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Across-the-Board Spending Cuts Are Very Inefficient: A Proof

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Across-the-Board Spending Cuts Are Very Inefficient: A Proof

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Abstract In many real-life situations, when there is a need for a spending cut, this cut is performed in an across-the-board way, so that each budget item is decreased by the same percentage. Such cuts are ubiquitous, they happen on all levels, from the US budget to the university budget cuts on the college and departmental levels. The main reason for the ubiquity of such cuts is that they are perceived as fair and, at the same time, economically reasonable. In this paper, we perform a quantitative analysis of this problem and show that, contrary to the widely spread positive opinion about across-the-board cuts, these cuts are, on average, very inefficient.

1 Formulation of the Problem: Are Across-the-Board Spending Cuts Economically Reasonable

Across-the-board spending cuts are ubiquitous. When a department or even a country faces an unexpected decrease in funding, it is necessary to balance the budget by making some spending cuts.

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In many such situations, what is implemented is an across-the board cut, when all the spending items are decreased by the same percentage. For example, all the salaries are decreased by the same percentage.

The ubiquity of such cuts is motivated largely by the fact that since they apply to everyone on the same basis, they are fair.

Across-the-board cuts may sound fair, but are they economically efficient? The fact that such cuts are fair do not necessarily mean that they are economically efficient. For example, if we consistently take all the wealth of a country and divide it equally between all its citizens, this may be a very fair division, but, because of its lack of motivations to work harder, this clearly will not be a very economically efficient idea.

Current impression. The current impression that across-the-board cuts may not be economically optimal, but they are economically reasonable; see, e.g., [1, 3, 4, 5, 6, 8, 9, 10, 11].

What we show in this paper. In this paper, we perform a quantitative analysis of the effect of across-the-board cuts, and our conclusion is that their economic effect is much worse than it is usually perceived.

Comment. To make our argument as convincing as possible, we tried our best to make this paper – and its mathematical arguments – as mathematically simple and easy-to-read as we could.

2 Let Us Formulate The Problem in Precise Terms

Formulation of the problem in precise terms. Let us start by formulating this problem in precise terms.

What is given. First, we need to describe what we had before the need appeared for budget cuts. Let us denote the overall spending amount by x, and the amount originally allocated to different spending categories by $x_1, x_2, \ldots, x_n$, so that

$$\sum_{i=1}^{n} x_i = x.$$

Sometimes, it turns out that the original estimate x for the spending amount was too optimistic, and instead we have a smaller amount $y < x$.

What we need to decide. Based on the decrease amount $y < x$, we need to select new allocations, i.e., select the values $y_1 \leq x_1, \ldots, y_n \leq x_n$ for which $\sum\limits_{i=1}^{n} x_i = x$.

What is an across-the-board spending cut. An across-the-board spending cut means that for each i, we take $y_i = (1-\delta) \cdot x_i$, where the common value $\delta > 0$ is determined by the condition that $(1-\delta) \cdot x = y$. Thus, this value δ is equal to

$$\delta = 1 - \frac{y}{x}.$$

What we plan to analyze. We want to check whether the across-the-board spending cut $y_i = (1-\delta)\cdot x_i$ is economically reasonable, e.g., to analyze how it compares with the optimal budget cut.

We need to describe, in precise terms, what is better and what is worse for the economy. To make a meaningful comparison between different alternative versions of budget cuts, we need to have a clear understanding of which economical situations are preferable. In other words, we need to be able to consistently compare any two different situations.

It is known that such a linear (total) order on the set of all possible alternatives can be, under reasonable conditions, described by a real-valued functions $f(y_1,\ldots,y_n)$ defined on the set of such alternatives: for every two alternatives $(y_1,\ldots,y_n)$ and $(y'_1,\ldots,y'_n)$, the one with the larger value of this function is preferable (see, e.g., [12]):

- if $f(y_1,\ldots,y_n) > f(y'_1,\ldots,y'_n)$, then the alternative $(y_1,\ldots,y_n)$ is preferable;
- on the other hand, if $f(y'_1,\ldots,y'_n) > f(y_1,\ldots,y_n)$, then the alternative $(y'_1,\ldots,y'_n)$ is preferable.

The objective function should be monotonic. The more money we allocate to each item i, the better. Thus, the objective function should be increasing in each of its variables: if $y_i < y'_i$ for some i and $y_i \le y'_i$ for all i, then we should have $f(y_1,\ldots,y_n) < f(y'_1,\ldots,y'_n)$

We consider the generic case. In this paper, we do not assume any specific form of the objective function $f(y_1,\ldots,y_n)$. Instead, we will show that the same result – that across-the-board cuts are not efficient – holds for all possible objective functions (of course, as long as they satisfy the above monotonicity condition). So, whether our main objective is:

- to increase the overall GDP,
- or to raise the average income of all the poor people,
- or, alternatively, to raise the average income of all the rich people,

no matter what is our goal, across-the-board cuts are a far-from-optimal optimal way to achieve this goal.

Resulting formulation of the problem. We assume that the objective function $f(y_1,\ldots,y_n)$ is given.

We have the initial amount x. Based on this amount, we selected the values $x_1,\ldots,x_n$ for which $f(x_1,\ldots,x_n)$ attains the largest possible value under the constraint that $\sum_{i=1}^{n} x_i = x$. Let us denote the value of the objective function corresponding to this original budget allocation by f_x.

Now, we are given a different amount $y < x$. Ideally, we should now select the values $y_1, \ldots, y_n$ for which $f(y_1, \ldots, y_n)$ attains the largest possible value under the constraint that $\sum\limits_{i=1}^{n} y_i = y$. Due to monotonicity, the resulting best-possible value f_y of the objective function $f(y_1, \ldots, y_n)$ is smaller than the original value f_x.

In the across-the-board arrangement, instead of selecting the optimal values y_i, we select the across-the-board values $y_i = (1-\delta)\cdot x_i$, where $\delta = 1 - \frac{y}{x}$. The resulting allocation of funds is, in general, not as good as the optimal one. Thus, the resulting value of the objective function f_δ is, in general, smaller than f_y.

To decide how economically reasonable are across-the-board cuts, we need to compare:

- the *optimal* decrease $f_x - f_y$ in the value of the objective function, with
- the decrease $f_x - f_\delta$ caused by using across-the-board spending cuts.

3 Analysis of the Problem

Possibility of linearization. Usually, the relative size of the overall cut does not exceed 10%; usually it is much smaller. By economic standards, a 10% cut is huge, but from the mathematical viewpoint, it is *small* – in the sense that terms which are quadratic in this cut can be safely ignored. Indeed, the square of $0.1 = 10\%$ is $0.01 = 1\% \ll 10\%$.

Thus, if we expand the dependence of the objective function $f(y_1, \ldots, y_n)$ in Taylor series around the point $(x_1, \ldots, x_n)$, i.e., if we consider the dependence

$$f(y_1, \ldots, y_n) = f(x_1, \ldots, x_n) - \sum_{i=1}^{n} c_i \cdot (x_i - y_i) + \ldots = f_x - \sum_{i=1}^{n} c_i \cdot (x_i - y_i) + \ldots, \quad (1)$$

where $c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial y_i}$, then we can safely ignore terms which are quadratic in terms of the differences and conclude that

$$f(y_1, \ldots, y_n) = f_x - \sum_{i=1}^{n} c_i \cdot \Delta y_i,$$

where we denoted $\Delta y_i \stackrel{\text{def}}{=} x_i - y_i \ge 0$, and thus, that:

$$f_x - f(y_1, \ldots, y_n) = \sum_{i=1}^{n} c_i \cdot \Delta y_i. \quad (2)$$

Comment. Since the objective function $f(x_1, \ldots, x_n)$ is monotonic in each of the variables, all the partial derivatives c_i are non-negative: $c_i \ge 0$.

Linearization simplifies the problem: general idea. Let us describe how the use of linearization simplifies the computation of the two differences $f_x - f_y$ and $f_x - f_\delta$.

Linearization simplifies the problem: case of optimal spending cuts. Let us start with the computation of the difference $f_x - f_y$ corresponding to the optimal spending cuts. The optimal arrangement $(y_1, \ldots, y_n)$ is the one that maximizes the value of the objective function $f(y_1, \ldots, y_n)$ under the constraint

$$\sum_{i=1}^{n} y_i = y.$$

Maximizing the value of the objective function $f(y_1, \ldots, y_n)$ is equivalent to minimizing the difference $f_x - f(y_1, \ldots, y_n)$, which, according to the formula (2), is equivalent to minimizing the sum $\sum\limits_{i=1}^{n} c_i \cdot \Delta y_i$.

To make the problem easier to solver, let us also describe the constraint $\sum\limits_{i=1}^{n} y_i = y$ in terms of the new variables Δy_i. This can be achieved if we subtract this constraint from the formula $\sum\limits_{i=1}^{n} x_i = x$. As a result, we get an equality $\sum\limits_{i=1}^{n} \Delta y_i = \Delta y$, where we denoted $\Delta y \stackrel{\text{def}}{=} x - y$.

Thus, due to the possibility of linearization, the corresponding optimization problem takes the following form: minimize the sum $\sum\limits_{i=1}^{n} c_i \cdot \Delta y_i$ under the constraint

$$\sum_{i=1}^{n} \Delta y_i = \Delta y.$$

Let us prove that this minimum is attained when $\Delta y_{i_0} = \Delta y$ for the index i_0 corresponding to the smallest possible value of the derivative c_i, and $\Delta y_i = 0$ for all other indices $i \neq i_0$.

Indeed, for the arrangement when $\Delta y_{i_0} = \Delta y$ and $\Delta y_i = 0$ for all $i \neq i_0$, the minimized sum attains the value

$$\sum_{i=1}^{n} \Delta y_i = c_{i_0} \cdot \Delta y = \left(\min_i c_i \right) \cdot \Delta y.$$

Let us prove that for every other arrangement, we have a larger (or equal) value of the difference $f_x - f(y_1, \ldots, y_n)$. Indeed, by our choice of i_0, we have $c_i \geq c_{i_0}$ for all i. Thus, due to $\Delta y_i \geq 0$, we have $c_i \cdot \Delta y_i \geq c_{i_0} \cdot \Delta y_i$, and therefore,

$$\sum_{i=1}^{n} c_i \cdot \Delta y_i \geq \sum_{i=1}^{n} c_{i_0} \cdot \Delta y_i = c_{i_0} \cdot \left(\sum_{i=1}^{n} \Delta y_i \right) = c_{i_0} \cdot \Delta y.$$

Thus, the difference $f_x - f_y$ corresponding to the optimal spending cuts is equal to

$$f_x - f_y = \left(\min_i c_i\right) \cdot \Delta y. \tag{3}$$

Linearization simplifies the problem: case of across-the-board spending cuts. For across-the-board spending cuts, we have $y_i = (1-\delta) \cdot x_i$ and hence,

$$\Delta y_i = x_i - y_i = \delta \cdot x_i.$$

The coefficient δ can be obtained from the condition that $(1-\delta) \cdot x = y$, i.e., that $\Delta y = x - y = \delta \cdot x$, thus $\delta = \dfrac{\Delta y}{x}$.

Substituting the corresponding values Δy_i into the linearized expression for the objective function, we conclude that

$$\sum_{i=1}^{n} c_i \cdot \Delta y_i = \sum_{i=1}^{n} c_i \cdot \delta \cdot x_i = \delta \cdot \sum_{i=1}^{n} c_i \cdot x_i = \frac{\Delta y}{x} \cdot \sum_{i=1}^{n} c_i \cdot x_i = \Delta y \cdot \sum_{i=1}^{n} c_i \cdot \delta x_i,$$

where we denoted $\delta x_i \stackrel{\text{def}}{=} \dfrac{x_i}{x}$. From the constraint $\sum\limits_{i=1}^{n} x_i = x$, one can conclude that $\sum\limits_{i=1}^{n} \delta x_i = 1$. Thus, the resulting decrease $f_x - f_\delta$ is equal to:

$$f_x - f_\delta = \Delta y \cdot \sum_{i=1}^{n} c_i \cdot \delta x_i. \tag{4}$$

What we need to compare. To compare the decreases in the value of the objective function corresponding to the optimal cuts and to the across-the-board cuts, we therefore need to compare the expressions (3) and (4).

Let us treat the values c_i and δx_i as random variables. The values of c_i and Δx_i depend on many factors which we do not know beforehand, so it makes sense to treat them as random variables. In this case, both expressions (3) and (4) become random variables.

How we compare the random variables. Because of the related uncertainty, sometimes, the difference $f_x - f_\delta$ may be almost optimal, and sometimes, it may be much larger than the optimal difference $f_x - f_y$.

A reasonable way to compare two random variables is to compare their mean values. This is what we mean, e.g., when we say that Swedes are, on average taller than Americans: that the average height of a Swede is larger than the average height of an American.

It is reasonable to assume that the variables c_i and δx_i are all independent. Since we have no reason to believe that the variables c_i corresponding to different budget items and/or the variables δx_j are correlated, it makes sense to assume that these variables are independent. This conclusion is in line with the general Maximum Entropy approach to dealing with probabilistic knowledge: if there are several possible probability distributions consistent with our knowledge, it makes sense to select the one which has the largest uncertainty (entropy; see, e.g., [2, 7]), i.e., to select a distribution for which the entropy

$$S = -\int \rho(x) \cdot \ln(\rho(x))\,dx$$

attains the largest possible value, where $\rho(x)$ is the probability density function (pdf).

In particular, for the case when for two random variables, we only know their marginal distributions, with probability densities $\rho_1(x_1)$ and $\rho_2(x_2)$, the Maximum Entropy approach selects the joint probability distribution with the probability density $\rho(x_1,x_2) = \rho_1(x_1) \cdot \rho_2(x_2)$ that corresponds exactly to the case when these two random variables are independent.

Consequence of independence. In general, the mean $E[X+Y]$ of the sum is equal to the sum $E[X]+E[Y]$ of the means $E[X]$ and $E[Y]$. So, from the formula (4), we conclude that

$$E[f_x - f_\delta] = \Delta y \cdot \sum_{i=1}^{n} E[c_i \cdot \delta x_i].$$

Since we assume that for each i, the variables c_i and δx_i are independent, we conclude that

$$E[f_x - f_\delta] = \Delta y \cdot \sum_{i=1}^{n} E[c_i] \cdot E[\delta x_i]. \quad (5)$$

Here, we have no reason to believe that some values δx_i are larger, so it makes sense to assume that they have the same value of $E[\delta x_i]$. From the fact that $\sum\limits_{i=1}^{n} \delta x_i = 1$, we conclude that $\sum\limits_{i=1}^{n} E[\delta x_i] = 1$, i.e., that $n \cdot E[\delta x_i] = 1$. Thus, $E[\delta x_i] = \dfrac{1}{n}$, and the formula (5) takes the form

$$E[f_x - f_\delta] = \Delta y \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} E[c_i]. \quad (6)$$

Let us select distributions for c_i. Now, we need to compare:

- the value (6) corresponding to across-the-board cuts with
- the expected value of the optimal difference (3):

$$E[f_x - f_y] = \Delta y \cdot E\left[\min_i c_i\right]. \quad (7)$$

In both cases, the only remaining random variables are c_i, so to estimate these expressions, we need to select appropriate probability distributions for these variables.

We do not have much information about the values c_i. We know that $c_i \geq 0$. We also know that these values cannot be too large. Thus, we usually know an upper bound c on these values. Thus, for each i, the only information that we have about the corresponding random variable c_i is that it is located on the interval $[0,c]$.

Under this information, the Maximum Entropy approach recommends that we select the uniform distribution on this interval. This recommendation is in perfect accordance with common sense: if we have no reason to believe that some values from this interval are more probable or less probable then others, then it is reasonable to assume that all these values have the exact same probability, i.e., that the distribution is indeed unform.

Let us use the selected distributions to estimate the desired mean decreases (6) and (7). For the uniform distribution on the interval $[0,c]$, the mean value is known to be equal to the midpoint $\frac{c}{2}$ of this interval. Substituting $E[c_i]=\frac{c}{2}$ into the formula (6), we conclude that

$$E[f_x - f_\delta] = \frac{1}{2}\cdot \Delta y \cdot c. \tag{8}$$

To compute the estimate (7), let us first find the probability distribution for the minimum $m \stackrel{\text{def}}{=} \min\limits_i c_i$. This distribution can be deduced from the fact that for each value v, the minimum m is greater than v if and only if each of the coefficients c_i is greater than v:

$$m \ge v \Leftrightarrow (c_1 > v)\,\&\ldots\&\,(c_n > v).$$

Thus,

$$\text{Prob}(m > v) = \text{Prob}((c_1 > v)\,\&\ldots\&\,(c_n > v)).$$

Since the variables $c_1,\ldots,c_n$ are all independent, we have

$$\text{Prob}(m > v) = \text{Prob}(c_1 > v)\cdot\ldots\cdot\text{Prob}(c_n > v).$$

For each i, the random variable c_i is uniformly distributed on the interval $[0,c]$, so $\text{Prob}(c_i > v) = \frac{c-v}{c}$, and thus,

$$\text{Prob}(m > v) = \left(\frac{c-v}{c}\right)^n.$$

So, the cumulative distribution function (cdf)

$$F_m(v) = \text{Prob}(m \le v) = 1 - \text{Prob}(m > v)$$

is equal to:

$$F_m(v) = 1 - \left(\frac{c-v}{c}\right)^n.$$

By differentiating the cdf, we can get the formula for the corresponding probability density function (pdf)

$$\rho_m(v) = \frac{dF_m(v)}{dv} = \frac{n}{c^n}\cdot(c-v)^{n-1}.$$

Based on this pdf, we can compute the desired mean value:

$$E[m] = \int_0^c v \cdot \rho_m(v)\,dv = \int_0^c v \cdot \frac{n}{c^n} \cdot (c-v)^{n-1}\,dv.$$

By moving the constant factor outside the integral and by introducing a new auxiliary variable $w = c - v$ for which $v = c - w$ and $dv = -dw$, we can reduce this integral expression to a simpler-to-integrate form

$$E[m] = \frac{n}{c^n} \cdot \int_0^c (c-w) \cdot w^{n-1}\,dw = \frac{n}{c^n} \cdot \left(c \cdot \int_0^c w^{n-1}\,dw - \int_0^c w^n\,dw \right) =$$

$$\frac{n}{c^n} \cdot \left(c \cdot \frac{c^n}{n} - \frac{c^{n+1}}{n+1} \right) = \frac{n}{c^n} \cdot c^{n+1} \cdot \left(\frac{1}{n} - \frac{1}{n+1} \right) = c \cdot n \cdot \frac{1}{n \cdot (n+1)} = \frac{c}{n+1}.$$

Substituting the resulting expression

$$E\left[\min_i c_i\right] = \frac{c}{n+1}$$

into the formula (7), we conclude that

$$E[f_x - f_\delta] = \frac{1}{n+1} \cdot \Delta y \cdot c, \tag{9}$$

which is indeed much smaller than the expression (8).

Conclusion: across-the-board spending cuts are indeed very inefficient. In this paper, we compared the decreases in the value of the objective function for two possible ways of distributing the spending cuts:

- the optimal spending cuts, and
- the across-the-board spending cuts.

The resulting mean decreases are provided by the expressions (8) and (9). By comparing these expressions, we can conclude that the average decrease caused by the across-the-board cuts is $\frac{n+1}{2}$ larger than what is optimally possible, where n is the overall number of different budget items.

This result shows that on average, across-the-board cuts are indeed very inefficient.

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