combine these pieces relative to this morphism (by using a cross-product construction).

6 Conclusion

In general, different parts of information are expressed in different forms, such as probabilistic information, fuzzy information, etc. To combine ("fuse") this information, we must describe all types of uncertainty in terms of a single general formalism. In this paper, we have described a new general scheme for data fusion based on the notion of Chu spaces, and presented the corresponding results.

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