Complex Fuzzy Sets: Towards New Foundations

Hung T. Nguyen
Mathematics, New Mexico State University, Las Cruces, NM 88003, USA, hunguyen@nmsu.edu

Abraham Kandel
Computer Science & Engineering, University of South Florida, Tampa, FL 33620, USA

Vladik Kreinovich
Computer Science, University of Texas, El Paso, TX 79968, USA, vladik@cs.utep.edu

***Abstract*—Uncertainty of complex-valued physical quantities $z = x + \mathrm{i}y$ can be described by complex fuzzy sets. Such sets can be described by membership functions $\mu(x, y)$ which map the universe of discourse (complex plane) into the interval $[0, 1]$. The problem with this description is that it is difficult to directly translate into words from natural language. To make this translation easier, several authors have proposed to use, instead of a single membership function for describing the complex number, several membership functions which describe different real-valued characteristics of this numbers, such as its real part, its imaginary part, its absolute value, etc. The quality of this new description strongly depends on the choice of these real-valued functions, so it is important to choose them optimally. In this paper, we formulate the problem of optimal choice of these functions and show that, for all reasonable optimality criteria, the level sets of optimal functions are straight lines and circles. This theoretical result is in good accordance with our numerical experiments, according to which such functions indeed lead to a good description of complex fuzzy sets.**

Many practical problems lead to complex fuzzy sets. Many physical quantities are complex-valued: wave function in quantum mechanics, complex amplitude and impedance in electrical engineering, etc.

In all these problems, expert uncertainty means that we do not know the exact value of the corresponding complex number; instead, we have a fuzzy knowledge about this number.

From complex fuzzy numbers of Kaufmann and Gupta to Buckley's membership function description. In order to describe a complex number $z = x + \mathrm{i}y$, we must describe two real numbers: its real part x and its imaginary part y. Thus, a natural idea is to represent a complex fuzzy number by describing two real fuzzy numbers: x and y (see, e.g., [2]) characterized by the corresponding membership functions $\mu_1(x)$ and $\mu_2(y)$. In this approach, for every complex value $x + \mathrm{i}y$, i.e., for every pair (x, y), the degree $\mu(x, y)$ with which this complex value is possible can be defined as

$$\mu(x, y) = \min(\mu_1(x), \mu_2(y)). \qquad (1)$$

Then, for each $\alpha \in [0, 1]$, the α-cut for the real part x is an interval $[x^-(\alpha), x^+(\alpha)]$, the α-cut for the imaginary part is also an interval $[y^-(\alpha), y^+(\alpha)]$, and hence, the α-cut for the resulting 2-D membership function (1) is a rectangular "box" $[x^-(\alpha), x^+(\alpha)] \times [y^-(\alpha), y^+(\alpha)]$. The boundary of this box consists of two straight line segments which are parallel to the x axis, and of two straight line segments which are parallel to the y axis.

In some practical problems, e.g., when the analyzed complex fuzzy number z is the result of applying an exponential function to some other complex number, its α-cuts may have a more complicated shape than a rectangle.

Some such situations can be described by using the fact that in many practical problems, it is more convenient to represent a complex number not in the form $z = x + \mathrm{i}y$ (which corresponds to Cartesian coordinates in the plane (x, y)), but in the form $z = \rho \cdot \exp(\mathrm{i}\theta)$ (which corresponds to polar coordinates (ρ, θ) in this plane. For such practical problems, A. Kaufmann and M. Gupta proposed, in [2], to use a *goniometric* representation in which a complex fuzzy number is represented by a pair of real fuzzy numbers ρ and θ, with membership functions $\mu_1(\rho)$ and $\mu_2(\theta)$. In this approach, for every complex value $x + \mathrm{i}y$, the degree $\mu(x, y)$ with which this complex value is possible can be defined as

$$\mu(x, y) = \min(\mu_1(\rho), \mu_2(\theta)), \qquad (2)$$

where $\rho = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ are the values of polar coordinates of the point whose Cartesian coordinates are x and y. For this representation, the α-cut is a set of all vectors for which ρ is between $\rho^-(\alpha)$ and $\rho^+(\alpha)$, and the angle θ is between $\theta^-(\alpha)$ and $\theta^+(\alpha)$. On a (x, y)-plane, this α-cut is no longer a rectangle: it is a rather complicated geometric figure which is bounded by two radial straight line segments (corresponding to $\theta = \theta^-(\alpha)$ and $\theta = \theta^+(\alpha)$) and by two circular segments (corresponding to $\rho = \rho^-(\alpha)$ and $\rho = \rho^+(\alpha)$).

Some complex fuzzy numbers have even more complicated α-cuts, or, in other words, membership functions which cannot be described by the expressions (1) and (2). To describe such complex fuzzy numbers, it is natural to use a general membership function $\mu(x, y)$ which maps a complex plane C into the interval $[0, 1]$. This approach was sketched in the above-mentioned book [2] and thoroughly developed in [1] (for the latest overview, see, e.g., [5]).

Why membership function description is sometimes not sufficient. From the purely mathematical viewpoint, this very general approach, in which use a general 2-D membership function, is very natural. However, there is one problem with this approach: A membership function is not something which is natural for a human to understand and to use. It was invented as a way of representing human fuzzy knowledge in a language which is understandable for a computer. From this viewpoint, after we get the desired membership function, we must perform one more step: we must translate it into the natural language.

This translation is difficult even for real variables; however, for real numbers, we have accumulated a lot of intuition, and we are often able to describe different 1-D membership functions by natural-language words such as "small", "close to 0", etc. Unfortunately, complex numbers are much less intuitive, and there are few terms of natural language which can be naturally used to described the knowledge about complex numbers.

A new approach to describing complex fuzzy sets which combines the generality of Buckley's approach with the intuitiveness of Kaufmann-Gupta descriptions. Since the original membership function $\mu : C \to [0,1]$ is difficult to interpret directly, the authors of [3] proposed a new approach. In this approach, to describe a fuzzy knowledge about a complex number, we use, instead of *single* membership function which describes the number $z = x + \mathrm{i}y$, *several* (two or more) membership functions which describe real-valued quantities which are *functions* of this complex number, such as its real part $\mathrm{Re}(z)$, its imaginary part $\mathrm{Im}(z)$, its absolute value $\rho = |z|$, its phase θ, etc.

In other words, instead of describing a single 2-D membership function $\mu(x,y)$, we describe several (two or more) membership functions $\mu_1(t_1), \ldots, \mu_k(t_k)$ corresponding to different real-valued characteristics $t_1 = f_1(x,y), \ldots, t_k = f_k(x,y)$ of this complex number $z = x + \mathrm{i}y$. In this approach, for every complex value $x + \mathrm{i}y$, the degree $\mu(x,y)$ with which this complex value is possible can be defined as

$$\mu(x,y) = \min(\mu_1(t_1), \ldots, \mu_k(t_k)), \qquad (3)$$

where $t_i = f_i(x,y)$.

If we use two characteristics $t_1 = f_1(x,y) = x$ and $t_2 = f_2(x,y) = y$, then we get complex numbers of type (1), in which α-cuts are rectangles.

If we use two characteristics $t_1 = f_1(x,y) = \rho = \sqrt{x^2 + y^2}$ and $t_2 = f_2(x,y) = \theta = \arctan(y/x)$, then we get complex numbers of type (2), in which α-cuts are above-described segments.

It turns out that in many practical problems, it is useful to use three or four different characteristics. For example, we can combine Cartesian and polar ones into a single 4-characteristic set, with $t_1 = f_1(x,y) = x$, $t_2 = f_2(x,y) = y$, $t_3 = f_3(x,y) = \rho = \sqrt{x^2 + y^2}$, and $t_4 = f_4(x,y) = \theta = \arctan(y/x)$. In this case, the α-cut is an intersection of a rectangle (corresponding to (1)) and a segment (corresponding to (2)), i.e., a set whose boundary consists partly of straight line circular arcs, partly of radial straight line segments, and partly of segments which are parallel to x or y axes.

Which real-valued characteristics of complex numbers should we use in this description of complex fuzzy numbers? As shown in [3], the efficiency of the new description in solving practical problems with complex numbers strongly depends on the appropriate choice of the real-valued characteristics which are used to describe the corresponding fuzzy set: a good choice can drastically improve the quality of the result. It is therefore important to find out which functions are the best here. This is the problem that we will be solving in the present paper.

Preliminary step: reformulation in terms of sets. The membership function $\mu_f(t)$ corresponding to a characteristic $f : C \to R$ can be described by the extension principle:

$$\mu_f(t) = \max_{z:\ f(z)=t} \mu(t).$$

Thus, to be able to compute all the values $\mu_f(t)$, we do not need to compute know the exact characteristics $f(t)$; it is sufficient to be able to describe their level sets $\{z \mid f(z) = t\}$. So, instead of choosing the best characteristics, we can choose a family of sets.

For the above characteristics $f_i(x,y)$, these level sets are straight lines and circles.

Of course, the more parameters we allow in the description of a family, the more elements this family contains, and therefore, the better the representation. So, the question can be reformulated as follows: for a given number of parameters (i.e., for a given dimension of approximating family of sets), which is the best family? In this paper, we formalize and solve this problem.

Formalizing the problem. All proposed families of sets have analytical (or piece-wise analytical) boundaries, so it is natural to restrict ourselves to such families. By definition, when we say that a piece of a boundary is analytical, we mean that it can be described by an equation $F(x,y) = 0$ for some analytical function

$$F(x,y) = a + b \cdot x + c \cdot y + d \cdot x^2 + e \cdot x \cdot y + f \cdot y^2 + \ldots$$

So, in order to describe a family, we must describe the corresponding class of analytical functions $F(x,y)$.

For example, the level set $f_1(x,y)=x=t$ can be described an equation $F(x,y)=0$ for an analytical function $F(x,y)=x-t$; the level set $f_2(x,y)=y=t$ can be described by an analytical function $F(x,y)=y-t$; the level set $f_3(x,y)=\sqrt{x^2+y^2}=t$ can be described by an analytical function $F(x,y)=x^2+y^2-t^2$; and the level set $f_4(x,y)=\arctan(y/x)=t$ can be described by an analytical function $F(x,y)=y-x\cdot\tan(t)$.

Since we are interested in finite-dimensional families of sets, it is natural to consider finite-dimensional families of functions, i.e., families of the type

$$\{C_1\cdot F^{(1)}(x,y)+\ldots+C_d\cdot F^{(d)}(x,y)\},$$

where $F^{(i)}(z)$ are given analytical functions, and $C_1,\ldots,C_d$ are arbitrary (real) constants. So, the question is: which of such families is the best?

When we say "the best", we mean that on the set of all such families, there must be a relation $\geq$ describing which family is better or equal in quality. This relation must be transitive (if A is better than B, and B is better than C, then A is better than C). This relation is not necessarily asymmetric, because we can have two approximating families of the same quality. However, we would like to require that this relation be *final* in the sense that it should define a unique *best* family $A_{\rm opt}$ (i.e., the unique family for which $\forall B\,(A_{\rm opt}\geq B)$. Indeed:

- If none of the families is the best, then this criterion is of no use, so there should be *at least one* optimal family.
- If *several* different families are equally best, then we can use this ambiguity to optimize something else: e.g., if we have two families with the same approximating quality, then we choose the one which is easier to compute. As a result, the original criterion was not final: we get a new criterion ($A\geq_{\rm new}B$ if either A gives a better approximation, or if $A\sim_{\rm old}B$ and A is easier to compute), for which the class of optimal families is narrower. We can repeat this procedure until we get a final criterion for which there is only one optimal family.

It is reasonable to require that the relation $A\geq B$ should not change if we add or multiply all elements of A and B by a complex number; in geometric terms, the relation $A\geq B$ should be shift-, rotation- and scale-invariant.

Now, we are ready for the formal definitions.

Definition 1. *Let $d>0$ be an integer. By a d-dimensional family, we mean a family A of all functions of the type $\{C_1\cdot F^{(1)}(x,y)+\ldots+C_d\cdot F^{(d)}(x,y)\}$, where $F^{(i)}(z)$ are given analytical functions, and $C_1,\ldots,C_d$ are arbitrary (real) constants. We say that a set is defined by this family A if its border consists of pieces described by equations $F(x,y)=0$, with $F\in A$.*

Definition 2. *By an optimality criterion, we mean a transitive relation $\geq$ on the set of all d-dimensional families. We say that a criterion is final if there exists one and only one optimal family, i.e., a family $A_{\rm opt}$ for which $\forall B\,(A_{\rm opt}\geq B)$. We say that a criterion $\geq$ is shift- (corr., rotation- and scale-invariant) if for every two families A and B, $A\geq B$ implies $TA\geq TB$, where TA is a shift (rotation, scaling) of the family A.*

Theorem. ($d\leq 4$) *For every final optimality criterion $\geq$ which is shift-, rotation-, and scale-invariant, the border of every set defined by the optimal family $A_{\rm opt}$ consists of straight line intervals and circular arcs.*

General comment. This result is in good accordance with the fact that for the above-described characteristics – which have actually been used to describe complex fuzzy numbers – the boundary is indeed of this type. In particular, we get a new theoretical justification of the goniometric approach developed in [2].

This result is also in good accordance with the experiments described in [3], according to which such sets indeed provide a good description of complex fuzzy sets.

Practical comment. In *practical* terms, our conclusion is that we should use characteristics whose level sets are general straight line intervals and circular arcs; in other words, in addition to the above characteristics $x={\rm Re}(z)$, $y={\rm Im}(z)$, $\rho=|z|$ and $\theta(z)$, we should also use $f(z)={\rm Re}(z\cdot z_1+z_2)$ and $f(z)=|z-z_2|$ for arbitrary complex-valued constants z_1 and z_2 with $|z_1|=1$; these characteristics describe arbitrary straight lines and circles. The resulting general description of a complex fuzzy number can include several 1-D membership functions corresponding to such characteristics.

Proof of the Theorem. This proof is similar to the ones from [4].

1. Let us first show that the optimal family $A_{\rm opt}$ is itself shift-, rotation-, and scale-invariant.

Indeed, let T be an arbitrary shift, rotation, or scaling. Since $A_{\rm opt}$ is optimal, for every other family B, we have $A_{\rm opt}\geq T^{-1}B$ (where T^{-1} means the inverse transformation). Since the optimality criterion $\geq$ is invariant, we conclude that $TA_{\rm opt}\geq T(T^{-1}B)=B$. Since this is true for every family B, the family $TA_{\rm opt}$ is also optimal. But since our criterion is final, there is only one optimal family and therefore, $TA_{\rm opt}=A_{\rm opt}$. In other words, the optimal family is indeed invariant.

2. Let us now show that all functions from $A_{\rm opt}$ are polynomials.

Indeed, every function $F\in A_{\rm opt}$ is analytical, i.e., can be represented as a Taylor series (sum of monomials). Let us combine together monomials $c\cdot x^a\cdot y^b$ of the same degree $a+b$; then we get

$$F(z)=F_0(z)+F_1(z)+\ldots+F_k(z)+\ldots,$$

where $F_k(z)$ is the sum of all monomials of degree k. Let us show, by induction over k, that for every k, the function $F_k(z)$ also belongs to $A_{\rm opt}$.

Let us first prove that $F_0(z) \in A_{\rm opt}$. Since the family $A_{\rm opt}$ is scale-invariant, we conclude that for every $\lambda > 0$, the function $F(\lambda \cdot z)$ also belongs to $A_{\rm opt}$. For each term $F_k(z)$, we have $F_k(\lambda z) = \lambda^k \cdot F_k(z)$, so

$$F(\lambda z) = F_0(z) + \lambda \cdot F_1(z) + \ldots \in A_{\rm opt}.$$

When $\lambda \to 0$, we get $F(\lambda \cdot z) \to F_0(z)$. The family $A_{\rm opt}$ is finite-dimensional hence closed; so, the limit $F_0(z)$ also belongs to $A_{\rm opt}$. The induction base is proven.

Let us now suppose that we have already proven that for all $k < s$, $F_k(z) \in A_{\rm opt}$. Let us prove that $F_s(z) \in A_{\rm opt}$. For that, let us take

$$G(z) = F(z) - F_1(z) - \ldots - F_{s-1}(z).$$

We already know that $F_1, \ldots, F_{s-1} \in A_{\rm opt}$; so, since $A_{\rm opt}$ is a linear space, we conclude that

$$G(z) = F_s(z) + F_{s+1}(z) + \ldots \in A_{\rm opt}.$$

The family $A_{\rm opt}$ is scale-invariant, so, for every $\lambda > 0$, the function $G(\lambda \cdot z) = \lambda^s \cdot F_s(z) + \lambda^{s+1} \cdot F_{s+1}(z) + \ldots$ also belongs to $A_{\rm opt}$. Since $A_{\rm opt}$ is a linear space, the function $H_\lambda(z) = \lambda^{-s} \cdot G(\lambda z) = F_s(z) + \lambda \cdot F_{s+1}(z) + \lambda^2 \cdot F_{s+2}(z) + \ldots$ also belongs to $A_{\rm opt}$.

When $\lambda \to 0$, we get $H_\lambda(z) \to F_s(z)$. The family $A_{\rm opt}$ is finite-dimensional hence closed; so, the limit $F_s(z)$ also belongs to $A_{\rm opt}$. The induction is proven.

Now, monomials of different degree are linearly independent; therefore, if we have infinitely many non-zero terms $F_k(z)$, we would have infinitely many linearly independent functions in a finite-dimensional family $A_{\rm opt}$ – a contradiction. Thus, only finitely many monomials $F_k(z)$ are different from 0, and so, $F(z)$ is a sum of finitely many monomials, i.e., a polynomial.

3. Let us prove that if a function $F(x, y)$ belongs to $A_{\rm opt}$, then its partial derivatives $F_{,x}(x, y)$ and $F_{,y}(x, y)$ also belong to $A_{\rm opt}$.

Indeed, since the family $A_{\rm opt}$ is shift-invariant, for every $h > 0$, we get $F(x + h, y) \in A_{\rm opt}$. Since this family is a linear space, we conclude that a linear combination $h^{-1}(F(x + h, y) - F(x, y))$ of two functions from $A_{\rm opt}$ also belongs to $A_{\rm opt}$. Since the family $A_{\rm opt}$ is finite-dimensional, it is closed and therefore, the limit $F_{,x}(x, y)$ of such linear combinations also belongs to $A_{\rm opt}$. (For $F_{,y}$, the proof is similar).

4. Due to Parts 2 and 3 of this proof, if any polynomial from $A_{\rm opt}$ has a non-zero part F_k of degree $k > 0$, then it also has a non-zero part ($(F_k)_{,x}$ or $(F_k)_{,y}$) of degree $k - 1$. Similarly, it has non-zero parts of degrees $k - 2, \ldots, 1, 0$.

So, in all cases, $A_{\rm opt}$ contains a non-zero constant and a non-zero linear function $F_1(x, y) = b \cdot x + c \cdot y$. We can now use the fact that the family $A_{\rm opt}$ is rotation-invariant; let T be a rotation which transforms (b, c) into the x-axis, then we conclude that $F_1(Tz) = b'x \in A_{\rm opt}$, and hence $x \in A_{\rm opt}$. Similarly, $y \in A_{\rm opt}$. So, the family $A_{\rm opt}$ contains at least 3 linearly independent functions: a non-zero constant, x, and y.

If $d = 3$, then the 3-D family $A_{\rm opt}$ cannot contain anything else, and all the pieces of borders $F(x, y) = 0$ of all the sets defined by this family are straight lines.

If $d = 4$, then we cannot have any cubic or higher order terms in $A_{\rm opt}$, because then, due to Part 3, we would have both this cubic part *and* a (linearly independent) quadratic part, and the total dimension of $A_{\rm opt}$ would be at least $3 + 2 = 5$. So, all functions from $A_{\rm opt}$ are quadratic. Since $\dim(A_{\rm opt}) = 4$, and the dimension of 0- and 1-D parts is 3, the dimension of possible parts of second degree is 1. Since $A_{\rm opt}$ is rotation-invariant, the quadratic part $d \cdot x^2 + e \cdot x \cdot y + f \cdot y^2$ must be also rotation-invariant (else, we would have two linearly independent quadratic terms in $A_{\rm opt}$: the original expression and its rotated version). Thus, this quadratic part must be proportional to $x^2 + y^2$.

Hence, every function $F \in A_{\rm opt}$ has the form $F(x, y) = a + b \cdot x + c \cdot y + d \cdot (x^2 + y^2)$, and therefore, all the pieces of borders $F(x, y) = 0$ of all the sets defined by this family are either straight lines or circular arcs. The proposition is proven.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, by the National Security Agency under Grant No. MDA904-98-1-0561.

The authors are thankful to anonymous referees for their valuable comments.

References

[1] J. J. Buckley, "Fuzzy complex numbers", *Fuzzy Sets and Systems*, 1989, Vol. 33, pp. 333-345.

[2] A. Kaufmann and M. M. Gupta, *Introduction to fuzzy arithmetic*, Van Nostrand Reinhold Co., N.Y., 1985.

[3] D. Moses, O. Degani, H.-N. Teodorescu, M. Friedman, and A. Kandel, "Linguistic coordinate transformations for complex fuzzy sets", *Proc. FUZZ-IEEE'99*, Seoul, Korea, Vol. III, pp. 1340–1345.

[4] H. T. Nguyen and V. Kreinovich, *Applications of continuous mathematics to computer science*, Kluwer, Dordrecht, 1997.

[5] C. Wu and J. Qiu, "Some remarks for fuzzy complex analysis", *Fuzzy Sets Syst.*, 1999, Vol. 106, No. 2, pp. 231–238.